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ON GENERALIZED ϕ-RECURRENT SASAKIAN MANIFOLDS

(DEDICATED IN OCCASION OF THE 65-YEARS OF PROFESSOR R.K. RAINA)

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Abstract. The object of the present paper is to study generalized ϕ-recurrent Sasakian manifolds. Here it is proved that a generalized ϕ-recurrent Sasakian manifold is an Einstein manifold. We also find a relation between the associated 1-forms A and B for a generalized ϕ -recurrent and generalized concircular ϕ-recurrent Sasakian manifolds. Finally, we proved that a three dimensional locally generalized ϕ-recurrent Sasakian manifold is of constant curvature.

1. Introduction

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi[10] introduced the notion of local ϕ-symmetry on a Sasakian manifold. Generalizing the notion of ϕ-symmetry, the authors U.C. De, A.A. Shaikh and Sudipta Biswas introduced the notion of ϕ-recurrent Sasakian manifolds in[4]. This notion has been studied by many authors for different types of Riemannain manifolds([7, 6, 5, 11]).

A Sasakian manifold is said to be a $\phi-$recurrent manifold if there exists a nonzero $1-$form A such that

$$\phi^2((\nabla_X R)(Y,Z)W) = A(X)R(Y,Z)W$$

for arbitrary vector fields X, Y, Z, W.
If the $1-$form A vanishes, then the manifold reduces to a $\phi-$symmetric manifold.

The notion of generalized recurrent manifolds was introduced by U.C.De and N.Guha[3]. A Riemannian manifold (M^{2n+1}, g) is called generalized recurrrent if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(X)[g(Z,W)Y - g(Y,W)Z]$$

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where, A and B are two 1-forms, B is non-zero and these are defined by

$$A(X) = g(X, \rho_1), \;\; B(X) = g(X, \rho_2),$$

ρ_1 and ρ_2 are vector fields associated with 1-froms A and B, respectively.

Generalizing the notion of ϕ-recurrency, the authors A. Basari and C. Murathan[1] introduced the notion of generalized ϕ-recurrency to Kenmotsu manifolds. Motivated by the above studies, in this paper we extend the study of generalized ϕ-recurrency to Sasakian manifolds and obtain some interesting results.

A Sasakian manifold (M^{2n+1}, g) is said to be an Einstein manifold is its Ricci tensor S is of the form

$$S(X, Y) = kg(X, Y) \tag{1.1}$$

for any vector fields X, Y and where k is any constant.

The paper is organized as follows. In preliminaries, we give a brief account of Sasakian manifolds. In section 3, it is proved that a generalized ϕ-recurrent Sasakian manifold is an Einstein manifold. We also find some relations between the associated 1-forms A and B for a generalized ϕ-recurrent and genralized concircular ϕ-recurrent sasakian manifolds. In the last section, we proved that a three dimensional locally generalized ϕ-recurrent Sasakian manifold is of constant curvature.

2. Sasakian manifolds

Let (M^{2n+1}, g) be a contact Riemannian manifold with a contact form η, the associated vector field ξ, $(1-1)$ tensor field ϕ and the associated Riemannian metric g. If ξ is a killing vector field, then M^{2n+1} is called a K-contact Riemannian manifold([2], [9]). A K-contact Riemannian manifold is called a sasakian manifold if

$$(\nabla_X \phi)(X, Y) = g(X, Y)\xi - \eta(Y)X \tag{2.1}$$

holds, where ∇ denotes the operator of covariant differentiation with respect to g.

Let S and r denote, the Ricci tensors of type $(0, 2)$ and of type $(1, 1)$ of M^{2n+1} respectively. It is known that in a Sasakian manifold M^{2n+1}, besides the relation (2.1), the following relations also hold (see [2], [9]):

$$\phi^2 = -I + \eta \otimes \xi, \tag{2.2}$$

$$(a)\eta(\xi) = 1, \quad (b)\phi\xi = 0, \quad (c)\eta \circ \phi = 0, \quad (d)g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

$$(a)\nabla_X \xi = -\phi X, \quad (b)(\nabla_X \eta)Y = g(X, \phi Y), \tag{2.5}$$

$$R(\xi, X)Y = (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.6}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.8}$$

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.9}$$

$$S(X, \xi) = 2n\eta(X), \tag{2.10}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2n\eta(X)\eta(Y), \tag{2.11}$$

for all vector fields X, Y, Z.

The above results will be used in the next sections.

3. ON GENERALIZED ϕ-RECURRENT SASAKIAN MANIFOLDS

Definition 3.1. *Sasakian manifold (M^{2n+1}, g) is called generalized ϕ-recurrent if its curvature tensor R satisfies the condition*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y] \tag{3.1}$$

where A and B are two 1-forms, B is non-zero and these are defined by:

$$\alpha(W) = g(W,\rho_1), \beta(W) = g(W,\rho_2) \tag{3.2}$$

and ρ_1, ρ_2 are vector fields associated with 1-forms A and B,respectively.

Let us consider a generalized ϕ-recurrent Sasakian manifold. Then by virtue of (2.2) and (3.1) we have

$$\begin{aligned} & -(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi \\ = \; & A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.3}$$

From which it follows that

$$\begin{aligned} & -g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) \\ = \; & A(W)g(R(X,Y)Z,U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{3.4}$$

Let $\{e_i\}$, $i = 1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.4) and taking summation over i, $1 \le i \le 2n+1$, we get

$$\begin{aligned} & -(\nabla_W S)(Y,Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) \\ = \; & A(W)S(Y,Z) + 2nB(W)g(Y,Z). \end{aligned} \tag{3.5}$$

The second term of left hand side of (3.5) by putting $Z = \xi$ takes the form $g((\nabla_W R)(e_i,Y)\xi,\xi)$, which is zero in this case. So, by replacing Z by ξ in (3.5) and using (2.10) we get

$$(\nabla_W S)(Y,\xi) = -A(W)2n\eta(Y) - 2nB(W)\eta(Y). \tag{3.6}$$

Now we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi).$$

Using (2.5)(a) and (2.9) in the above relation, then it follows that

$$(\nabla_W S)(Y,\xi) = -2ng(\phi W,Y) + S(Y,\phi W). \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$-2ng(\phi W,Y) + S(Y,\phi W) = -2n\eta(Y)(A(W) + B(W)). \tag{3.8}$$

Replacing $Y = \xi$ in (3.8) then using (2.9) and (2.2) we get

$$A(W) = -B(W). \tag{3.9}$$

Thus the 1-forms A and B are related as $\alpha + \beta = 0$.
Next using (3.9) in (3.8), we obtain

$$S(Y,\phi W) = 2ng(Y,\phi W). \tag{3.10}$$

That is, the manifold is an Einstein manifold. This leads to the following result:

Theorem 3.2. *A generalized ϕ-recurrent Sasakian manifold (M^{2n+1}, g) is an Einstein manifold and moreover; the 1-forms A and B are related as $A + B = 0$.*

Now from (3.1) we have

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z - B(W)[g((Y,Z)X - g(X,Z)Y]. \tag{3.11}$$

Changing W, X, Y cyclically in (3.11) and then adding the results, we obtain

$$\begin{aligned}&(\nabla_W R)(X,Y)Z + (\nabla_X R)(Y,W)Z + (\nabla_Y R)(W,X)Z \\ = \; & \eta((\nabla_W R)(X,Y)Z)\xi + \eta((\nabla_X R)(Y,W)Z)\xi + \eta((\nabla_Y R)(W,X)Z)\xi \\ & -A(W)R(X,Y)Z - B(W)[g((Y,Z)X - g(X,Z)Y] \\ & -A(X)R(Y,W)Z - B(X)[g((W,Z)Y - g(Y,Z)W] \\ & -A(Y)R(W,X)Z - B(Y)[g((X,Z)W - g(W,Z)X] = 0.\end{aligned} \tag{3.12}$$

Then by the use of second Bianchi identity and (3.9) we have

$$\begin{aligned}& A(W)R(X,Y)Z - B(W)[g((Y,Z)X - g(X,Z)Y] \\ & +A(X)R(Y,W)Z - B(X)[g((W,Z)Y - g(Y,Z)W] \\ & +A(Y)R(W,X)Z - B(Y)[g((X,Z)W - g(W,Z)X] = 0.\end{aligned}$$

so by a suitable contraction from (3.12) we get

$$\begin{aligned}& A(W)S(X,U) - 2nA(W)g(X,U) - A(X)S(W,U) + 2nA(X)g(W,U) \\ & -A(R(W,X)U) - A(X)g(W,U) + A(W)g(X,U) = 0.\end{aligned} \tag{3.13}$$

Using (3.10) in above, we get

$$-g(R(W,X)U,\rho_1) - A(X)g(W,U) + A(W)g(X,U) = 0. \tag{3.14}$$

Replacing $X = U = e_i$ in (3.14) we get

$$S(W,\rho_1) = 2nA(W). \tag{3.15}$$

This leads to the following result:

Theorem 3.3. *In a generalized ϕ-recurrent Sasakian manifold (M^{2n+1}, g), $2n$ is the eigen value of the ricci tensor corresponding to the eigen vector ρ_1, where ρ_1 is the associated vector field of the 1-form A.*

Definition 3.4. *A Sasakian manifold (M^{2n+1}, g) is called generalized concircular ϕ-recurrent if its concircular curvature tensor $\overline{C}$ (Yano, K., Kon, M., 1984)*

$$\overline{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{3.16}$$

satisfies the condition [8]

$$\phi^2(\nabla_W \overline{C}(X,Y)Z) = A(W)\overline{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y] \tag{3.17}$$

where $A(W)$ and $B(W)$ are defined as in (3.2) and r is the scalar curvature of the manifold (M^{2n+1}, g).

Let us consider a generalized concircular ϕ-recurrent Sasakian manifold. Then by virtue of (2.2) we have

$$\begin{aligned}& -(\nabla_W \overline{C}(X,Y)Z) + \eta((\nabla_W \overline{C}(X,Y)Z))\xi \\ = \; & A(W)\overline{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{3.18}$$

From which, it follows that

$$\begin{aligned} & -g((\nabla_W \overline{C}(X,Y)Z),U) + \eta((\nabla_W \overline{C}(X,Y)Z))\eta(U) \\ = \;& A(W)g(\overline{C}(X,Y)Z,U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{3.19}$$

Let $\{e_i\}$, $i = 1, 2, ..., 2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (3.19) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned} & -(\nabla_W S)(X,U) + \frac{\nabla_W r}{(2n+1)} g(X,U) + (\nabla_W S)(X,\xi)\eta(U) - \frac{\nabla_W r}{2n+1}\eta(X)\eta(U) \\ = \;& A(W)\left[S(X,U) - \frac{r}{2n+1}g(X,U)\right] + 2nB(W)g(X,U). \end{aligned} \tag{3.20}$$

Replacing U by ξ in (3.20) and using (2.3d) and (2.10), we have

$$A(W)\left[2n - \frac{r}{2n+1}\right]\eta(X) + 2nB(W)\eta(X) = 0. \tag{3.21}$$

Putting $X = \xi$ in (3.21), we obtain

$$B(W) = \left(\frac{r}{2n(2n+1)} - 1\right)A(W). \tag{3.22}$$

This leads to the following result:

Theorem 3.5. *In a generalized concircular ϕ-recurrent Sasakian manifold (M^{2n+1}, g), the 1-forms A and B are related as in (3.22).*

4. Three Dimensional Locally Generalized ϕ-recurrent Sasakian Manifolds

In a three-dimensional Riemannian manifold (M^3, g), we have

$$\begin{aligned} R(X,Y)Z \;=\;& g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ & -S(X,Z)Y + \frac{r}{2}[g(X,Z)Y - g(Y,Z)X], \end{aligned} \tag{4.1}$$

where Q is the Ricci operator, that is, S(X, Y) = g(QX, Y) and r is the scalar curvature of the manifold. Now putting $Z = \xi$ in (4.1) and using (2.10), we get

$$\begin{aligned} R(X,Y)\xi \;=\;& \eta(Y)QX - \eta(X)QY \\ & +2[\eta(Y)X - \eta(X)Y] + \frac{r}{2}[\eta(X)Y - \eta(Y)X]. \end{aligned} \tag{4.2}$$

Using (2.7) in (4.2), we have

$$\left(1 - \frac{r}{2}\right)[\eta(Y)X - \eta(X)Y] = \eta(X)QY - \eta(Y)QX. \tag{4.3}$$

Putting $Y = \xi$ in (4.3) and using (2.10), we get

$$QX = \left(\frac{r}{2} - 1\right)X + \left(3 - \frac{r}{2}\right)\eta(X)\xi. \tag{4.4}$$

Therefore, it follows from (4.4) that

$$S(X,Y) = \left(\frac{r}{2} - 1\right)g(X,Y) + \left(3 - \frac{r}{2}\right)\eta(X)\eta(Y). \tag{4.5}$$

Thus from (4.1), (4.4) and (4.5), we get

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2}-2\right)[g(Y,Z)X-g(X,Z)Y] \\ &+\left(3-\frac{r}{2}\right)[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ &+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y]. \end{aligned} \tag{4.6}$$

Taking the covariant differentiation to the both sides of the equation (4.6), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)Y-g(Y,Z)\eta(X)\xi \\ &+g(X,Z)\eta(Y)\xi-\eta(Y)\eta(Z)X+\eta(X)\eta(Z)Y] \\ &+\left(3-\frac{r}{2}\right)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\nabla_W\xi \\ &+\left(3-\frac{r}{2}\right)[\eta(Y)X-\eta(X)Y](\nabla_W\eta)(Z) \\ &+\left(3-\frac{r}{2}\right)[g(Y,Z)\xi-\eta(Z)Y](\nabla_W\eta)(X) \\ &-\left(3-\frac{r}{2}\right)[g(X,Z)\xi-\eta(Z)X](\nabla_W\eta)(Y). \end{aligned} \tag{4.7}$$

Noting that we may assume that all vector fields X, Y, Z, W are orthogonal to ξ and using (2.2), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)Y] \\ +\left(3-\frac{r}{2}\right)[g(Y,Z)(\nabla_W\eta)(X)-g(X,Z)(\nabla_W\eta)(Y)]\xi. \end{aligned} \tag{4.8}$$

Applying ϕ^2 to the both sides of (4.8) and using (2.2) and (2.3), we get

$$\phi^2((\nabla_W R)(X,Y)Z) = \frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)X]. \tag{4.9}$$

By (3.1) the equation (4.9) reduces to

$$A(W)R(X,Y)Z = \left[\frac{dr(W)}{2}-B(W)\right][g(Y,Z)X-g(X,Z)X].$$

Putting $W = \{e_i\}$, where $\{e_i\}, i = 1, 2, 3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq 3$, we obtain

$$R(X,Y)Z = \lambda[g(Y,Z)X-g(X,Z)X].$$

where $\lambda = \left[\frac{dr(e_i)}{2A(e_i)} - \frac{Be_i}{A(e_i)}\right]$ is a scalar, since A is a non-zero 1-form. Then by Schur's theorem λ will be a constant on the manifold. Therefore, (M^3, g) is of constant curvature λ. Thus we get the following theorem:

Theorem 4.1. *A three dimensional locally generalized ϕ-recurrent Sasakian manifold (M^3, g) is of constant curvature.*

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