THE CARRIER GRAPH TOPOLOGY

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ABSTRACT. We define a new metric on the set of all closed linear operators between Hilbert spaces and investigate its properties. In particular, we show that the set of all closed operators with a closed range is an open subset of the set of all closed operators and the map $T \mapsto T^{\dagger}$ is an isometry in this metric. We also investigate the relationships between the topology induced by this metric and the gap metric.

1. Introduction

A large number of practical problems can be modeled by the operator equation of the type

$$Tx = y$$

where T is a bounded (or unbounded but closed) operator from a complex Hilbert space H_1 to a complex Hilbert space H_2 and y is a given element in H_2. Such a problem has a unique solution and this unique solution depends continuously on y if T has a continuous (bounded) inverse $T^{-1} : H_2 \to H_1$. Since in practical problems, T is known only approximately, it becomes important to know the answer to the following perturbation problem:

If an operator S is sufficiently close to T, does it follow that S also has a bounded inverse and S^{-1} is close to T^{-1}?. This question has a satisfactory affirmative answer because the set G of all bounded operators with bounded inverses is an open set in the norm topology and the map $T \to T^{-1}$ is continuous on G.

The situation is far from satisfactory if we consider similar question about Moore–Penrose inverses. The need to consider Moore–Penrose inverses arises when the operator equation $Tx = y$ does not have a solution and we have to consider least-square (best approximate) solutions instead. It is well known that the set $G^{\dagger}$ of those T that have a bounded Moore–Penrose inverse $T^{\dagger}$ is not an open set in the norm topology and the map $T \to T^{\dagger}$ is not continuous. This is so not only in the norm topology but also in many other topologies introduced by several authors including the well known gap topology.

The aim of this article is to introduce a new metric on the class of closed operators between Hilbert spaces H_1 and H_2 such that these deficiencies are removed. It turns out that this new metric has some properties that are similar to those of the gap metric.

Thus this new metric and the topology induced by it, called the carrier graph topology is more suitable to study perturbation questions about Moore–Penrose inverses.

Several authors [4, 8] have paid attention to study this map by defining metrics on the space of all bounded operators between H_1 and H_2. As these definitions involves operator norms, these cannot be extended to the case of unbounded closed operators. In many cases the operators which arise in applications are unbounded (see [7, 10, 16] for details), hence it is necessary to consider the case of unbounded operators in a different manner. Our approach can be compared with the approach of Labrousse and Mbekhta [12] who arrive at the same metric but from a different view point.

This paper is organized as follows: In section 2 we set up notations and terminology and also review some of the standard facts about the gap metric. In the third section we introduce a metric $\eta(\cdot, \cdot)$ on the class of closed operators between H_1 and H_2 and discuss some of its properties. In particular, we prove that the set $\mathcal{CR}(H_1, H_2)$ of all closed operators between H_1 and H_2 with closed range is an open set (Corollary 3.12) and the map $T \mapsto T^{\dagger}$ is an isometry (Theorem 3.9). In the fourth section we study the relations between the Carrier Graph Topology, gap metric and the norm topology on the space of bounded operators.

2. Notations and preliminary results

We denote Hilbert spaces over the complex field $\mathbb{C}$ by H, H_1, H_2 etc., and the corresponding inner product and the norm by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively.

If H_1 and H_2 are Hilbert spaces over $\mathbb{C}$, then $H_1 \times H_2$ is also a Hilbert space with respect to the inner product $[(x_1, y_1), (x_2, y_2)] = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle$ for all $(x_1, y_1), (x_2, y_2) \in H_1 \times H_2$. In this case the norm induced by this inner product is $\|(x, y)\| = \{\|x\|^2 + \|y\|^2\}^{\frac{1}{2}}$ for all $(x, y) \in H_1 \times H_2$.

$\mathcal{C}(H_1, H_2) :=$ the class of closed linear operators

$\mathcal{C}(H, H) = \mathcal{C}(H)$.

For $T \in \mathcal{C}(H_1, H_2)$, the domain, null space and range space will be denoted by $D(T), N(T)$ and $R(T)$ respectively.

$\mathcal{CR}(H_1, H_2) := \{T \in \mathcal{C}(H_1, H_2) : R(T) \text{ is closed}\}$.

If $T \in \mathcal{C}(H_1, H_2)$, then the graph $G(T) := \{(x, Tx) : x \in D(T)\}$ is a closed subspace of $H_1 \times H_2$.

$\mathcal{B}(H_1, H_2) :=$ the space of all bounded linear operators between H_1 and H_2.

$\mathcal{B}(H, H) = \mathcal{B}(H)$

By the closed graph theorem a closed operator $T \in \mathcal{C}(H_1, H_2)$ with $D(T) = H_1$ is bounded [17, Page 306].

If $M \subseteq H$, then the orthogonal complement of M is denoted by $M^\perp$ and the orthogonal projection onto M by P_M. If $u \in H$, then the distance between u and M is denoted by $\text{dist}(u, M)$.

If M and N are closed subspaces of H, then their direct sum will be denoted by $M \oplus N$ and orthogonal direct sum by $M \oplus^\perp N$.

If S and T are two linear operators such that $D(T) \subseteq D(S)$ and $Tx = Sx$ for all $x \in D(T)$, then T is called a *restriction* of S and S is called an *extension* of T. We denote this fact by $T \subseteq S$.

Definition 2.1. [2, Pages 308, 310] Let $T \in \mathcal{L}(H_1, H_2)$. If $\overline{D(T)} = H_1$, then T is called *densely defined*. The subspace $C(T) := D(T) \cap N(T)^\perp$ is called the *carrier* of T.

We denote the graph of the restriction operator $T|_{C(T)}$ by $G_C(T)$. Thus $G_C(T) := \{(x, Tx) : x \in C(T)\}$. We call this the *carrier graph* of T.

If $A \in \mathcal{C}(H_1, H_2)$ is densely defined, then $I + A^*A$ and $I + AA^*$ have bounded inverse (see [6, 7, 18] for details). We define $\check{A} := (I + A^*A)^{-1}$ and $\hat{A} := (I + AA^*)^{-1}$.

Note 2.2. If $T \in \mathcal{C}(H_1, H_2)$, then $D(T) = N(T) \oplus^\perp C(T)$ [2, Page 340].

Definition 2.3. (Moore–Penrose Inverse) [2, Pages 314, 318-320]
Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then there exists a unique densely defined operator $T^\dagger \in \mathcal{C}(H_2, H_1)$ with domain $D(T^\dagger) = R(T) \oplus^\perp R(T)^\perp$ and has the following properties:

(1) $TT^\dagger y = P_{\overline{R(T)}}\, y$, for all $y \in D(T^\dagger)$.
(2) $T^\dagger T x = P_{N(T)^\perp}\, x$, for all $x \in D(T)$.
(3) $N(T^\dagger) = R(T)^\perp$.

This unique operator $T^\dagger$ is called the *Moore–Penrose inverse* of T.
The following property of $T^\dagger$ is also well known. For every $y \in D(T^\dagger)$, let

$$L(y) := \left\{ x \in D(T) : \|Tx - y\| \leq \|Tu - y\| \quad \text{for all} \quad u \in D(T) \right\}.$$

Here any $u \in L(y)$ is called a *least square solution* of the operator equation $Tx = y$. The vector $T^\dagger y \in L(y)$, $\|T^\dagger y\| \leq \|x\|$ for all $x \in L(y)$ and it is called the *least square solution of minimal norm*. A different treatment of $T^\dagger$ is given in [2, Pages 336, 339, 341], where it is called "*the Maximal Tseng generalized Inverse*".

Definition 2.4. Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then

$$\gamma(T) := \inf \left\{ \|Tx\| : x \in C(T),\ \|x\| = 1 \right\}$$

is called the *reduced minimum modulus* of T and

$$m(T) := \inf \{\|Tx\| : x \in H_1, \|x\| = 1\}$$

is called the *minimum modulus* of T.

Definition 2.5 (spectrum). [18, Page 346] Let $T \in \mathcal{C}(H)$ be densely defined. The *resolvent* of T is defined by

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I : D(T) \to H \text{ is bijective and } (T - \lambda I)^{-1} \in \mathcal{B}(H)\}$$

and the spectrum $\sigma(T)$ is the complement of $\rho(T)$ in $\mathbb{C}$.

Proposition 2.6. *Let $T \in \mathcal{C}(H)$ be self-adjoint. Then $m(T) = \inf \{|\lambda| : \lambda \in \sigma(T)\}$.*

Proof. The proof goes along similar lines to that of [13, Theorem 3.5]. □

Proposition 2.7. [13, Proposition 2.12] *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then the following statements are equivalent.*

(1) $T \in \mathcal{CR}(H_1, H_2)$.
(2) $\gamma(T) > 0$.
(3) $T^\dagger$ *is bounded and* $\|T^\dagger\| = \dfrac{1}{\gamma(T)}$.
(4) $T^* \in \mathcal{CR}(H_2, H_1)$.

Remark 2.8. Since $(T^\dagger)^\dagger = T$ ([2, Theorem 2, Page 341]), by Proposition 2.7, T is bounded if and only if $R(T^\dagger)$ is closed.

First we give some preliminary definitions and results which will be used subsequently.

Definition 2.9. (Gap between subspaces) [10, Page 197] Let H be a Hilbert space and M, N be closed subspaces of H. Let S_M be the unit sphere of M. Define

$$\delta(M, N) := \begin{cases} \sup_{u \in S_M} dist(u, N), & \text{if } M \neq \{0\}, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\hat{\delta}(M, N) := \max \{\delta(M, N), \delta(N, M)\}.$$

The quantity $\hat{\delta}(M, N)$ is called the *gap* between the subspaces M and N.

Note 2.10. The following result is proved in [1, Page 70].

$$\hat{\delta}(M, N) = \|P_M - P_N\| = \max \left\{\|P_M(I - P_N)\|, \|P_N(I - P_M)\|\right\}.$$

For the properties of $\delta(\cdot, \cdot)$ and $\hat{\delta}(\cdot, \cdot)$ we refer to [10, Pages 197, 200, 201].

2.1. **The gap between closed operators.** For $S, T \in \mathcal{C}(H_1, H_2)$, $G(T), G(S) \subseteq H_1 \times H_2$ are closed subspaces. Define

$$\delta(T, S) = \delta(G(T), G(S)), \ \hat{\delta}(T, S) = \hat{\delta}(G(T), G(S)) = \max[\delta(T, S), \delta(S, T)].$$

Then $\hat{\delta}(T, S)$ is called the *gap* between T and S. The function $\hat{\delta}(\cdot, \cdot)$ defines a metric on $\mathcal{C}(H_1, H_2)$ called the gap metric and the topology induced by this metric is called the *Gap topology*. For a deeper discussion on these concepts we refer to [10, Chapter IV]. In the following theorem we collect some of the important properties of the gap metric on $\mathcal{C}(H_1, H_2)$.

Theorem 2.11. [10, Pages 202-205] *Let* $S, T \in \mathcal{C}(H_1, H_2)$ *be densely defined. Then*

(1) *If T is bounded and $\hat{\delta}(S, T) < \dfrac{1}{(1 + \|T\|^2)^{\frac{1}{2}}}$, then S is bounded.*

(2) $\hat{\delta}(S, T) = \hat{\delta}(S^*, T^*)$.

(3) $B(H_1, H_2)$ *is an open subset of $\mathcal{C}(H_1, H_2)$ with respect to the gap topology.*

(4) *If S, T are one-to-one, then $\hat{\delta}(S, T) = \hat{\delta}(S^{-1}, T^{-1})$.*

(5) $\mathcal{C}(H_1, H_2)$ *is not complete with respect to the gap metric.*

(6) *If $T^{-1} \in \mathcal{B}(H_2, H_1)$ and S is such that $\hat{\delta}(S, T) < \dfrac{1}{\sqrt{1 + \|T^{-1}\|^2}}$, then S is invertible and $S^{-1} \in \mathcal{B}(H_2, H_1)$.*

Theorem 2.12. [14, Theorem 2.5] *Let $A, B \in \mathcal{B}(H_1, H_2)$. Then*

$$\frac{\|A - B\|}{(1 + \|A\|^2)^{\frac{1}{2}}(1 + \|B\|^2)^{\frac{1}{2}}} \leq \hat{\delta}(A, B) \leq \|A - B\|.$$

In particular, the relative topology induced by the gap topology on $\mathcal{B}(H_1, H_2)$ coincides with the norm topology.

Theorem 2.13. [9, Theorem 3.5] *Let $A, B \in \mathcal{C}(H_1, H_2)$ be densely defined. Then the operators $B\check{B}^{\frac{1}{2}}\check{A}^{\frac{1}{2}}$, $\hat{B}^{\frac{1}{2}}A\check{A}^{\frac{1}{2}}$, $A\check{A}^{\frac{1}{2}}\check{B}^{\frac{1}{2}}$ and $\hat{A}^{\frac{1}{2}}B\check{B}^{\frac{1}{2}}$ are bounded and*

$$\hat{\delta}(A, B) = \max \left\{ \|B\check{B}^{\frac{1}{2}}\check{A}^{\frac{1}{2}} - \hat{B}^{\frac{1}{2}}A\check{A}^{\frac{1}{2}}\|, \ \|A\check{A}^{\frac{1}{2}}\check{B}^{\frac{1}{2}} - \hat{A}^{\frac{1}{2}}B\check{B}^{\frac{1}{2}}\| \right\}.$$

3. THE CARRIER GRAPH TOPOLOGY

First we consider two examples to illustrate that the map $T \mapsto T^\dagger$ is not continuous in the gap metric and the set $\mathcal{CR}(H_1, H_2)$ is not open in the gap topology.

Example 3.1. Let $T_n := \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$ and $T := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $T_n \to T$ in the norm topology of $\mathcal{B}(\mathbb{R}^2)$. Since for $\mathcal{B}(\mathbb{R}^2)$, the convergence in the norm topology and the convergence in the gap metric are same (Theorem 2.12), $T_n \to T$ in the gap metric.

But $T_n^\dagger := \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$ is not a Cauchy sequence. Thus the map $T \mapsto T^\dagger$ is not continuous.

Example 3.2. Let $T : \ell^2 \to \ell^2$ be given by

$$T(x_1, x_2, x_3, \dots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots) \quad \text{for all } (x_1, x_2, x_3, \dots) \in \ell^2.$$

Then $R(\alpha T)$ is not closed for each $\alpha \neq 0$. But $\hat{\delta}(\alpha T, 0) \leq \|\alpha T - 0\| = |\alpha|\|T\| \leq |\alpha|$. In other words, we can find an operator with non closed range in every neighbourhood of 0, which has a closed range. Thus $\mathcal{CR}(H_1, H_2)$ is not open in the gap topology.

Next, we define a new metric $\eta(\cdot, \cdot)$ on $\mathcal{C}(H_1, H_2)$ and show that with respect to this metric $\eta(\cdot, \cdot)$, the set $\mathcal{CR}(H_1, H_2)$ is open in $\mathcal{C}(H_1, H_2)$ and the map $T \mapsto T^\dagger$ is an isometry between $\mathcal{C}(H_1, H_2)$ and $\mathcal{C}(H_2, H_1)$. We also show that this metric $\eta(\cdot, \cdot)$ has properties similar to those of $\hat{\delta}(\cdot, \cdot)$ given in Theorem 2.11.

Lemma 3.3. *Let* $T \in \mathcal{C}(H_1, H_2)$ *be densely defined. Then* $\overline{C(T)} = N(T)^\perp$.

Proof. By definition $\overline{C(T)} \subseteq N(T)^\perp$. For the converse, let $x \in N(T)^\perp$. Choose $\{x_n\} \subseteq D(T)$ such that $x_n \to x$. Then by Note 2.2, $x_n = u_n + v_n$, where $u_n \in N(T), v_n \in C(T)$ and $\langle u_n, v_n \rangle = 0$, for all n. By the Pythagorean property, $u_n \to u \in N(T)$ and $v_n \to v \in \overline{C(T)}$. Hence $x = u + v$. That is $u = x - v \in N(T)^\perp$. Thus $u = 0$, concluding $x = v \in \overline{C(T)}$. $\square$

Definition 3.4. Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Define the *Carrier Graph of* T by

$$G_C(T) := \{(x, Tx) : x \in C(T)\} \subseteq H_1 \times H_2.$$

Note that for $T \in \mathcal{C}(H_1, H_2)$, $G_C(T)$ is a closed subspace of $H_1 \times H_2$. For $S, T \in \mathcal{C}(H_1, H_2)$, define

$$\eta(T, S) = \hat{\delta}(G_C(T), G_C(S)).$$

Note that if $N(T) = \{0\} = N(S)$, then $C(T) = H_1 = C(S)$. Hence $G_C(T) = G(T)$, $G_C(S) = G(S)$, thus $\eta(T, S) = \hat{\delta}(T, S)$.

Proposition 3.5. *The function* $\eta(\cdot, \cdot) : \mathcal{C}(H_1, H_2) \times \mathcal{C}(H_1, H_2) \to \mathbb{R}$ *defined above is a metric on* $\mathcal{C}(H_1, H_2)$.

Proof. It suffices to prove that if $\eta(T, S) = 0$, then $T = S$. If $\eta(T, S) = 0$, then $G_C(T) = G_C(S)$. Hence $C(T) = C(S)$ and $Tx = Sx$ for all $x \in C(T)$. Now

$$C(T) = C(S) \Rightarrow \overline{C(T)} = \overline{C(S)} \Rightarrow N(S)^\perp = N(T)^\perp \Rightarrow N(S) = N(T).$$

Since $S, T \in \mathcal{C}(H_1, H_2)$, $D(T) = N(T) \oplus^\perp C(T) = N(S) \oplus^\perp C(S) = D(S)$. Hence for $x \in D(T)$, $x = u + v$, $u \in N(T)$ and $v \in C(T)$. Then $Tx = Tv = Sv = Sx$.

All the other axioms of the metric follow from the fact that $\hat{\delta}(\cdot, \cdot)$ is a metric on the set of all closed subspaces of $H_1 \times H_2$ [10, Page 197]. $\square$

Remark 3.6. Let $T_n, T \in \mathcal{C}(H_1, H_2)$ be densely defined. We say $T_n \to T$ in the metric $\eta(\cdot, \cdot)$, if $\eta(T_n, T) \to 0$ as $n \to \infty$. We denote this by $T_n \xrightarrow{\eta} T$.

Remark 3.7. The function $\eta(\cdot, \cdot)$ satisfies all the properties of $\hat{\delta}(\cdot, \cdot)$ mentioned in [10, Pages 197, 200, 201]. In general, it is not complete as the following example shows.

Let $H_1 = H_2 = H$ and consider $T_n = nI$. Then $C(T_n) = H$, thus $G(T_n) = G_C(T_n)$ and $\eta(T_n, T_m) = \hat{\delta}(T_n, T_m)$. It is known that in the metric $\hat{\delta}(\cdot, \cdot)$, T_n is a Cauchy sequence, but not convergent ([10, Remark 2.10, page 202]).

Definition 3.8. The topology induced by the metric $\eta(\cdot, \cdot)$ on $\mathcal{C}(H_1, H_2)$ is called the *Carrier Graph Topology*.

Theorem 3.9. *Let $S, T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then $\eta(T^\dagger, S^\dagger) = \eta(T, S)$. In other words, the map $\mu : \left(\mathcal{C}(H_1, H_2), \eta(\cdot, \cdot) \right) \rightarrow \left(\mathcal{C}(H_2, H_1), \eta(\cdot, \cdot) \right)$ given by $\mu(T) = T^\dagger$ is continuous in the carrier graph topology. In fact, it is an isometry.*

Proof. The map $U : H_1 \times H_2 \rightarrow H_2 \times H_1$ given by $(x, y) \mapsto (y, x)$ is an onto isometry, it preserves the gap between closed subspaces. Further, $C(T^\dagger) = R(T)$. Hence $(x, Tx) \in G_C(T) \Leftrightarrow x \in C(T), Tx \in R(T) \Leftrightarrow (Tx, x) = (Tx, T^\dagger Tx) \in G_C(T^\dagger)$. Thus, $U(G_C(T)) = G_C(T^\dagger)$. Similarly, $U(G_C(S)) = G_C(S^\dagger)$. Therefore $\eta(T^\dagger, S^\dagger) = \hat{\delta}(G_C(T^\dagger), G_C(S^\dagger)) = \hat{\delta}(U(G_C(T)), U(G_C(S))) = \hat{\delta}(G_C(T), G_C(S)) = \eta(T, S)$. $\square$

Here is an immediate consequence of Theorem 3.9.

Theorem 3.10. *Let $T_n, T \in C(H_1, H_2)$ be densely defined. Then the following statements are equivalent.*

(1) $T_n \xrightarrow{\eta} T$.
(2) $T_n^\dagger \xrightarrow{\eta} T^\dagger$.

Next, we prove a very important Theorem about the Carrier Graph Topology. It makes use of the concept of the reduced minimum modulus (See Definition 2.4 and Proposition 2.7).

Theorem 3.11. *Let $S, T \in \mathcal{C}(H_1, H_2)$ be densely defined. If $T \in \mathcal{CR}(H_1, H_2)$, then*

$$\frac{1}{\sqrt{1 + \|T^\dagger\|^2}} \leq \eta(S, T) + \gamma(S)(1 + \eta(S, T)).$$

Further, if $\eta(S, T) < \dfrac{1}{\sqrt{1 + \|T^\dagger\|^2}}$, then $S \in \mathcal{CR}(H_1, H_2)$ and

$$\|S^\dagger\| \leq \frac{(1 + \|T^\dagger\|^2)^{\frac{1}{2}}(\eta(T, S) + 1)}{1 - (1 + \|T^\dagger\|^2)^{\frac{1}{2}}\eta(T, S)}.$$

Proof. There exists a sequence $(x_n) \subseteq C(S)$ with $\|x_n\| = 1$, for all n such that $\|Sx_n\| \rightarrow \gamma(S)$ as $n \rightarrow \infty$.

Let $\epsilon > 0$. Then $\dfrac{(x_n, Sx_n)}{(1 + \|Sx_n\|^2)^{\frac{1}{2}}} \in G_C(S)$ and its norm is one. There exists $(u_n, Tu_n) \in G_C(T)$ such that

$$\left\| \frac{(x_n, Sx_n)}{(1 + \|Sx_n\|^2)^{\frac{1}{2}}} - (u_n, Tu_n) \right\| \le \eta(S, T) + \epsilon.$$

Let $\beta_n := (1 + \|Sx_n\|^2)^{\frac{1}{2}}$. Then $\beta_n \to (1 + \gamma(S)^2)^{\frac{1}{2}}$ as $n \to \infty$. Also, from the above inequality, $\|(x_n - \beta_n u_n, Sx_n - \beta_n Tu_n)\| \le \beta_n(\eta(S, T) + \epsilon)$. That is

$$\|x_n - \beta_n u_n\|^2 + \|Sx_n - \beta_n Tu_n\|^2 \le \beta_n^2(\eta(S, T) + \epsilon)^2.$$

Note that since $u_n \in C(T)$, we have $T^\dagger Tu_n = u_n$.

Now

$$\begin{aligned}
1 = \|x_n\|^2 &\le \left(\|x_n - \beta_n u_n\| + \|\beta_n u_n\| \right)^2 \\
&\le \left(\|x_n - \beta_n u_n\| + \|\beta_n T^\dagger Tu_n\| \right)^2 \\
&\le \left(\|x_n - \beta_n u_n\| + \|T^\dagger\| \|\beta_n Tu_n\| \right)^2 \\
&\le (1 + \|T^\dagger\|^2)\left(\|x_n - \beta_n u_n\|^2 + \|\beta_n Tu_n\|^2 \right) \\
&\le (1 + \|T^\dagger\|^2)\left(\|x_n - \beta_n u_n\|^2 + (\|Sx_n - \beta_n Tu_n\| + \|Sx_n\|)^2 \right) \\
&\le (1 + \|T^\dagger\|^2)\Big(\|x_n - \beta_n u_n\|^2 + \|Sx_n - \beta_n Tu_n\|^2 + \|Sx_n\|^2 \\
&\quad + 2\|Sx_n\| \|Sx_n - \beta_n Tu_n\| \Big).
\end{aligned}$$

$$1 \le (1 + \|T^\dagger\|^2)\left[\beta_n^2(\eta(S, T) + \epsilon)^2 + \|Sx_n\|^2 + 2\|Sx_n\|\Big(\beta_n(\eta(S, T) + \epsilon)\Big) \right].$$

Letting $n \to \infty$, we get

$$1 \le (1 + \|T^\dagger\|^2)\left[(1 + \gamma(S)^2)(\eta(S, T) + \epsilon)^2 + \gamma(S)^2 + 2\gamma(S)(1 + \gamma(S)^2)^{\frac{1}{2}}\Big(\eta(S, T) + \epsilon\Big) \right].$$

Since ϵ was arbitrary, we have

$$\begin{aligned}
1 &\le (1 + \|T^\dagger\|^2)\left[(1 + \gamma(S)^2)\eta(S, T)^2 + \gamma(S)^2 + 2\gamma(S)(1 + \gamma(S)^2)^{\frac{1}{2}}\eta(S, T) \right] \\
&= (1 + \|T^\dagger\|^2)\left[(1 + \gamma(S)^2)^{\frac{1}{2}}\eta(S, T) + \gamma(S) \right]^2.
\end{aligned}$$

By taking square root and noting that $(1 + \gamma(S)^2)^{\frac{1}{2}} \le 1 + \gamma(S)$, we obtain,

$$1 \le (1 + \|T^\dagger\|^2)^{\frac{1}{2}}\left[\big(1 + \gamma(S)\big)\eta(S, T) + \gamma(S) \right].$$

Thus

$$\frac{1}{\sqrt{1 + \|T^\dagger\|^2}} \le \eta(S, T) + \gamma(S)\big(1 + \eta(S, T)\big).$$

Hence, if $\eta(S,T) < \dfrac{1}{\sqrt{1+\|T^\dagger\|^2}}$, then

$$\gamma(S) \geq \frac{1 - \eta(S,T)\,\sqrt{1+\|T^\dagger\|^2}}{\sqrt{1+\|T^\dagger\|^2}(1+\eta(S,T))} > 0.$$

That is $R(S)$ is closed and

$$\|S^\dagger\| = \frac{1}{\gamma(S)} \leq \frac{\sqrt{1+\|T^\dagger\|^2}(1+\eta(S,T))}{1 - \eta(S,T)\,\sqrt{1+\|T^\dagger\|^2}}. \quad \square$$

Corollary 3.12. *The set $\mathcal{CR}(H_1, H_2)$ is open in $\mathcal{C}(H_1, H_2)$ with respect to the carrier graph topology.*

Remark 3.13. Note that in Example 3.2, since $C(0) = \{0\}, G_C(0) = \{(0,0)\}$. For $\alpha \neq 0$, $G_C(\alpha T) \neq \{(0,0)\}$. Hence $\eta(\alpha T, 0) = \hat{\delta}(G_C(\alpha T), G_C(0)) = 1$. Thus even for small non zero α, αT is not close to 0 in the carrier graph topology.

Remark 3.14. If $T \in \mathcal{C}(H_1, H_2) \setminus \{0\}$, then $\eta(T, 0) = 1$. Hence

$$\eta(T, 0) < 1 \Leftrightarrow T = 0.$$

In other words, 0 is an isolated point in $\big(\mathcal{C}(H_1, H_2), \eta(\cdot, \cdot)\big)$.

Corollary 3.15. *Let $S \in \mathcal{C}(H)$ be such that $\eta(S, I) < \frac{1}{\sqrt{2}}$, then $R(S)$ is closed.*

Proof. Follows by taking $T = I$ and noting that $I^\dagger = I$. $\qquad\square$

Lemma 3.16. [5, 6, 15] *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then*

(1) $\check{T} \in \mathcal{B}(H_1)$, $\hat{T} \in \mathcal{B}(H_2)$
(2) $\hat{T}T \subseteq T\check{T}, \quad \|T\check{T}\| \leq \dfrac{1}{2}$ *and* $\check{T}T^* \subseteq T^*\hat{T}, \quad \|T^*\hat{T}\| \leq \dfrac{1}{2}$.

Proposition 3.17. *Let $T \in \mathcal{C}(H)$ be densely defined unbounded symmetruc operator (i.e., $T \subseteq T^*$). Then $\hat{\delta}(T, nI) = \dfrac{1}{\sqrt{1+n^2}}$. If T is one-to-one, then $\eta(T, nI) = \dfrac{1}{\sqrt{1+n^2}}$.*

Proof. We use the formula of 2.13. Let $S = nI$. Then $\check{S}^{\frac{1}{2}} = \dfrac{1}{\sqrt{1+n^2}}I$ and $S\check{S}^{\frac{1}{2}} = \dfrac{n}{\sqrt{1+n^2}}I$. Now $S\check{S}^{\frac{1}{2}}\check{T}^{\frac{1}{2}} - \hat{S}^{\frac{1}{2}}T\check{T}^{\frac{1}{2}} = \dfrac{1}{\sqrt{1+n^2}}\big(n\check{T}^{\frac{1}{2}} - T\check{T}^{\frac{1}{2}}\big)$ and $T\check{T}^{\frac{1}{2}}\check{S}^{\frac{1}{2}} - \hat{T}^{\frac{1}{2}}S\check{S}^{\frac{1}{2}} = \dfrac{1}{\sqrt{1+n^2}}\big(T\check{T}^{\frac{1}{2}} - n\hat{T}^{\frac{1}{2}}\big)$. Let $A := n\check{T}^{\frac{1}{2}} - T\check{T}^{\frac{1}{2}}$ and $B := T\check{T}^{\frac{1}{2}} - n\hat{T}^{\frac{1}{2}}$.

Then $\hat{\delta}(T, nI) = \dfrac{1}{\sqrt{1+n^2}}\max\{\|A\|, \|B\|\}$. To find $\|A\|$, consider

$$
\begin{aligned}
A^*A &= \left(n\check{T}^{\frac{1}{2}} - T^*\hat{T}^{\frac{1}{2}}\right)\left(n\check{T}^{\frac{1}{2}} - T\check{T}^{\frac{1}{2}}\right) \\
&= n^2\check{T} - nT\check{T} - nT^*\hat{T}^{\frac{1}{2}}\check{T}^{\frac{1}{2}} + T^*\hat{T}^{\frac{1}{2}}T\check{T}^{\frac{1}{2}} \\
&= n^2\check{T} - nT\check{T} - nT^*\hat{T}^{\frac{1}{2}}\check{T}^{\frac{1}{2}} + \check{T}^{\frac{1}{2}}T^*T\check{T}^{\frac{1}{2}} \\
&= n^2\check{T} - nT\check{T} - nT^*\hat{T}^{\frac{1}{2}}\check{T}^{\frac{1}{2}} + \check{T}^{\frac{1}{2}}TT\check{T}^{\frac{1}{2}} \quad \text{(since } T \subseteq T^*) \\
&= (n^2I + T^*T)\check{T} - 2nT\check{T}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|A\|^2 &= \sup_{x\in H,\,\|x\|=1}\|(n^2I + T^*T)\check{T}x\| - \inf_{x\in H,\,\|x\|=1}\|2nT\check{T}x\| \\
&= \|(n^2I + T^*T)\check{T}\| - m\big(2nT\check{T}\big) \\
&= \|(n^2I + T^*T)\check{T}\| - m\big(4n^2T^*T\check{T}^2\big)^{\frac{1}{2}} \\
&= \sup\left\{\frac{n^2+\lambda}{1+\lambda} : \lambda \in \sigma(T^*T)\right\} - \left(\inf\left\{\frac{4n^2\mu}{(1+\mu)^2} : \mu \in \sigma(T^*T)\right\}\right)^{\frac{1}{2}} \\
&= 1.
\end{aligned}
$$

Similarly we can show that $\|B\| = 1$. Hence $\hat{\delta}(T, nI) = \dfrac{1}{\sqrt{1+n^2}}$.

If T is one-to-one, we have $\eta(T, nI) = \hat{\delta}(T, nI)$. $\qquad\qquad\square$

Example 3.18. Let $H = L^2[0,1]$, relative to Lebesgue measure. Let

$$
\begin{aligned}
D(T_1) &= \{f \in H : f \text{ is absolutely continuous and } f' \in H\} \\
D(T_2) &= D(T_1) \cap \{f : f(0) = 0 = f(1)\}.
\end{aligned}
$$

Define $T_k f = i\,f'$ for $f \in D(T_k)$, $k = 1, 2$. It can be shown that T_k $(k = 1, 2)$ is densely defined closed operator such that $T_1^* = T_2$. That is T_2 is symmetric. Also $R(T_k)$ $(k = 1, 2)$ is closed (see [18, Example 13.4, page 331] for details). Also note that T_2 is one-to-one. Hence by Proposition 3.17,

$$
\eta(T_1, 2I) = \eta(T_2, 2I) = \hat{\delta}(T_2, 2I) = \hat{\delta}(T_1, 2I) = \frac{1}{\sqrt{5}} < \frac{2}{\sqrt{5}} = \frac{1}{\sqrt{1 + \|(2I)^\dagger\|^2}}.
$$

We have $\gamma(2I) = 2$. Hence T_2 and $2I$ satisfy hypotheses in Theorem 3.11 and it can be shown that all the estimtes in Theorem 3.11 are true. On the other hand, $\eta(T_1, I) = \eta(T_2, I) = \frac{1}{\sqrt{2}}$. This shows that the condition in Theorem 3.11 is not necessary.

Theorem 3.19. *Let $S, T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then*

$$
\eta(S, T) = \left\|\begin{bmatrix} T^\dagger T\check{T} - S^\dagger S\check{S} & T^*\hat{T} - S^*\hat{S} \\ T\check{T} - S\check{S} & \hat{T} - \hat{S} \end{bmatrix}\right\|.
$$

Proof. Let $(x_1, x_2) \in H_1 \times H_2$ and $P := P_{G_C(T)}$. Then $P(x_1, x_2) \in G_C(T)$. Hence there exists a $z \in C(T)$ such that $P(x_1, x_2) = (z, Tz)$. Now $(x_1 - z, x_2 - Tz) \in (G_C(T))^\perp$.

Note that $G_C(T)^\perp = \{(-T^*y + u, y) : y \in D(T^*), u \in N(T)\}$. Hence $(x_1 - z, x_2 - Tz) = (-T^*y + u, y)$ for some $y \in D(T^*)$ and $u \in N(T)$. That is

$$
\begin{aligned}
z &= x_1 + T^*y - u \\
Tz &= x_2 - y.
\end{aligned}
$$

Since $\check{T} \in \mathcal{B}(H_1)$ and $\hat{T} \in \mathcal{B}(H_2)$, we have

$$\check{T}z = \check{T}x_1 + \check{T}T^*y - \check{T}u$$

$$\hat{T}Tz = \hat{T}x_2 - \hat{T}y. \tag{3.1}$$

From Equation 3.1, we can get that

$$T^*\hat{T}Tz = T^*\hat{T}x_2 - T^*\hat{T}y.$$

By Lemma 3.16, $\hat{T}Tx = T\check{T}x$ for all $x \in D(T)$, the above equation can be written as

$$
\begin{aligned}
T^*T\check{T}z &= T^*\hat{T}x_2 - T^*\hat{T}y \\
&= T^*\hat{T}x_2 - \check{T}T^*y \\
z &= \check{T}x_1 + T^*\hat{T}x_2 - \check{T}u.
\end{aligned}
$$

Hence $Tz = T\check{T}x_1 + TT^*\hat{T}x_2$.

Since $z \in C(T)$, we have $T^\dagger Tz = z$. Hence

$$
\begin{aligned}
z &= T^\dagger T\check{T}x_1 + T^\dagger TT^*\hat{T}x_2 - T^\dagger T\check{T}u \\
&= T^\dagger T\check{T}x_1 + T^\dagger TT^*\hat{T}x_2 - T^\dagger\check{T}Tu \\
&= T^\dagger T\check{T}x_1 + T^\dagger TT^*\hat{T}x_2.
\end{aligned}
$$

Hence the matrix of P is given by

$$
P = \begin{bmatrix} T^\dagger T\check{T} & T^\dagger TT^*\hat{T} \\ T\check{T} & TT^*\hat{T} \end{bmatrix} = \begin{bmatrix} T^\dagger T\check{T} & T^*\hat{T} \\ T\check{T} & TT^*\hat{T} \end{bmatrix}. \tag{3.2}
$$

Similarly if $Q := P_{G(S)}$, then we have

$$
Q = \begin{bmatrix} S^\dagger S\check{S} & S^*\hat{S} \\ S\check{S} & SS^*\hat{S} \end{bmatrix}.
$$

Hence

$$
P - Q = \begin{bmatrix} T^\dagger T\check{T} - S^\dagger S\check{S} & T^*\hat{T} - S^*\hat{S} \\ T\check{T} - S\check{S} & \hat{T} - \hat{S} \end{bmatrix}
$$

and

$$
\eta(T, S) = \|P - Q\| = \left\| \begin{bmatrix} T^\dagger T\check{T} - S^\dagger S\check{S} & T^*\hat{T} - S^*\hat{S} \\ T\check{T} - S\check{S} & \hat{T} - \hat{S} \end{bmatrix} \right\|. \quad \square
$$

Remark 3.20. We note that the metric $f(\cdot, \cdot)$ defined by Labrousse and Mbekhta (see [12] for details) on $\mathcal{C}(H_1, H_2)$ is the same as the metric $\eta(\cdot, \cdot)$ above. To see this, let

$$F_T := \begin{bmatrix} \check{T} - P_{N(T)} & T^*\hat{T} \\ T\check{T} & I - \hat{T} \end{bmatrix}.$$

Then the metric $f(\cdot, \cdot)$ is defined by $f(T, S) := \|F_T - F_S\|$. First we note that F_T is the same as P in Equation 3.2 above. To see this it is enough to show that $\check{T} - P_{N(T)} = T^\dagger T \check{T}$. For $z \in H_1$, write $z = x + y$ such that $x \in N(T)$ and $y \in N(T)^\perp$. Then

$$\begin{aligned} T^\dagger T \check{T} z &= T^\dagger T \check{T} x + T^\dagger T \check{T} y \\ &= T^\dagger \hat{T} T x + T^\dagger T \check{T} y \\ &= T^\dagger T \check{T} y \\ &= P_{N(T^\perp)} \check{T} y \\ &= (I - P_{N(T)}) \check{T}(z - x) \\ &= \check{T} z - P_{N(T)} z. \end{aligned}$$

Thus $F_T = P$. Similarly $F_S = Q$. Hence $f(T, S) = \|F_T - F_S\| = \|P - Q\| = \eta(T, S)$.

Since the metrics $\eta(\cdot, \cdot)$ and $f(\cdot, \cdot)$ are equal, Theorems 3.9 and 3.10 can be proved using the properties of $f(\cdot, \cdot)$ (see [12] for details).

Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then $G(T) = G_C(T) \oplus^\perp J(N(T))$, where $J : H_1 \to H_1 \times \{0\}$ is the isometry given by $J(x) = (x, 0)$ for all $x \in H_1$. Since isometry preserves the distances, it preserves the gap between two subspaces.

Theorem 3.21. *Let $T, S \in \mathcal{C}(H_1, H_2)$ be densely defined. Then*

$$|\eta(T, S) - \hat{\delta}(N(T), N(S))| \le \hat{\delta}(T, S) \le \eta(T, S) + \hat{\delta}(N(T), N(S)).$$

Proof. The proof follows from Remark 2.8 of [12]. $\square$

Remark 3.22. If $R(T) = R(S)$, then $N(T^\dagger) = N(S^\dagger)$. Hence $\hat{\delta}(T^\dagger, S^\dagger) = \eta(T^\dagger, S^\dagger)$. Now by Theorem 3.9, it follows that $\hat{\delta}(T^\dagger, S^\dagger) = \eta(T, S)$.

4. BOUNDED OPERATORS

In this section, we obtain some results for bounded operators from the point of view of the Carrier Graph Topology .

Theorem 4.1. *Let $S \in \mathcal{C}(H_1, H_2)$ be densely defined. If $T \in \mathcal{B}(H_1, H_2)$, then*

$$\frac{1}{\sqrt{1 + \|T\|^2}} \le \eta(S, T) + \gamma(S^\dagger)(1 + \eta(S, T)).$$

Further, if $\eta(S, T) < \dfrac{1}{\sqrt{1 + \|T\|^2}}$, then $S \in \mathcal{B}(H_1, H_2)$ and

$$\|S\| \le \frac{(1 + \|T\|^2)^{\frac{1}{2}}(\eta(T, S) + 1)}{1 - (1 + \|T\|^2)^{\frac{1}{2}}\eta(T, S)}.$$

Proof. Let $T \in \mathcal{B}(H_1, H_2)$ and $S \in \mathcal{C}(H_1, H_2)$ be such that $\eta(S, T) < \dfrac{1}{(1 + \|T\|^2)^{\frac{1}{2}}}$.

Let $A = T^\dagger$. Since $A^\dagger = T$ is bounded, $R(A)$ is closed. Now by Theorem 3.9, $\eta(A, S^\dagger) = \eta(T, S) < \dfrac{1}{(1 + \|A^\dagger\|)^2}$. Hence by Theorem 3.11, $R(S^\dagger)$ is closed. Hence S is bounded by Remark 2.8.

Replacing $S^\dagger, T^\dagger$ by S, T respectively in Theorem 3.11 and using the fact that $\eta(S, T) = \eta(S^\dagger, T^\dagger)$, we can obtain the bound for $\|S\|$.   □

Corollary 4.2. $\mathcal{B}(H_1, H_2)$ *is an open subset of* $\mathcal{C}(H_1, H_2)$ *with respect to the Carrier Graph Topology.*

The following example shows that unlike the gap topology, the relative topology induced by $\eta(\cdot, \cdot)$ on $\mathcal{B}(H_1, H_2)$ is different from the norm topology on $\mathcal{B}(H_1, H_2)$.

Example 4.3. Let T_n and T be as in Example 3.1. We can note $N(T_n) = \{0\}$. Hence $C(T_n) = \mathbb{C}^2$. Now $G(T_n) = \{(x_1, x_2, x_1, x_2/n) : x_1, x_2 \in \mathbb{C}\} \subseteq \mathbb{C}^4$, $\dim G_C(T_n) = 2$. Also $N(T) = \{(0, x_2) : x_2 \in \mathbb{C}\}$, $C(T) = \{(x_1, 0) : x_1 \in \mathbb{C}\}$, $G_C(T) = \{(x_1, 0, x_1, 0) : x_1 \in \mathbb{C}\}$, $\dim G_C(T) = 1$. Therefore $\eta(T_n, T) = 1$ for all n by [10, Page 200, Corollary 2.6]. Then $\eta(T_n^\dagger, T^\dagger) = 1$ for all n by Theorem 3.9. Note that $T_n \to T$ in norm and also in the gap topology but not in the Carrier Graph Topology. Also $\eta(T_n, T_m) \leq \|T_n - T_m\| \leq |\frac{1}{n} - \frac{1}{m}|$ (since $N(T_n) = N(T_m)$). Thus $\{T_n\}$ is a Cauchy sequence but not convergent.

Theorem 4.4. *Let* $S, T \in \mathcal{B}(H_1, H_2)$. *Then*

(i)
$$\frac{\|T - S\|}{\sqrt{1 + \|T\|^2}\sqrt{1 + \|S\|^2}} - \hat{\delta}(N(T), N(S)) \leq \eta(S, T) \leq \|T - S\| + \hat{\delta}(N(S), N(T)).$$

(ii) *If* $R(T)$ *is closed and* $\|T - S\| + \hat{\delta}(N(S), N(T)) < \dfrac{1}{\sqrt{1 + \|T^\dagger\|^2}}$, *then* $R(S)$ *is closed and* $\|S^\dagger\| \leq \dfrac{(1 + \|T^\dagger\|^2)^{\frac{1}{2}}(\eta(T, S) + 1)}{1 - (1 + \|T^\dagger\|^2)^{\frac{1}{2}}\eta(T, S)}$.

(iii) *If* $T_n \in \mathcal{B}(H_1, H_2)$ *be such that* $\|T_n - T\| + \hat{\delta}(N(T_n), N(T)) \to 0$, *then* $\eta(T_n^\dagger, T^\dagger) \to 0$.

Proof. We know by Theorem 2.12 that $\|T - S\| \leq \sqrt{1 + \|T\|^2}\sqrt{1 + \|S\|^2}\,\hat{\delta}(S, T)$. Now by Theorem 3.21, $\hat{\delta}(S, T) \leq \eta(S, T) + \hat{\delta}(N(S), N(T))$. Combining these two relations we obtain $\dfrac{\|T - S\|}{\sqrt{1 + \|T\|^2}\sqrt{1 + \|S\|^2}} - \hat{\delta}(N(T), N(S)) \leq \eta(S, T)$. The second part of the inequality follows from Theorem 3.21 and Theorem 2.12.

Assume that $R(T)$ is closed and $\|T - S\| + \hat{\delta}(N(S), N(T)) < \dfrac{1}{\sqrt{1 + \|T^\dagger\|^2}}$.

Making use of (i), we get $\eta(S, T) < \dfrac{1}{\sqrt{1 + \|T^\dagger\|^2}}$. The conclusion follows from Theorem 3.11.

If $T_n \in \mathcal{B}(H_1, H_2)$ be such that $\|T_n - T\| + \hat{\delta}(N(T_n), T) \to 0$, it follows that $\eta(T_n, T) \to 0$. Hence by Theorem 3.9, $\eta(T_n^\dagger, T^\dagger) \to 0$. $\qquad\square$

Remark 4.5. The statement (ii) of Theorem 4.4 is similar to that of Ding and Huang [3, Theorem 3.1].

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