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ON COMMON FIXED POINTS FOR CONTRACTIVE TYPE MAPPINGS IN CONE METRIC SPACES

(DEDICATED IN OCCASION OF THE 65-YEARS OF PROFESSOR R.K. RAINA)

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Abstract. This paper presents some common fixed point theorems in complete cone metric spaces. Also discussed periodic point theorems in complete cone metric spaces.

1. Introduction

Huang and Zhang [4] introduced the notion of cone metric spaces and some fixed point theorems for contractive mappings were proved in these spaces. The results in [4] were generalized by Sh. Rezapour and R. Hamlbarani in [8]. Subsequently, Abbas and Jungck [2], Abbas and Rhoades [1], *Ilić* and *Rakočević* [5], Akbar Azam, Muhammad Arshad, and Ismat Beg [3] were investigated some common fixed point theorems for different types of contractive mappings in cone metric spaces. The purpose of this paper is to provide some common fixed point results in cone metric spaces.

Let E be a real Banach space and P a subset of E. P is called a cone if and only if:
(i) P is closed, non-empty and $P \neq 0$,
(ii)$ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in intP$, intP denotes the interior of P. Denote by $\|\cdot\|$ the norm on E. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|. \tag{1.1}$$

The least positive number K satisfying the above is called the normal constant of P, see [4]. In [8] the authors showed that there are no normal cones with normal constant $M < 1$ and for each $k > 1$ there are cones with normal constant $M > k$.

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The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

The cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

Lemma 1.1. [8] *Every regular cone is normal.*

In the following we always suppose that E is a real Banach space, P is a cone in E with $\text{int}P \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition 1.2. *Let X be a non-empty set and $d : X \times X \to E$ a mapping such that*
$(d_1) 0 \leq d(x,y)$ *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$,
$(d_2) d(x,y) = d(y,x)$ *for all* $x, y \in X$,
$(d_3) d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$.
Then d is called a cone metric on X, and (X,d) is called a cone metric space [4].

Example 1.3. *Let $E = R^2, P = \{(x,y) \in E : x, y \geq 0\}, X = R$ and $d : X \times X \to E$ defined by $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space* [4].

Definition 1.4. *(See* [4]*) Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then*
(i)$\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
(ii)$\{x_n\}_{n\geq 1}$ is said to Cauchy sequence if for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
(iii)(X,d) is called a complete cone metric space if every Cauchy sequence in X is convergent.

Lemma 1.5. [8] *There is not normal cone with normal constant $M < 1$.*

Example 1.6. [8] *Let $E = C_{\mathrm{R}}([0,1])$ endowed with the supremum norm and $P = \{f \in E : f \geq 0\}$. Then P is a cone with normal constant of $M = 1$. Consider the sequence $\{x \geq x^2 \geq x^3 \geq \cdots \geq 0\}$ of elements of E which is decreasing and bounded from below but it is not convergent in E. Therefore, the converse of Lemma 1.5 is not true.*

Example 1.7. [8] *Let $E = l^1, P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \text{for all } n\}$, (X,ρ) a metric space and $d : X \times X \to E$ defined by $d(x,y) = \{\frac{\rho(x,y)}{2^n}\}_{n\geq 1}$. Then (X,d) is a cone metric space and the normal constant of P is equal to $M = 1$. Moreover, this example shows that the category of cone metric spaces is bigger than the category of metric spaces.*

Proposition 1.8. [8] *For each $k > 1$, there is a normal cone with normal constant $M > k$.*

2. Results.

In this section we provide our main results. The first one is as follows.

Theorem 2.1. *Let (X, d) be a complete cone metric space, and P a normal cone with normal constant K. Suppose that the mappings $f, g : X \to X$ satisfy*

$$\alpha d(fx, gy) + \beta d(x, fx) + \gamma d(y, gy) \leq \delta d(x, y), \tag{2.1}$$

for all $x, y \in X, \alpha, \beta, \gamma, \delta \geq 0$, $\beta < \delta, \gamma < \delta$ and $\delta < \alpha$. Then f and g have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X, there is x_1 and x_2 in X such that $f(x_0) = x_1$ and $gx_1 = x_2$. In this way we have $f(x_{2n-2}) = x_{2n-1}, g(x_{2n-1}) = x_{2n}$. Put $x = x_{2n}, y = x_{2n+1}$ in (2), we have

$$\alpha d(fx_{2n}, gx_{2n+1}) + \beta d(x_{2n}, fx_{2n}) + \gamma d(x_{2n+1}, gx_{2n+1}) \leq \delta d(x_{2n}, x_{2n+1}),$$

This implies that

$$d(x_{2n+1}, x_{2n+2}) \leq \eta d(x_{2n}, x_{2n+1}).$$

where $\eta = \dfrac{\delta - \beta}{\alpha + \gamma}$.

Again we put $x = x_{2n}, y = x_{2n-1}$ in (2), we have

$$\begin{aligned}&\alpha d(fx_{2n}, gx_{2n-1}) + \beta d(x_{2n}, fx_{2n}) + \gamma d(x_{2n-1}, gx_{2n-1})\\ &\quad \leq \delta d(x_{2n}, x_{2n-1})\end{aligned}$$

implies that

$$d(x_{2n}, x_{2n+1}) \leq \theta d(x_{2n-1}, x_{2n})$$

where $\theta = \dfrac{\delta - \gamma}{\alpha + \beta}$. In this way we have

$$d(x_{2n+1}, x_{2n+2}) \leq \eta^n \theta^n d(x_1, x_2),$$

and

$$d(x_{2n}, x_{2n+1}) \leq \eta^n \theta^n d(x_0, x_1).$$

For $n < m$, we have

$$\begin{aligned} d(x_{2n}, x_{2m}) &\leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \cdots + d(x_{2m-1}, x_{2m}) \\ &\leq (\eta^n\theta^n + \eta^{n+1}\theta^{n+1} + \cdots + \eta^{m-1}\theta^{m-1}) d(x_0, x_1) \\ &\quad + (\eta^n\theta^n + \eta^{n+1}\theta^{n+1} + \cdots + \eta^{m-1}\theta^{m-1}) d(x_1, x_2) \\ &\leq \frac{(\eta\theta)^n}{1 - (\eta\theta)} (d(x_0, x_1) + d(x_1, x_2)). \end{aligned}$$

From (1)

$$\begin{aligned} \|d(x_{2n}, x_{2m})\| &\leq \frac{(\eta\theta)^n}{1 - (\eta\theta)} K \|(d(x_0, x_1) + d(x_1, x_2))\| \\ &\to 0, \end{aligned}$$

as $n \to \infty$, since $\eta\theta < 1$. Similarly $d(x_{2n}, x_{2m+1}), d(x_{2n+1}, x_{2m+1}) \to 0$ as $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete cone metric space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Thus $fx_{2n} \to u$ and $gx_{2n+1} \to u$ as $n \to \infty$.

Put $x = u$ and $y = x_{2n+1}$ in (2), then

$$\alpha d(fu, gx_{2n+1}) + \beta d(u, fu) + \gamma d(x_{2n+1}, gx_{2n+1}) \leq \delta d(u, x_{2n+1})$$
$$\alpha d(fu, x_{2n+2}) + \beta d(u, fu) + \gamma d(x_{2n+1}, x_{2n+2}) \leq \delta d(u, x_{2n+1}).$$

Thus

$$d(u, fu) \leq \frac{\delta}{\beta} d(u, x_{2n+1}).$$

Now the right hand side of the above inequality approaches zero as $n \to \infty$. Hence $\|d(u, fu)\| = 0$ and $u = fu$. Now put $x = u, y = u$ in (2)

$$\alpha d(fu, gu) + \beta d(u, fu) + \gamma d(u, gu) \leq \delta d(u, u)$$
$$\alpha d(u, gu) + \beta d(u, u) + \gamma d(u, gu) \leq \delta d(u, u)$$

implies that $d(u, gu) \leq 0$. Hence $u = gu$. Thus u is a common fixed point of f and g. If v is a common fixed point of f and g other then u, then putting $x = u, y = v$ in (2)

$$\alpha d(fu, gv) + \beta d(u, fu) + \gamma d(v, gv) \leq \delta d(u, v)$$
$$\alpha d(u, v) + \beta d(u, u) + \gamma d(v, v) \leq \delta d(u, v)$$

implies that

$$d(u, v) \leq \frac{\delta}{\alpha} d(u, v).$$

which gives $d(u, v) = 0$ and $u = v$.

Corollary 2.2. *Let (X, d) be a complete cone metric space, and P a normal cone with normal constant K. Suppose that the mapping $f : X \to X$ satisfies*

$$\alpha d(f^p x, f^q y) + \beta d(x, f^p x) + \gamma d(y, f^q y) \leq \delta d(x, y), \tag{2.2}$$

for all $x, y \in X, \alpha, \beta, \gamma, \delta \geq 0, \delta < \alpha$ and p and q are fixed positive integers. Then f has a unique fixed point in X.

Proof. The inequality (3) is obtained from (2) by setting $f = f^p$ and $g = f^q$. The results follows from Theorem 2.1.

Corollary 2.3. *Let (X, d) be a complete cone metric space, and P a normal cone with normal constant K. Suppose that the mapping $f : X \to X$ satisfies*

$$\alpha d(fx, fy) + \beta d(x, fx) + \gamma d(y, fy) \leq \delta d(x, y), \tag{2.3}$$

for all $x, y \in X, x \neq y, \alpha, \beta, \gamma, \delta \geq 0$ and $1 + \delta < \alpha$. Then f has a unique fixed point in X.

Proof. Put $p = q = 1$ in Corollary 2.2.

It is clear that if f is a map which has a fixed point p then p is also a fixed point of f^n for every natural number n. But the converse is not true, see example [1]. If a map satisfies $F(f) = F(f^n)$ for each $n \in N$, where $F(f)$ denotes a set of all fixed points of f , then it is said to have property P, see [7]. Moreover, f and g are said to have property Q[1] if $F(f) \cap F(g) = F(f^n) \cap F(g^n)$.

Theorem 2.4. *Let (X, d) be a complete cone metric space, and P a normal cone with normal constant K. Suppose that the mappings $f, g : X \to X$ satisfy (2). Then f and g have property Q.*

Proof. From Theorem 2.1, f and g have a common fixed point in X. Let $u \in F(f^n) \cap F(g^n)$. Set $x = f^{n-1}u, y = g^n u$ in (2), we have

$$\alpha d(f^n u, g^{n+1}u) + \beta d(f^{n-1}u, f^n u) + \gamma d(g^n u, g^{n+1}u) \leq \delta d(f^{n-1}u, g^n u)$$
$$\alpha d(u, gu) + \beta d(f^{n-1}u, u) + \gamma d(u, gu) \leq \delta d(f^{n-1}u, u)$$

which implies that

$$d(u, gu) \leq hd(f^{n-1}u, u) = hd(f^{n-1}u, f^n u).$$

where $h = \frac{\delta-\beta}{\alpha+\gamma}$. From (2) we have $d(f^{n-1}u, f^n u) \leq hd(f^{n-2}u, f^{n-1}u)$. Thus

$$d(u, gu) \leq hd(f^{n-1}u, f^n u) \leq h^2 d(f^{n-2}u, f^{n-1}u) \leq \cdots \leq h^n d(u, fu).$$

From (1)

$$\|d(u, gu)\| \leq h^n K \|d(u, fu)\|.$$

Now the right hand side of the above inequality approaches zero as $n \to \infty$. Hence $\|d(u, gu)\| = 0$, and $u = gu$, which, from Theorem 2.1, implies that $u = fu$.

Theorem 2.5. *Let (X, d) be a complete cone metric space, and P a normal cone with normal constant K. Suppose that the mapping $f : X \to X$ satisfies (4). Then f has property P.*

Proof. From Corollary 2.3, f has a unique fixed point. Let $u \in F(f^n)$. Put $x = f^{n-1}u, y = f^n u$ in (4), then

$$\alpha d(f^n u, f^{n+1}u) + \beta d(f^{n-1}u, f^n u) + \gamma d(f^n u, f^{n+1}u) \leq \delta d(f^{n-1}u, f^n u)$$
$$\alpha d(u, fu) + \beta d(f^{n-1}u, u) + \gamma d(u, fu) \leq \delta d(f^{n-1}u, u)$$

which implies that

$$d(u, fu) \leq hd(f^{n-1}u, u) = hd(f^{n-1}, f^n u)$$

where $h = \frac{\delta-\beta}{\alpha+\gamma}$. From (4) we have $d(f^{n-1}, f^n u) \leq hd(f^{n-2}, f^{n-1}u)$. Thus

$$d(u, fu) \leq hd(f^{n-1}u, f^n u) \leq h^n d(u, fu).$$

From (1)

$$\|d(u, fu)\| \leq h^n K \|d(u, fu)\|.$$

Now the right hand side of the above inequality approaches zero as $n \to \infty$. Hence $\|d(u, fu)\| = 0$, and $u = fu$.

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