Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 2 Issue 2 (2010), Pages 1-17.

ON THE GROWTH ESTIMATE OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

S. KR. DATTA AND T. BISWAS

Abstract. In this paper we study the growth properties of composite entire and meromorphic functions which improve some earlier results.

1. Introduction

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [7] and [4]. In the sequel we use the following notations:

$$\begin{aligned}\log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots \text{ and}\\ \log^{[0]} x &= x;\end{aligned}$$

and

$$\begin{aligned}\exp^{[k]} x &= \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots \text{ and}\\ \exp^{[0]} x &= x.\end{aligned}$$

The following definition is well known.

Definition. The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Juneja, Kapoor and Bajpai[5] defined the (p,q) th order and (p,q) th lower order of an entire function f respectively as follows :

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r},$$

where p, q are positive integers with $p > q$.

2000 *Mathematics Subject Classification.* 30D30, 30D35.
Key words and phrases. Order, lower order, (p,q)-th order, (p,q)-th lower order, entire function, meromorphic function, composition, growth.

Submitted November, 2009. Published April, 2010.

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.
If $p = 2$ and $q = 1$ then we write $\rho_f(2,1) = \rho_f$ and $\lambda_f(2,1) = \lambda_f$.

In this paper we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of (p,q) th order ((p,q) th lower order) improving some earlier results where p, q are any two positive integers with $p > q$.

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [1] *If f is a meromorphic function and g is an entire function then for all sufficiently large values of r,*

$$T(r, f\circ g) \leqslant \{1+o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma 2.2. [2] *Let f be a meromorphic function and g be an entire function and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T(r, f\circ g) \geq T(\exp(r^\mu), f).$$

Lemma 2.3. [3] *If f and g are entire functions then for all sufficiently large values of r,*

$$M(r, f\circ g) \geq M(\frac{1}{8} M(\frac{r}{2}, g) - |g(0)|, f).$$

3. Theorems.

In this section we present the main results of the paper.

Theorem 3.1. *Let g be entire function and h, k be two transcendental entire functions such that $\lambda_h(a,b) > 0$, $\lambda_k(c,d) > 0$ and $\rho_g(m,n) < \lambda_k(c,d)$ where m, n, a, b, c, d are all positive integers with $m > n$, $a > b$ and $c > d$. Then for every meromorphic function f with $0 < \rho_f(p,q) < \infty$ and for any two positive integers p, q with $p > q$*

$$(i) \lim_{r\to\infty} \frac{\log^{[a-1]} T(\exp^{[d-1]} r, h\circ k)}{\log^{[p-1]} T(r, f\circ g)} = \infty \textit{ if } q \geqslant m \textit{ and } b < c,$$

$$(ii) \lim_{r\to\infty} \frac{\log^{[a-1]} T(\exp^{[d-1]} r, h\circ k)}{\log^{[p+m-q-2]} T(r, f\circ g)} = \infty \textit{ if } q < m \textit{ and } b < c,$$

$$(iii) \lim_{r\to\infty} \frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h\circ k)}{\log^{[p-1]} T(r, f\circ g)} = \infty \textit{ if } q \geqslant m \textit{ and } b \geqslant c,$$

$$\textit{and } (iv) \lim_{r\to\infty} \frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h\circ k)}{\log^{[p+m-q-2]} T(r, f\circ g)} = \infty \textit{ if } q < m \textit{ and } b \geqslant c.$$

Proof. *In view of Lemma 1 and the inequality* $T(r,g) \leqslant \log^{+} M(r,g)$ *we obtain for all sufficiently large values of* r *that*

$$\log^{[p-1]} T(r, f\circ g) \leqslant \log^{[p-1]} T(M(r,g), f) + O(1)$$
$$i.e.,\ \log^{[p-1]} T(r, f\circ g) \leqslant (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r,g) + O(1). \tag{3.1}$$

Now the following cases may arise :

Case I. *Let* $q \geqslant m$. *Then we have from* (3.1) *for all sufficiently large values of* r,

$$\log^{[p-1]} T(r, f\circ g) \leqslant (\rho_f(p,q)+\varepsilon)\log^{[m-1]} M(r,g) + O(1). \tag{3.2}$$

Now from the definition of (m,n) *th order of* g *we get for arbitrary positive* ε *and for all sufficiently large values of* r,

$$\log^{[m]} M(r,g) \leqslant (\rho_g(m,n)+\varepsilon)\log^{[n]} r$$
$$i.e.,\ \log^{[m]} M(r,g) \leqslant (\rho_g(m,n)+\varepsilon)\log r. \tag{3.3}$$

Also for all sufficiently large values of r *it follows from* (3.3) *that*

$$\log^{[m-1]} M(r,g) \leqslant r^{(\rho_g(m,n)+\varepsilon)}. \tag{3.4}$$

So from (3.2) *and* (3.4) *it follows for all sufficiently large values of* r *that*

$$\log^{[p-1]} T(r, f\circ g) \leqslant (\rho_f(p,q)+\varepsilon)\, r^{(\rho_g(m,n)+\varepsilon)} + O(1). \tag{3.5}$$

Case II. *Let* $q < m$. *Then we get from*(3.1) *for all sufficiently large values of* r *that*

$$\log^{[p-1]} T(r, f\circ g) \leqslant (\rho_f(p,q)+\varepsilon)\exp^{[m-q]}\log^{[m]} M(r,g) + O(1). \tag{3.6}$$

Again from (3.3) *for all sufficiently large values of* r,

$$\exp^{[m-q]}\log^{[m]} M(r,g) \leqslant \exp^{[m-q]}\log r^{(\rho_g(m,n)+\varepsilon)}$$
$$i.e.,\ \exp^{[m-q]}\log^{[m]} M(r,g) \leqslant \exp^{[m-q-1]} r^{(\rho_g(m,n)+\varepsilon)}. \tag{3.7}$$

Now from (3.6) *and* (3.7) *we obtain for all sufficiently large values of* r *that*

$$\log^{[p-1]} T(r, f\circ g) \leqslant (\rho_f(p,q)+\varepsilon)\exp^{[m-q-1]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$
$$i.e.,\ \log^{[p]} T(r, f\circ g) \leqslant \exp^{[m-q-2]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$

$$i.e.,\ \log^{[p+m-q-2]} T(r, f\circ g) \leqslant \log^{[m-q-2]}\exp^{[m-q-2]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$
$$i.e.,\ \log^{[p+m-q-2]} T(r, f\circ g) \leqslant r^{(\rho_g(m,n)+\varepsilon)} + O(1). \tag{3.8}$$

Since $\rho_g(m,n) < \lambda_k(c,d)$ *we can choose* $\varepsilon\,(>0)$ *in such a way that*

$$\rho_g(m,n) + \varepsilon < \lambda_k(c,d) - \varepsilon. \tag{3.9}$$

Now using the inequality $T(r, h\circ k) \geqslant \frac{1}{3}\log\left\{\frac{1}{8}M(\frac{r}{4},k) + \circ(1), h\right\}$ $\{cf.$ [6]$\}$ *we obtain for all large values of* r *that*

$$\log^{[a-1]} T(\exp^{[d-1]} r, h\circ k)$$
$$\geqslant \log^{[a]}\left\{\frac{1}{8}M(\frac{\exp^{[d-1]} r}{4},k) + \circ(1), h\right\} + O(1)$$
$$i.e.,\ \log^{[a-1]} T(\exp^{[d-1]} r, h\circ k)$$
$$\geqslant (\lambda_h(a,b)-\varepsilon)\log^{[b]}\left\{\frac{1}{9}M(\frac{\exp^{[d-1]} r}{4},k)\right\} + O(1)$$

$$\begin{aligned} & i.e.,\ \log^{[a-1]} T(\exp^{[d-1]} r, h \circ k) \\ \geqslant & \ (\lambda_h(a,b) - \varepsilon) \log^{[b]} M(\frac{\exp^{[d-1]} r}{4}, k) + O(1). \end{aligned} \tag{3.10}$$

Case III. *Let $b < c$. Then from (3.10) it follows for all sufficiently large values of r that*

$$\begin{aligned} & \log^{[a-1]} T(\exp^{[d-1]} r, h \circ k) \\ & \geqslant (\lambda_h(a,b) - \varepsilon) \exp^{[c-b-1]} \log^{[c-1]} M(\frac{\exp^{[d-1]} r}{4}, k) + O(1). \end{aligned} \tag{3.11}$$

Now from the definition of (c,d) th lower order of k we obtain for arbitrary positive $\varepsilon\,(>0)$ and for all sufficiently large values of r that

$$\begin{aligned} \log^{[c]} M(\frac{\exp^{[d-1]} r}{4}, k) & \geqslant (\lambda_k(c,d) - \varepsilon) \log^{[d]}(\frac{\exp^{[d-1]} r}{4}) \\ i.e., \quad \log^{[c]} M(\frac{\exp^{[d-1]} r}{4}, k) & \geqslant (\lambda_k(c,d) - \varepsilon) \log r + O(1) \\ i.e., \quad \log^{[c]} M(\frac{\exp^{[d-1]} r}{4}, k) & \geqslant \log r^{(\lambda_k(c,d) - \varepsilon)} + O(1). \end{aligned} \tag{3.12}$$

Also for all large values of r we get from (3.12) that

$$\log^{[c-1]} M(\frac{\exp^{[d-1]} r}{4}, k) \geqslant r^{(\lambda_k(c,d) - \varepsilon)} + O(1). \tag{3.13}$$

Now from (3.11) and (3.13) it follows for all sufficiently large values of r that

$$\log^{[a-1]} T(\exp^{[d-1]} r, h \circ k) \geqslant (\lambda_h(a,b) - \varepsilon) \exp^{[c-b-1]} r^{(\lambda_k(c,d) - \varepsilon)} + O(1). \tag{3.14}$$

Case IV. *Let $b \geqslant c$. Then from (3.10) we obtain for all sufficiently large values of r,*

$$\begin{aligned} & \log^{[a-1]} T(\exp^{[d-1]} r, h \circ k) \\ & \qquad \geqslant (\lambda_h(a,b) - \varepsilon) \log^{[b-c]} \log^{[c]} \left\{ M(\frac{\exp^{[d-1]} r}{4}, k) \right\} + O(1). \end{aligned} \tag{3.15}$$

Now from (3.12) and (3.15) we have for all sufficiently large values of r,

$$\begin{aligned} & \log^{[a-1]} T(\exp^{[d-1]} r, h \circ k) \\ & \qquad \geqslant (\lambda_h(a,b) - \varepsilon) \log^{[b-c]} \log r^{(\lambda_k(c,d) - \varepsilon)} + O(1) \\ & i.e.,\ \log^{[a-1]} T(\exp^{[d-1]} r, h \circ k) \\ & \qquad \geqslant (\lambda_h(a,b) - \varepsilon) \log^{[b-c+1]} r^{(\lambda_k(c,d) - \varepsilon)} + O(1) \\ & i.e.,\ \log^{[a]} T(\exp^{[d-1]} r, h \circ k) \geqslant \log^{[b-c+2]} r^{(\lambda_k(c,d) - \varepsilon)} + O(1) \\ & i.e.,\ \log^{[a+c-b-2]} T(\exp^{[d-1]} r, h \circ k) \geqslant r^{(\lambda_k(c,d) - \varepsilon)} + O(1). \end{aligned} \tag{3.16}$$

Now combining (3.5) of Case I and (3.14) of Case III it follows for all sufficiently large values of r that

$$\frac{\log^{[a-1]} T(\exp^{[d-1]} r, h \circ k)}{\log^{[p-1]} T(r, f \circ g)} \geq \frac{(\lambda_h(a,b) - \varepsilon) \exp^{[c-b-1]} r^{(\lambda_k(c,d) - \varepsilon)} + O(1)}{(\rho_f(p,q) + \varepsilon) r^{(\rho_g(m,n) + \varepsilon)} + O(1)}$$

$$i.e.,\ \liminf_{r \to \infty} \frac{\log^{[a-1]} T(\exp^{[d-1]} r, h \circ k)}{\log^{[p-1]} T(r, f \circ g)} = \infty,$$

from which the first part of the theorem follows.
Again combining (3.8) *of Case II and* (3.14) *of Case III we obtain for all sufficiently large values of* r *that*

$$\frac{\log^{[a-1]} T(\exp^{[d-1]} r, h \circ k)}{\log^{[p+m-q-2]} T\left(r, f \circ g\right)} \geq \frac{(\lambda_h(a,b) - \varepsilon) \exp^{[c-b-1]} r^{(\lambda_k(c,d)-\varepsilon)} + O(1)}{r^{(\rho_g(m,n)+\varepsilon)} + O(1)}$$

$$i.e., \liminf_{r\to\infty} \frac{\log^{[a-1]} T(\exp^{[d-1]} r, h \circ k)}{\log^{[p+m-q-2]} T\left(r, f \circ g\right)} = \infty$$

$$i.e., \lim_{r\to\infty} \frac{\log^{[a-1]} T(\exp^{[d-1]} r, h \circ k)}{\log^{[p+m-q-2]} T\left(r, f \circ g\right)} = \infty.$$

This establishes the second part of the theorem.
Now in view of (3.5) *of Case I and* (3.16) *Case IV we get for all sufficiently large values of* r *that*

$$\frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h \circ k)}{\log^{[p-1]} T\left(r, f \circ g\right)} \geq \frac{r^{(\lambda_k(c,d)-\varepsilon)} + O(1)}{\left(\rho_f\left(p,q\right) + \varepsilon\right) r^{(\rho_g(m,n)+\varepsilon)} + O(1)}. \qquad (3.17)$$

So from (3.9) *and* (3.17) *we obtain that*

$$\liminf_{r\to\infty} \frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h \circ k)}{\log^{[p-1]} T\left(r, f \circ g\right)} = \infty,$$

from which the third part of the theorem follows.
Again combining (3.8) *of Case II and* (3.16) *of Case IV it follows for all sufficiently large values of* r *that*

$$\frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h \circ k)}{\log^{[p+m-q-2]} T\left(r, f \circ g\right)} \geq \frac{r^{(\lambda_k(c,d)-\varepsilon)} + O(1)}{r^{(\rho_g(m,n)+\varepsilon)} + O(1)}. \qquad (3.18)$$

Now in view of (3.9) *we obtain from* (3.18) *that*

$$\liminf_{r\to\infty} \frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h \circ k)}{\log^{[p+m-q-2]} T\left(r, f \circ g\right)} = \infty$$

$$i.e., \lim_{r\to\infty} \frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h \circ k)}{\log^{[p+m-q-2]} T\left(r, f \circ g\right)} = \infty.$$

This proves the fourth part of the theorem.
Thus the theorem follows.

Remark. The conditions $\lambda_h\left(a,b\right) > 0$, $\rho_g\left(m,n\right) < \lambda_k\left(c,d\right)$ and $\rho_f\left(p,q\right) < \infty$ in Theorem 1 are necessary which are evident from the following examples.

Example. Let

$$f = g = h = \exp z \text{ and } k = \exp\left(z^2\right).$$

Also let

$$a = 3\ ,\ p = m = c = 2 \text{ and } q = n = b = d = 1.$$

Then

$$\rho_f = 1, \rho_g = 1 < 2 = \lambda_k \text{ and } \bar{\lambda}_h = \lambda_h\left(3,1\right) = 0.$$

Now

$$T(r, h \circ k) \leq \log M(r, h \circ k) = \log \exp^{[2]} r^2 = \exp r^2$$

and

$$T(r, f\circ g) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}.$$

So

$$\begin{aligned}\frac{\log^{[2]} T(r, h\circ k)}{\log T(r, f\circ g)} &\leq \frac{2\log r}{r - \frac{1}{2}\log r + O(1)} \\ i.e.,\ \lim_{r\to\infty} \frac{\log^{[2]} T(r, h\circ k)}{\log T(r, f\circ g)} &= 0.\end{aligned}$$

Example. Let

$$f = h = k = \exp z \text{ and } g = \exp\left(z^2\right).$$

Also let

$$p = m = a = c = 2 \text{ and } q = n = b = d = 1.$$

Then

$$\rho_f = 1,\ \rho_g = 2 > 1 = \lambda_k \text{ and } \lambda_h = 1.$$

Now

$$\begin{aligned}T(r, h\circ k) &\sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \\ i.e.,\ \log T(r, h\circ k) &\sim r - \frac{1}{2}\log r + O(1)\end{aligned}$$

and

$$\begin{aligned}T(r, f\circ g) &\geq \frac{1}{3}\log M(\frac{r}{2}, f\circ g) \\ i.e.,\ \log T(r, f\circ g) &\geq \log^{[2]}\exp^{[2]}\left(\frac{r^2}{4}\right) + O(1) \\ i.e.,\ \log T(r, f\circ g) &\geq \frac{r^2}{4} + O(1).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} &\leq \frac{r - \frac{1}{2}\log r + O(1)}{\frac{r^2}{4} + O(1)} \\ i.e.,\ \lim_{r\to\infty} \frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} &= 0,\end{aligned}$$

which is contrary to Theorem 1.

Example. Let

$$f = g = h = k = \exp z \text{ and } p = m = a = c = 2 \text{ and } q = n = b = d = 1.$$

Then

$$\rho_f = 1, \rho_g = 1 = \lambda_k \text{ and } \lambda_h = 1.$$

Now

$$\begin{aligned}T(r, h\circ k) &\sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \\ i.e.,\ \log T(r, h\circ k) &\sim r - \frac{1}{2}\log r + O(1)\end{aligned}$$

and
$$\begin{aligned} T(r, f\circ g) &\geq \frac{1}{3}\log M(\frac{r}{2}, f\circ g) \\ i.e.,\ \log T(r, f\circ g) &\geq \log^{[2]} M(\frac{r}{2}, f\circ g) + O(1) \\ i.e.,\ \log T(r, f\circ g) &\geq \log^{[2]} \exp^{[2]}\left(\frac{r}{2}\right) + O(1) \\ i.e.,\ \log T(r, f\circ g) &\geq \frac{r}{2} + O(1). \end{aligned}$$

So we get that
$$\begin{aligned} \frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} &\leq \frac{r - \frac{1}{2}\log r + O(1)}{\frac{r}{2} + O(1)} \\ i.e.,\ \lim_{r\to\infty} \frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} &\leq 2. \end{aligned}$$

Example. Let
$$f = \exp^{[2]} z,\ \ g = h = \exp z,\ k = \exp\left(z^2\right)$$
and
$$p = m = a = c = 2 \text{ and } q = n = b = d = 1.$$
Then
$$\rho_f = \infty, \rho_g = 1 < 2 = \lambda_k \text{ and } \lambda_h = 1.$$
Now
$$\begin{aligned} &T(r, h\circ k) \leq \log M(r, h\circ k) = \log\exp^{[2]}\left(r^2\right) = \exp\left(r^2\right) \\ &i.e.,\ \log T(r, h\circ k) \leq r^2 \end{aligned}$$
and
$$\begin{aligned} T(r, f\circ g) &\geq \frac{1}{3}\log M(\frac{r}{2}, f\circ g) \\ i.e.,\ \log T(r, f\circ g) &\geq \log^{[2]} M(\frac{r}{2}, f\circ g) + O(1) \\ i.e.,\ \log T(r, f\circ g) &\geq \log^{[2]} \exp^{[3]}\left(\frac{r}{2}\right) + O(1) \\ i.e.,\ \log T(r, f\circ g) &\geq \exp\left(\frac{r}{2}\right) + O(1). \end{aligned}$$

Hence
$$\begin{aligned} \frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} &\leq \frac{r^2}{\exp\left(\frac{r}{2}\right) + O(1)} \\ i.e.,\ \lim_{r\to\infty} \frac{\log T(r, h\circ k)}{\log T(r, f\circ g)} &= 0, \end{aligned}$$
which contradicts Theorem 1.

Remark. The condition $\rho_g(m,n) < \lambda_k(c,d)$ in Theorem 1 is necessary, which is true in general only if $\rho_f(p,q) > 0$ otherwise the condition $\rho_g(m,n) < \lambda_k(c,d)$ will be violated.The following two examples strengthen this comment.

Example. Let
$$g = \exp\left(z^2\right) \text{ and } f = h = k = \exp z.$$
Also let
$$p = 3\ ,\ m = a = c = 2 \text{ and } q = n = b = d = 1.$$

Then

$$\bar{\rho}_f = \rho_f(3,1) = 0 < \infty, \ \rho_g = 2 > 1 = \lambda_k \text{ and } \lambda_h = 1.$$

Now

$$\begin{aligned} T(r, h \circ k) &\sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \\ i.e., \ \log T(r, h \circ k) &\sim r - \frac{1}{2}\log r + O(1) \end{aligned}$$

and

$$\begin{aligned} T(r, f \circ g) &\leq \log M(r, f \circ g) = \exp r^2 \\ i.e., \ \log^{[2]} T(r, f \circ g) &\leq 2\log r. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\log T(r, h \circ k)}{\log^{[2]} T(r, f \circ g)} &\geq \frac{r - \frac{1}{2}\log r + O(1)}{2\log r} \\ i.e., \lim_{r\to\infty} \frac{\log T(r, h \circ k)}{\log^{[2]} T(r, f \circ g)} &= \infty. \end{aligned}$$

Example. Let

$$f = g = h = k = \exp z.$$

Also let

$$p = 3\ , \ m = a = c = 2 \text{ and } q = n = b = d = 1.$$

Then

$$\bar{\rho}_f = \rho_f(3,1) = 0 < \infty, \ \rho_g = 1 = \lambda_k \text{ and } \lambda_h = 1.$$

Now

$$\begin{aligned} T(r, h \circ k) &\sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \\ i.e., \ \log T(r, h \circ k) &\sim r - \frac{1}{2}\log r + O(1) \end{aligned}$$

and

$$\begin{aligned} T(r, f \circ g) &\leq \log M(r, f \circ g) = \exp r \\ i.e., \ \log^{[2]} T(r, f \circ g) &\leq \log r. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\log T(r, h \circ k)}{\log^{[2]} T(r, f \circ g)} &= \frac{r - \frac{1}{2}\log r + O(1)}{\log r} \\ i.e., \lim_{r\to\infty} \frac{\log T(r, h \circ k)}{\log^{[2]} T(r, f \circ g)} &= \infty. \end{aligned}$$

Theorem 3.2. *Let h be meromorphic and g, k be entire such that $\lambda_h(a,b) > 0$, $0 < \rho_k < \infty$ and $\rho_g(m,n) < \rho_k$ where m, n, a, b are all positive integers with $m > n$*

and $a > b$. Then for every meromorphic function f with $0 < \rho_f(p,q) < \infty$ and for any two positive integers p, q with $p > q$

$$(i)\quad \limsup_{r\to\infty} \frac{\log^{[a-1]} T(r, h\circ k)}{\log^{[p-1]} T(r, f\circ g) + \log^{[m]} M(r,g)} = \infty \quad \text{if } b = 1 \text{ and } q \geqslant m;$$

$$(ii)\quad \limsup_{r\to\infty} \frac{\log^{[a-1]} T(r, h\circ k)}{\log^{[p+m-q-2]} T(r, f\circ g) + \log^{[m]} M(r,g)} = \infty \quad \text{if } b = 1 \text{ and } q < m;$$

$$(iii)\quad \limsup_{r\to\infty} \frac{\log^{[a-1]} T(r, h\circ k)}{\log^{[p-1]} T(r, f\circ g) + \log^{[m]} M(r,g)} = \infty \quad \text{if } q \geqslant m \text{ and } 1 < b < n+1;$$

$$(iv)\quad \limsup_{r\to\infty} \frac{\log^{[a-1]} T(r, h\circ k)}{\log^{[p-1]} T(r, f\circ g) + \log^{[m]} M(r,g)} \geq \frac{\mu\lambda_h(a,b)}{(\rho_f(p,q)+1)\,\rho_g(m,n)} \quad \text{if } q \geq m,\ b = n = 2 \text{ and } 0 < \mu < \rho_k;$$

$$(v)\quad \limsup_{r\to\infty} \frac{\log^{[a-1]} T(r, h\circ k)}{\log^{[p-1]} T(r, f\circ g) + \log^{[m]} M(r,g)} \geq \frac{\lambda_h(a,b)}{(\rho_f(p,q)+1)\,\rho_g(m,n)} \quad \text{if } q \geq m \text{ and } b = n > 2;$$

$$(vi)\quad \limsup_{r\to\infty} \frac{\log^{[a-1]} T(r, h\circ k)}{\log^{[p+m+n-q-2]} T(r, f\circ g) + \log^{[m]} M(r,g)} = \infty \quad \text{if } q < m \text{ and } 1 < b < n+1;$$

$$(vii)\quad \limsup_{r\to\infty} \frac{\log^{[a-1]} T(r, h\circ k)}{\log^{[p+m+n-q-2]} T(r, f\circ g) + \log^{[m]} M(r,g)} \geq \frac{\mu\lambda_h(a,b)}{1+\rho_g(m,n)} \quad \text{if } q < m\ ,\ b = n = 2 \text{ and } 0 < \mu < \rho_k$$

and

$$(viii)\quad \limsup_{r\to\infty} \frac{\log^{[a-1]} T(r, h\circ k)}{\log^{[p+m+n-q-2]} T(r, f\circ g) + \log^{[m]} M(r,g)} \geq \frac{\lambda_h(a,b)}{1+\rho_g(m,n)} \quad \text{if } q < m \text{ and } b = n > 2.$$

Proof. Since $\rho_g(m,n) < \rho_k$ we can choose $\varepsilon(> 0)$ in such a way that

$$\rho_g(m,n) + \varepsilon < \mu < \rho_k - \varepsilon. \tag{3.19}$$

By Lemma 2 we obtain for a sequence of values of r tending to infinity,

$$T(r, h\circ k) \geq T(\exp(r^\mu), h), \text{ where } 0 < \mu < \rho_k \leq \infty$$

$$\begin{aligned} i.e.,\ \log^{[a-1]} T(r, h\circ k) &\geq \log^{[a-1]} T(\exp(r^\mu), h) \\ i.e.,\ \log^{[a-1]} T(r, h\circ k) &\geq (\lambda_h(a,b) - \varepsilon) \log^{[b]} \exp(r^\mu) \\ i.e.,\ \log^{[a-1]} T(r, h\circ k) &\geq (\lambda_h(a,b) - \varepsilon) \log^{[b-1]}(r^\mu). \end{aligned} \tag{3.20}$$

Now the following two cases may arise :

Case I. Let $b = 1$. Then from (3.20) we get for a sequence of values of r tending to infinity that

$$\log^{[a-1]} T(r, h\circ k) \geq (\lambda_h(a,b) - \varepsilon)(r^\mu). \tag{3.21}$$

Case II. Let $b-1=l>0$. Then from (3.20) it follows for a sequence of values of r tending to infinity that

$$\log^{[a-1]} T(r,h\circ k) \geq (\lambda_h(a,b)-\varepsilon)\log^{[l]}(r^{\mu}). \tag{3.22}$$

Now from the definition of (m,n) th order of g we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[m]} M(r,g) \leqslant (\rho_g(m,n)+\varepsilon)\log^{[n]} r. \tag{3.23}$$

Let $q\geqslant m$. Then we have from (3.1) and (3.23) for all sufficiently large values of r,

$$\log^{[p-1]} T(r,f\circ g) \leq (\rho_f(p,q)+\varepsilon)(\rho_g(m,n)+\varepsilon)\log^{[n]} r. \tag{3.24}$$

Now if $b=1$ and $q\geqslant m$, we get from (3.5) , (3.21) , (3.23) and in view of (3.19) for a sequence of values of r tending to infinity,

$$\frac{\log^{[a-1]} T(r,h\circ k)}{\log^{[p-1]} T(r,f\circ g)+\log^{[m]} M(r,g)}$$
$$\geq \frac{(\lambda_h(a,b)-\varepsilon)(r^{\mu})}{(\rho_f(p,q)+\varepsilon)\, r^{(\rho_g(m,n)+\varepsilon)}+(\rho_g(m,n)+\varepsilon)\log^{[n]} r+O(1)}$$
$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[a-1]} T(r,h\circ k)}{\log^{[p-1]} T(r,f\circ g)+\log^{[m]} M(r,g)}=\infty,$$

which proves the first part of the theorem.
Again we obtain from (3.8) , (3.19) ,(3.22) and (3.23) for a sequence of values of r tending to infinity when $b=1$ and $q<m$

$$\frac{\log^{[a-1]} T(r,h\circ k)}{\log^{[p+m-q-2]} T(r,f\circ g)+\log^{[m]} M(r,g)}$$
$$\geq \frac{(\lambda_h(a,b)-\varepsilon)(r^{\mu})}{r^{(\rho_g(m,n)+\varepsilon)}+(\rho_g(m,n)+\varepsilon)\log^{[n]} r+O(1)}$$
$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[a-1]} T(r,h\circ k)}{\log^{[p+m-q-2]} T(r,f\circ g)+\log^{[m]} M(r,g)}=\infty.$$

This proves the second part of the theorem.
When $b>1$ and $q\geq m$, from (3.22) , (3.23) , and (3.24) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[a-1]} T(r,h\circ k)}{\log^{[p-1]} T(r,f\circ g)+\log^{[m]} M(r,g)}$$
$$\geq \frac{(\lambda_h(a,b)-\varepsilon)\log^{[l]}(r^{\mu})}{(\rho_f(p,q)+\varepsilon)(\rho_g(m,n)+\varepsilon)\log^{[n]} r+(\rho_g(m,n)+\varepsilon)\log^{[n]} r+O(1)}$$
$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[a-1]} T(r,h\circ k)}{\log^{[p-1]} T(r,f\circ g)+\log^{[m]} M(r,g)}=\infty \text{ if } 1<b<n+1;$$

again

$$\limsup_{r\to\infty}\frac{\log^{[a-1]} T(r,h\circ k)}{\log^{[p-1]} T(r,f\circ g)+\log^{[m]} M(r,g)} \geq \frac{\mu\lambda_h(a,b)}{(\rho_f(p,q)+1)\,\rho_g(m,n)}$$
$$\text{if } b=n=2 \text{ and } 0<\mu<\rho_k$$

and also

$$\limsup_{r\to\infty}\frac{\log^{[a-1]}T(r,h\circ k)}{\log^{[p-1]}T(r,f\circ g)+\log^{[m]}M(r,g)}\geq\frac{\lambda_h(a,b)}{(\rho_f(p,q)+1)\rho_g(m,n)}$$

$$\text{if } b=n>2 \text{ and } 0<\mu<\rho_k.$$

This respectively proves the third , fourth and fifth part of the theorem.
Again when $b>1$ and $q<m$, combining (3.8) , (3.22) and (3.23) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[a-1]}T(r,h\circ k)}{\log^{[p+m+n-q-2]}T(r,f\circ g)+\log^{[m]}M(r,g)}$$

$$\geq\frac{(\lambda_h(a,b)-\varepsilon)\log^{[l]}(r)^{\mu}}{\log^{[n]}r+(\rho_g(m,n)+\varepsilon)\log^{[n]}r+O(1)}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[a-1]}T(r,h\circ k)}{\log^{[p+m+n-q-2]}T(r,f\circ g)+\log^{[m]}M(r,g)}=\infty \text{ if } 1<b<n+1;$$

also

$$\limsup_{r\to\infty}\frac{\log^{[a-1]}T(r,h\circ k)}{\log^{[p+m+n-q-2]}T(r,f\circ g)+\log^{[m]}M(r,g)}\geq\frac{\mu\lambda_h(a,b)}{1+\rho_g(m,n)}$$

$$\text{if } b=n=2 \text{ and } 0<\mu<\rho_k$$

and again

$$\limsup_{r\to\infty}\frac{\log^{[a-1]}T(r,h\circ k)}{\log^{[p+m+n-q-2]}T(r,f\circ g)+\log^{[m]}M(r,g)}\geq\frac{\lambda_h(a,b)}{1+\rho_g(m,n)}$$

$$\text{if } b=n>2 \text{ and } 0<\mu<\rho_k.$$

from which the sixth , seventh and eighth part of the theorem respectively follows.

Remark. The condition $\rho_g(m,n)<\rho_k$ and $\rho_f(p,q)<\infty$ in Theorem 2 are essential as we see in the following examples.

Example. Let

$$f=g=h=k=\exp z.$$

Also let

$$p=m=a=2 \text{ and } q=n=b=1.$$

Then

$$\rho_f=1,\ \rho_g=1=\rho_k \text{ and } \lambda_h=1.$$

Now

$$\begin{aligned}T(r,f\circ g)&=T(r,h\circ k)\sim\frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\\ i.e., \log T(r,f\circ g)&=\log T(r,h\circ k)\sim r-\frac{1}{2}\log r+O(1)\end{aligned}$$

So

$$\limsup_{r\to\infty}\frac{\log T(r,h\circ k)}{\log T(r,f\circ g)+\log^{[2]}M(r,g)}=\limsup_{r\to\infty}\frac{r-\frac{1}{2}\log r+O(1)}{r-\frac{1}{2}\log r+O(1)+\log r}$$

$$
\begin{aligned}
& i.e.,\ \limsup_{r\to\infty}\frac{\log T(r,h\circ k)}{\log T(r,f\circ g)+\log^{[2]}M(r,g)} \\
= & \limsup_{r\to\infty}\frac{r-\frac{1}{2}\log r+O(1)}{r+\frac{1}{2}\log r+O(1)}=1,
\end{aligned}
$$

which is contrary to Theorem 2.

Example. Let

$$f=\exp^{[2]}z\ ,g=h=\exp z \text{ and } k=\exp\left(z^2\right).$$

and

$$p=m=a=2 \text{ and } q=n=b=1.$$

Then

$$\rho_f=\infty,\ \rho_g=1<2=\rho_k \text{ and } \lambda_h=1.$$

Now

$$
\begin{aligned}
T(r,h\circ k) &\leq \log M(r,h\circ k)=\log\exp^{[2]}\left(r^2\right) \\
i.e.,\ T(r,h\circ k) &\leq \exp\left(r^2\right), \\
\text{and } T(r,f\circ g) &\geq \frac{1}{3}\log M\left(\frac{r}{2},f\circ g\right) \\
i.e.,\ T(r,f\circ g) &\geq \frac{1}{3}\log\exp^{[3]}\left(\frac{r}{2}\right)=\frac{1}{3}\exp^{[2]}\left(\frac{r}{2}\right) \\
\text{and } \log^{[2]}M(r,g) &= \log r.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{\log T(r,h\circ k)}{\log T(r,f\circ g)+\log^{[2]}M(r,g)} &\leq \frac{\log\exp\left(r^2\right)}{\log\exp^{[2]}\left(\frac{r}{2}\right)+O(1)+\log^{[2]}\exp r} \\
i.e.,\ \frac{\log T(r,h\circ k)}{\log T(r,f\circ g)+\log^{[2]}M(r,g)} &\leq \frac{r^2}{\exp\left(\frac{r}{2}\right)+\log r+O(1)} \\
i.e.,\ \limsup_{r\to\infty}\frac{\log T(r,h\circ k)}{\log T(r,f\circ g)+\log^{[2]}M(r,g)} &= 0,
\end{aligned}
$$

which is contrary to Theorem 2.

Remark. The condition $\rho_g\left(m,n\right)<\lambda_k$ in Theorem 2 is necessary which is true in general only if $\rho_f\left(p,q\right)>0$ otherwise the condition $\rho_g\left(m,n\right)<\rho_k$ will be violated.The following example ensure this comment.

Example. Let

$$f=h=k=\exp z \text{ and } g=\exp\left(z^3\right).$$

Also let

$$p=3\ ,\ m=a=2 \text{ and } q=n=b=1.$$

Then

$$\bar{\rho}_f=\rho_f\left(3,1\right)=0<\infty,\ \rho_g=3>1=\rho_k \text{ and } \lambda_h=1.$$

Now

$$
\begin{aligned}
T(r,h\circ k) &\sim \frac{\exp r}{\left(2\pi^3 r\right)^{\frac{1}{2}}} \\
i.e.,\ \log T(r,h\circ k) &\sim r-\frac{1}{2}\log r+O(1),
\end{aligned}
$$

$$\begin{aligned} T(r, f\circ g) &\leq \log M(r, f\circ g) = \exp r^3 \\ i.e.,\ \log^{[2]} T(r, f\circ g) &\leq 3\log r. \\ \text{and } \log^{[2]} M(r,g) &= 3\log r. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\log T(r, h\circ k)}{\log^{[2]} T(r, f\circ g) + \log^{[2]} M(r,g)} &\geq \frac{r - \frac{1}{2}\log r + O(1)}{6\log r} \\ i.e., \lim_{r\to\infty} \frac{\log T(r, h\circ k)}{\log^{[2]} T(r, f\circ g) + \log^{[2]} M(r,g)} &= \infty. \end{aligned}$$

Theorem 3.3. *Let f, g be entire functions such that $0 < \lambda_f(p,q) \leq \rho_f(p,q) < \infty$ and $\lambda_g(m,n) > 0$ where p, q, m, n are positive integers with $p > q$ and $m > n$. Then for any positive integer l,*

$$(i)\qquad \lim_{r\to\infty} \frac{\log^{[p]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p+1]} M\left(\exp^{[l]} r, f\right)} = \infty \text{ if } q < m \text{ and } q \geqslant l;$$

$$(ii)\qquad \limsup_{r\to\infty} \frac{\log^{[p]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p+1]} M\left(\exp^{[l]} r, f\right)} \geq \lambda_f(p,q)\,\lambda_g(m,n) \quad \text{if } q = m \text{ and } q \geqslant l;$$

$$(iii)\qquad \lim_{r\to\infty} \frac{\log^{[p]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p-q-l+1]} M\left(\exp^{[l]} r, f\right)} = \infty \text{ if } q < m \text{ and } q < l;$$

$$(iv)\qquad \limsup_{r\to\infty} \frac{\log^{[p]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p-q-l+1]} M\left(\exp^{[l]} r, f\right)} \geq \lambda_f(p,q)\,\lambda_g(m,n) \quad \text{if } q = m \text{ and } q < l;$$

$$(v)\qquad \lim_{r\to\infty} \frac{\log^{[p+m-q-1]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p+1]} M\left(\exp^{[l]} r, f\right)} = \infty \text{ if } q > m \text{ and } q < l;$$

and

$$(vi)\qquad \lim_{r\to\infty} \frac{\log^{[p+m-q-1]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p+1]} M\left(\exp^{[l]} r, f\right)} = \infty \text{ if } q > m \text{ and } q \geqslant l.$$

Proof. Let us choose $0 < \varepsilon < \min\{\lambda_f(p,q), \lambda_g(m,n)\}$.
Now for all sufficiently large values of r we get from Lemma 3,

$$\begin{aligned} M(\exp^{[n-1]} r, f\circ g) &\geqslant M\left\{\frac{1}{16}M(\frac{\exp^{[n-1]} r}{2}, g), f\right\} \\ i.e.,\ \log^{[p]} M(\exp^{[n-1]} r, f\circ g) &\geqslant \log^{[p]} M\left\{\frac{1}{16}M(\frac{\exp^{[n-1]} r}{2}, g), f\right\} \\ i.e.,\ \log^{[p]} M(\exp^{[n-1]} r, f\circ g) &\geqslant (\lambda_f(p,q) - \varepsilon)\log^{[q]}\left\{\frac{1}{16}M(\frac{\exp^{[n-1]} r}{2}, g)\right\} \end{aligned}$$

$i.e.,\ \log^{[p]} M(\exp^{[n-1]} r, f\circ g)$

$$\geqslant (\lambda_f(p,q) - \varepsilon)\log^{[q]} M(\frac{\exp^{[n-1]} r}{2}, g) + O(1). \tag{3.25}$$

Now the following two cases may arise.
Case I. Let $q \leq m$. Then from (3.25) we obtain for all sufficiently large values of r that

$$\log^{[p]} M(\exp^{[n-1]} r, f \circ g)$$
$$\geqslant (\lambda_f(p,q) - \varepsilon) \exp^{[m-q]} \log^{[m]} M(\frac{\exp^{[n-1]} r}{2}, g). \tag{3.26}$$

Now from the definition of (m,n) th lower order of g we have for all sufficiently large values of r,

$$\begin{aligned} \log^{[m]} M(\frac{\exp^{[n-1]} r}{2}, g) &\geqslant (\lambda_g(m,n) - \varepsilon) \log^{[n]}(\frac{\exp^{[n-1]} r}{2}) \\ i.e.,\ \log^{[m]} M(\frac{\exp^{[n-1]} r}{2}, g) &\geqslant (\lambda_g(m,n) - \varepsilon) \log r + O(1) \\ i.e.,\ \log^{[m]} M(\frac{\exp^{[n-1]} r}{2}, g) &\geqslant \log r^{(\lambda_g(m,n)-\varepsilon)} + O(1). \end{aligned} \tag{3.27}$$

Now from(3.26)and(3.27)we get for all sufficiently large values of r that

$$\log^{[p]} M(\exp^{[n-1]} r, f \circ g) \geqslant (\lambda_f(p,q) - \varepsilon) \exp^{[m-q]} \log r^{(\lambda_g(m,n)-\varepsilon)} + O(1)$$
$$i.e.,\ \log^{[p]} M(\exp^{[n-1]} r, f \circ g)$$
$$\geqslant (\lambda_f(p,q) - \varepsilon) \exp^{[m-q-1]} r^{(\lambda_g(m,n)-\varepsilon)} + O(1). \tag{3.28}$$

Case II. Let $q > m$. Then from (3.25) and (3.27) it follows for all sufficiently large values of r that

$$\log^{[p]} M(\exp^{[n-1]} r, f \circ g)$$
$$\geqslant (\lambda_f(p,q) - \varepsilon) \log^{[q-m]} . \log r^{(\lambda_g(m,n)-\varepsilon)} + O(1)$$
$$i.e.,\quad \log^{[p+1]} M(\exp^{[n-1]} r, f \circ g) \geqslant \log^{[q-m+2]} r^{(\lambda_g(m,n)-\varepsilon)} + O(1)$$
$$i.e.,\ \log^{[p+m-q-1]} M(\exp^{[n-1]} r, f \circ g) \geqslant r^{(\lambda_g(m,n)-\varepsilon)} + O(1). \tag{3.29}$$

Again from the definition of $\rho_f(p,q)$ we get for all large values of r that

$$\log^{[p]} M\left(\exp^{[l]} r, f\right) \leq (\rho_f(p,q) + \varepsilon) \log^{[q]} \left\{\exp^{[l]} r\right\}. \tag{3.30}$$

Now the following two cases may arise.
Case III. Let $q \geqslant l$. Then we have from (3.30) for all sufficiently large values of r,

$$\begin{aligned} \log^{[p]} M\left(\exp^{[l]} r, f\right) &\leq (\rho_f(p,q) + \varepsilon) \log^{[l]} \left\{\exp^{[l]} r\right\} \\ i.e.,\ \log^{[p]} M\left(\exp^{[l]} r, f\right) &\leq (\rho_f(p,q) + \varepsilon) r \\ i.e.,\ \log^{[p+1]} M\left(\exp^{[l]} r, f\right) &\leq \log r + O(1). \end{aligned} \tag{3.31}$$

Case IV. Let $q < l$. Then we have from (3.30) for all sufficiently large values of r that

$$\begin{aligned} \log^{[p]} M\left(\exp^{[l]} r, f\right) &\leq (\rho_f(p,q) + \varepsilon) \exp^{[l-q]} r \\ i.e.,\quad \log^{[p+1]} M\left(\exp^{[l]} r, f\right) &\leq \exp^{[l-q-1]} r + O(1) \\ i.e.,\ \log^{[p-q+l+1]} M\left(\exp^{[l]} r, f\right) &\leq \log r + O(1). \end{aligned} \tag{3.32}$$

Now combining (3.28) of Case I and (3.31) of Case III it follows for all sufficiently large values of r that

$$\frac{\log^{[p]} M(\exp^{[n-1]} r, f \circ g)}{\log^{[p+1]} M\left(\exp^{[l]} r, f\right)} \geq \frac{(\lambda_f(p,q) - \varepsilon) \exp^{[m-q-1]} r^{(\lambda g(m,n)-\varepsilon)} + O(1)}{\log r + O(1)}. \tag{3.33}$$

If $q < m$ then from (3.33) we get that

$$\liminf_{r\to\infty} \frac{\log^{[p]} M(\exp^{[n-1]} r, f \circ g)}{\log^{[p+1]} M\left(\exp^{[l]} r, f\right)} = \infty$$

$$i.e., \lim_{r\to\infty} \frac{\log^{[p]} M(\exp^{[n-1]} r, f \circ g)}{\log^{[p+1]} M\left(\exp^{[l]} r, f\right)} = \infty.$$

This proves the first part of the theorem.
If $q = m$ then from (3.33) it follows for all sufficiently large values of r that

$$\frac{\log^{[p]} M(\exp^{[n-1]} r, f \circ g)}{\log^{[p+1]} M\left(\exp^{[l]} r, f\right)} \geq \frac{(\lambda_f(p,q) - \varepsilon)(\lambda g(m,n) - \varepsilon) \log r + O(1)}{\log r + O(1)}.$$

As $\varepsilon\,(> 0)$ is arbitrary we obtain from above that

$$\limsup_{r\to\infty} \frac{\log^{[p]} M(\exp^{[n-1]} r, f \circ g)}{\log^{[p+1]} M\left(\exp^{[l]} r, f\right)} \geq \lambda_f(p,q)\, \lambda g(m,n)\,.$$

Thus the second part of the theorem follows.
Again in view of(3.28) of Case I and (3.32) of Case IV we have for all sufficiently large values of r,

$$\frac{\log^{[p]} M(\exp^{[n-1]} r, f \circ g)}{\log^{[p-q+l+1]} M\left(\exp^{[l]} r, f\right)} \geq \frac{(\lambda_f(p,q) - \varepsilon) \exp^{[m-q-1]} r^{(\lambda_f(m,n)-\varepsilon)} + O(1)}{\log r + O(1)}. \tag{3.34}$$

When $q < m$ and $q < l$ then we get from (3.34) that

$$\liminf_{r\to\infty} \frac{\log^{[p]} M(\exp^{[n-1]} r, f \circ g)}{\log^{[p-q+l+1]} M\left(\exp^{[l]} r, f\right)} = \infty$$

$$i.e., \lim_{r\to\infty} \frac{\log^{[p]} M(\exp^{[n-1]} r, f \circ g)}{\log^{[p-q+l+1]} M\left(\exp^{[l]} r, f\right)} = \infty.$$

This establishes the third part of the theorem.
Again when $q = m$ and $q < l$ then it follows from (3.34) for all sufficiently large values of r that

$$\frac{\log^{[p]} M(\exp^{[n-1]} r, f \circ g)}{\log^{[p-q+l+1]} M\left(\exp^{[l]} r, f\right)} \geq \frac{(\lambda_f(p,q) - \varepsilon)(\lambda g(m,n) - \varepsilon) \log r + O(1)}{\log r + O(1)}.$$

As $\varepsilon\,(>0)$ is arbitrary it follows from above that

$$\limsup_{r\to\infty}\frac{\log^{[p]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p-q+l+1]} M\left(\exp^{[l]} r, f\right)} \geq \lambda_f\,(p,q)\,\lambda g\,(m,n)\,.$$

Thus the fourth part of the theorem is proved.
Now in view of (3.29) of Case II and (3.31) of Case III we get for all sufficiently large values of r that

$$\begin{aligned}\frac{\log^{[p+m-q-1]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p+1]} M\left(\exp^{[l]} r, f\right)} &\geq \frac{r^{(\lambda_g(m,n)-\varepsilon)}+O(1)}{\log r+O(1)}\\ i.e.,\ \liminf_{r\to\infty}\frac{\log^{[p+m-q-1]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p+1]} M\left(\exp^{[l]} r, f\right)} &= \infty,\end{aligned}$$

from which the fifth part of the theorem follows.
Again from (3.29) of Case II and (3.32) of Case IV we have for all sufficiently large values of r that

$$\begin{aligned}\frac{\log^{[p+m-q-1]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p-q+l+1]} M\left(\exp^{[l]} r, f\right)} &\geq \frac{r^{(\lambda_g(m,n)-\varepsilon)}+O(1)}{\log r+O(1)}\\ i.e.,\ \liminf_{r\to\infty}\frac{\log^{[p+m-q-1]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p-q+l+1]} M\left(\exp^{[l]} r, f\right)} &= \infty\\ i.e.,\ \lim_{r\to\infty}\frac{\log^{[p+m-q-1]} M(\exp^{[n-1]} r, f\circ g)}{\log^{[p-q+l+1]} M\left(\exp^{[l]} r, f\right)} &= \infty.\end{aligned}$$

This proves the sixth part of the theorem .
Thus the theorem follows.

Acknowledgments. The authors are thankful to the referee for his / her valuable suggestions towards the improvement of the paper.

References

[1] W. Bergweiler, On the Nevanlinna Characteristic of a composite function,Complex Variables **10**(1988), 225-236.

[2] W. Bergweiler, On the growth rate of composite meromorphic functions, Complex Variables **14** (1990), 187-196.

[3] J. Clunie, The composition of entire and meromorphic functions, Mathematical essays dedicated to A.J. Macintyre, Ohio University Press, 1970, 75-92.

[4] W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[5] O. P. Juneja, G. P. Kapoor, and S. K. Bajpai, On the (p,q)-order and lower (p,q)-order of an entire function, J. Reine Angew. Math. **282**(1976), 53-67.

[6] K.Niino and C.C.Yang , Some growth relationships on factors of two composite entire functions, Factorization Theory of Meromorphic Functions and Related Topics, Marcel Dekker, Inc.(New York and Basel), (1982), 95-99.

[7] G. Valiron, Lectures on the General Theory of Integral Functions, Chelsea Publishing Company, 1949.

Sanjib Kumar Datta
Department of Mathematics, University of North Bengal, P.O. North Bengal University, Raja Rammohunpur, Dist. Darjeeling PIN- 734013, West Bengal,India.
E-mail address: sanjib_kr_datta@yahoo.co.in

Tanmay Biswas
Department of Mathematics, University of North Bengal, P.O. North Bengal University, Raja Rammohunpur, Dist. Darjeeling, PIN- 734013, West Bengal, India.
E-mail address: Tanmaybiswas_math@rediffmail.com