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ELEMENTARY OPERATORS AND SUBHOMOGENEOUS C^*-ALGEBRAS II

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Abstract. Let A be a separable unital C^*-algebra and let Θ_A be the canonical contraction from the Haagerup tensor product of A with itself to the space of completely bounded maps on A. In our previous paper we showed that if A satisfies (a) the lengths of elementary operators on A are uniformly bounded, or (b) the image of Θ_A equals the set of all elementary operators on A, then A is necessarily SFT (subhomogeneous of finite type). In this paper we extend this result; we show that if A satisfies (a) or (b) then the codimensions of 2-primal ideals of A are also finite and uniformly bounded. Using this, we provide an example of a unital separable SFT algebra which does not satisfy (a) nor (b). However, if the primitive spectrum of a unital SFT algebra A is Hausdorff, we show that such an A satisfies (a) and (b).

1. Introduction and Preliminaries

Through this paper A will denote a C^*-algebra. The center of A is denoted by $Z(A)$, and the set of all ideals of A is denoted by $\mathrm{Id}(A)$ (in this paper ideal means closed two-sided ideal). By $\mathrm{Prim}(A)$ we denote the primitive spectrum of A (i.e. the set of all primitive ideals of A), equipped with the Jacobson topology.

Let $A \otimes_h A$ be the Haagerup tensor product of A with itself. If $M(A)$ denotes the multiplier algebra of A, and $\mathrm{ICB}(A)$ the space of all completely bounded maps $T : A \to A$ which preserve every ideal of A (i.e. $T(J) \subseteq J$, for each $J \in \mathrm{Id}(A)$), there is a canonical contraction $\Theta_A : M(A) \otimes_h M(A) \to \mathrm{ICB}(A)$

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given on elementary tensors by

$$\Theta_A(a \otimes b)(x) := axb \quad (a, b \in M(A),\ x \in A).$$

The subset $\Theta_A(M(A) \otimes M(A))$ of $\operatorname{Im}\Theta_A$(=the image of Θ_A) is denoted by $\mathrm{E}(A)$ and its elements are called *elementary operators* on A. The *length* $\ell(T)$ of $T \in \mathrm{E}(A)$ is defined as the smallest number d such that $T = \Theta_A(\sum_{k=1}^d a_k \otimes b_k)$, for some $a_k, b_k \in M(A)$. If

$$\ell(\mathrm{E}(A)) := \sup\{\ell(T) : \ T \in \mathrm{E}(A)\} < \infty,$$

we say that $\mathrm{E}(A)$ is of *finite length*.

In our previous paper [10] we considered the following two conditions on A:

(a) $\mathrm{E}(A)$ is of finite length,
(b) $\operatorname{Im}\Theta_A = \mathrm{E}(A)$.

In [10, 2.13] we showed that if a separable A satisfies (a) or (b) then A is necessarily *subhomogeneous of finite type* (shorter, *SFT algebra*). Recall from [10] (see also [13]) that A is said to be an n-SFT algebra if A is n-subhomogeneous (that is, the supremum of dimensions of all irreducible representations of A equals $n < \infty$) and the C^*-bundles corresponding to the homogeneous sub-quotients of A must be of finite type. If A is an n-SFT algebra for some n then we say that A is a SFT algebra.

We first give another description of separable SFT algebras:

Proposition 1.1. *Let A be a separable* C^**-algebra. Then the following conditions are equivalent:*

(i) *A is a SFT algebra,*
(ii) *There exists $m \in \mathbb{N}$ and elements $x_1, \dots, x_m \in A$ such that*

$$\operatorname{span}\{x_1 + P, \dots, x_m + P\} = A/P, \quad \text{for all } P \in \operatorname{Prim}(A).$$

Proof. (i) $\Rightarrow$ (ii). Suppose that A is an n-SFT algebra, and let

$$\{0\} = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_p = A \tag{1.1}$$

be the standard composition series of A [12, 6.2.5]. By assumption, each homogeneous quotient J_i/J_{i-1} is of finite type. Let $J := J_1$ (J is called the *n-homogeneous ideal* of A), and first assume that $p = 2$. Then A/J is homogeneous (of finite type). In this case, by [11, Section 1] there exist elements $a_1, \dots, a_r \in J$ and $\dot{b}_1, \dots, \dot{b}_s \in A/J$ such that

$$\operatorname{span}\{a_1 + P, \dots, a_r + P\} = J/P, \quad \text{for all } P \in \operatorname{Prim}(J),$$

and

$$\operatorname{span}\{\dot{b}_1 + \dot{P}, \dots, \dot{b}_s + \dot{P}\} = (A/J)/\dot{P}, \quad \text{for all } \dot{P} \in \operatorname{Prim}(A/J).$$

Choose $b_1, \dots, b_s \in A$ such that $\dot{b}_i = b_i + J$ ($1 \le i \le s$). Then it is easy to check that

$$\operatorname{span}\{a_1 + P, \dots, a_r + P, b_1 + P, \dots, b_s + P\} = A/P, \quad \text{for all } P \in \operatorname{Prim}(A).$$

Indeed, let $P \in \operatorname{Prim}(A)$. If $J \subseteq P$, then $\dot{P} := P/J \in \operatorname{Prim}(A/J)$, and since $A/P \cong (A/J)/\dot{P}$, we have $\operatorname{span}\{b_1 + P, \dots, b_s + P\} = A/P$. If $J \nsubseteq P$ then

$P \cap J \in \text{Prim}(J)$. Since A is subhomogeneous (hence liminal), primitive ideals of A are maximal, so $A/P = (J+P)/P \cong J/(P \cap J)$. It follows that $\text{span}\{a_1 + P, \ldots, a_r + P\} = A/P$. If $p > 2$, we proceed by induction, using the same arguments as above.

(ii) $\Rightarrow$ (i) If A satisfies (ii) then A is obviously (say n-)subhomogeneous. Let p be the length of the corresponding composition series (1.1) of A. If $p = 1$ then A is homogeneous. In this case, by [11, Section 1], A is of finite type. Suppose that $p > 1$ and that the conclusion holds for all separable C^*-algebras which satisfy (ii) and whose length of the corresponding composition series equals $p-1$. Note that if $J \in \text{Id}(A)$ then A/J satisfies (ii) whenever A satisfies (ii). Specifically, if J is the n-homogeneous ideal of A then our induction hypothesis implies that A/J is of finite type (since the length of the corresponding composition series of A/J equals $p-1$). We claim that J is also of finite type. Indeed, since A is separable, so is the center $Z(J)$ of J, and hence there exits a strictly positive element $z \in Z(J)$. Since $Z(J) \subseteq Z(A)$, we have $zx_i \in J$ $(1 \le i \le m)$, and since the homogeneous C^*-algebras are quasicentral (see Remark 3.5 and the proof of Proposition 3.7) we have $z \notin P$ for all $P \in \text{Prim}(J) \subseteq \text{Prim}(A)$. It follows that

$$\text{span}\{zx_1 + P, \ldots, zx_m + P\} = J/P \quad \text{for all } P \in \text{Prim}(J).$$

By [11, Section 1], J is of finite type. Since A is of finite type if and only if J and A/J are of finite type, the proof is finished.

□

In this paper we shall also see that, besides the necessity of the SFT condition, the conditions (a) and (b) are also affected by some topological obstructions on the primitive spectrum of $M(A)$. Since the primitive spectrum of $M(A)$ can be much more complicated than that of A (even if $\text{Prim}(A)$ is Hausdorff, see [5, Example 12]) we shall restrict ourselves to the class of unital C^*-algebras.

In Section 2 we show that if a unital C^*-algebra A satisfies (a) or (b) then the supremum

$$\sup\{\dim(A/R) \ : \ R \in \text{Primal}_2(A)\} \tag{1.2}$$

is also finite, where by $\text{Primal}_2(A)$ we denote the set of all 2-primal ideals of A. Recall, an ideal R of A is said to be 2-*primal* if whenever J_1 and J_2 are ideals of A with zero-product, then $J_1 \subseteq R$ or $J_2 \subseteq R$. Note that every prime (hence primitive) ideal of A is 2-primal. By [3, 3.2], $R \in \text{Id}(A)$ is 2-primal if and only if for every two primitive ideals $P_1, P_2 \in \text{Prim}(A/R) = \{P \in \text{Prim}(A) \ : \ R \subseteq P\}$ there exists a net (P_α) in $\text{Prim}(A)$ which converges simultaneously to P_1 and P_2 in $\text{Prim}(A)$. Hence, if A admits a 2-primal ideal which is not maximal then $\text{Prim}(A)$ is certainly non-Hausdorff. At the end of Section 2 we give an example of a unital separable 2-SFT algebra for which the supremum (1.2) is infinite. It follows that such A cannot satisfy (a) nor (b).

In Section 3, we consider the central Haagerup tensor product $A \otimes_{Z,h} A$ of a unital SFT algebra A. By definition, $A \otimes_{Z,h} A$ is the quotient of the Haagerup tensor product $A \otimes_h A$ by the closure of the linear span of tensors of the form

$az \otimes b - a \otimes zb$, where $a, b \in A$ and $z \in Z(A)$. We now introduce the notion of central Haagerup rank of A:

Definition 1.2. For $t \in A \otimes_h A$ we define a *central Haagerup rank* of t, denoted by $\mathrm{rank}_{Z,h}(t)$, as the smallest nonnegative integer m for which there exists a rank m tensor $u \in A \otimes A$ such that $t_Z = u_Z$ in $A \otimes_{Z,h} A$ (here, as usual, t_Z denotes the canonical image of t in $A \otimes_{Z,h} A$). If such m does not exist we define $\mathrm{rank}_{Z,h}(t) := \infty$. The *central Haagerup rank* of A is defined as

$$\mathrm{rank}_{Z,h}(A) := \sup\{\mathrm{rank}_{Z,h}(t) \ : \ t \in A \otimes_h A\} \in \mathbb{N} \cup \{\infty\}. \tag{1.3}$$

If a unital separable C^*-algebra A satisfies $\mathrm{rank}_{Z,h}(A) < \infty$ then obviously A satisfies (a) and (b) as well (since the canonical contraction Θ_A factorizes through $A \otimes_{Z,h} A$), and hence, by [10, 2.13] such A is necessarily SFT. We also give an example which shows that the converse does not hold in general. However, if the primitive spectrum of a unital (not necessarily separable) SFT algebra A is Hausdorff we show that $\mathrm{rank}_{Z,h}(A) < \infty$.

2. Another reduction

We now show that if a unital C^*-algebra A satisfies $\mathrm{Im}\,\Theta_A = \mathrm{E}(A)$ or $\ell(\mathrm{E}(A)) < \infty$ then the supremum (1.2) is also finite. Moreover, if a separable A satisfies $\mathrm{Im}\,\Theta_A = \mathrm{E}(A)$ we show that the set $\mathrm{Prim}(A)$ in Proposition 1.1 (ii) can be replaced by the larger set $\mathrm{Primal}_2(A)$.

To do this, we first recall the following two facts. Let $J \in \mathrm{Id}(A)$ and let $q_J : A \to A/J$ be the quotient map. Then by [1, 2.8] the induced contraction $q_J \otimes q_J : A \otimes_h A \to (A/J) \otimes_h (A/J)$ is in fact a (complete) quotient map with kernel

$$\ker(q_J \otimes q_J) = J \otimes_h A + A \otimes_h J.$$

Thus, we have

$$(A \otimes_h A)/(J \otimes_h A + A \otimes_h J) \cong (A/J) \otimes_h (A/J)$$

(completely) isometrically. Also, by [14, Corollary 6] we have the following description of the kernel of Θ_A:

$$\ker \Theta_A = \bigcap \{R \otimes_h A + A \otimes_h R \ : \ R \in \mathrm{Primal}_2(A)\}. \tag{2.1}$$

The proof of the next lemma is omitted since it is almost identical to that of [10, 2.5].

Lemma 2.1. *Let A be a C^*-algebra. Let (a_k), (b_k), and (e_k) be sequences in A such that $e_k^* = e_k$ for all $k \in \mathbb{N}$, and such that the series $\sum_{k=1}^\infty a_k a_k^*$, $\sum_{k=1}^\infty b_k^* b_k$ and $\sum_{k=1}^\infty e_k^2$ are norm convergent. Let t and u be the tensors in $A \otimes_h A$ defined by $t := \sum_{k=1}^\infty e_k \otimes e_k$ and $u := \sum_{k=1}^\infty a_k \otimes b_k$. If $t - u \in J \otimes_h A + A \otimes_h J$ for some $J \in \mathrm{Id}(A)$ then*

$$\overline{\mathrm{span}}\{e_k + J \ : \ k \in \mathbb{N}\} \subseteq \overline{\mathrm{span}}\{b_k + J \ : \ k \in \mathbb{N}\},$$

where $\overline{\mathrm{span}}$ denotes the closed linear span.

We also introduce the following notation. For $m \in \mathbb{N}$ by $A \overset{m}{\otimes} A$ we denote the set of all tensors in $A \otimes A$ of rank at most m.

Lemma 2.2. *Let A be a* C**-algebra and let $\mathcal{F} \subseteq \mathrm{Id}(A)$ be some non void family of ideals of A. Let*

$$\Lambda(\mathcal{F}) := \bigcap \{J \otimes_h A + A \otimes_h J : \ J \in \mathcal{F}\} \in \mathrm{Id}(A \otimes_h A).$$

Suppose that A satisfies one of the following two conditions:

(i) *There exists $m \in \mathbb{N}$ such that $A \overset{m}{\otimes} A + \Lambda(\mathcal{F}) = A \otimes A + \Lambda(\mathcal{F})$,*
(ii) *$A \otimes_h A + \Lambda(\mathcal{F}) = A \otimes A + \Lambda(\mathcal{F})$.*

Then $\sup\{\dim(A/J) \ : \ J \in \mathcal{F}\} < \infty$. Moreover, if a separable A satisfies (ii) then there exist $m \in \mathbb{N}$ and elements $x_1, \ldots, x_m \in A$ such that

$$\mathrm{span}\{x_1 + J, \ldots, x_m + J\} = A/J, \quad \text{for all } J \in \mathcal{F}.$$

Proof. The proof of (i) and (ii) is also omitted since it is almost identical to that of [10, 2.6] (here $\mathrm{Prim}(A)$ should be replaced by $\mathcal{F}$, and instead of using [10, 2.5] we use Lemma 2.1).

Now suppose that a separable A satisfies (ii). Let (e_k) be a sequence of norm one self-adjoint elements of A whose linear span is dense in A. We define a tensor $t \in A \otimes_h A$ by

$$t := \sum_{k=1}^{\infty} \frac{1}{k^2} e_k \otimes e_k = \sum_{k=1}^{\infty} \frac{1}{k} e_k \otimes \frac{1}{k} e_k.$$

By assumption, there exist the elements $a_1, \ldots, a_m, \ b_1, \ldots, b_m \in A$ such that for $u := \sum_{i=1}^{m} a_i \otimes b_i$ we have $t - u \in \Lambda(\mathcal{F})$. By Lemma 2.2 we have

$$\overline{\mathrm{span}}\{e_k + J : \ k \in \mathbb{N}\} \subseteq \mathrm{span}\{b_i + J \ : \ 1 \leq i \leq m\} \tag{2.2}$$

for all $J \in \mathcal{F}$. Since the linear span of (e_k) is dense in A, so is the linear span of $(e_k + J)$ in A/J. Hence, (2.2) implies

$$\mathrm{span}\{b_i + J \ : \ 1 \leq i \leq m\} = A/J, \quad \text{for all } J \in \mathrm{Id}(A).$$

Letting $x_i := b_i$ $(1 \leq i \leq m)$ we obtain the desired elements. □

Theorem 2.3. *Let A be a unital separable* C**-algebra.*

(i) *If $\mathrm{Im}\,\Theta_A = \mathrm{E}(A)$ then there exists $m \in \mathbb{N}$ and the elements $x_1, \ldots, x_m \in A$ such that*

$$\mathrm{span}\{x_1 + R, \ldots, x_m + R\} = A/R, \quad \text{for all } R \in \mathrm{Primal}_2(A). \tag{2.3}$$

(ii) *If $\ell(\mathrm{E}(A)) < \infty$ then A is SFT and*

$$\sup\{\dim(A/R) \ : \ R \in \mathrm{Primal}_2(A)\} < \infty. \tag{2.4}$$

Remark 2.4. Note that (by Proposition 1.1) the condition (2.3) immediately implies that A is a SFT algebra.

Proof. (i). Using the same notation as in Lemma 2.2, (2.1) implies $\ker \Theta_A = \Lambda(\mathrm{Primal}_2(A))$. Then the condition $\mathrm{Im}\,\Theta_A = \mathrm{E}(A)$ may be rewritten in the following form

$$(A \otimes_h A) + \Lambda(\mathrm{Primal}_2(A)) = A \otimes A + \Lambda(\mathrm{Primal}_2(A)).$$

Now the conclusion follows from Lemma 2.2.

(ii). By Corollary [10, 2.13] A is a SFT algebra. Furthermore, by (2.1), the condition $k := \ell(\mathrm{E}(A)) < \infty$ is equivalent to

$$A \otimes A + \Lambda(\mathrm{Primal}_2(A)) = A \overset{k}{\otimes} A + \Lambda(\mathrm{Primal}_2(A)).$$

Again, the conclusion now follows from Lemma 2.2. □

Remark 2.5. It is still unknown to us if the conditions $\ell(\mathrm{E}(A)) < \infty$ and $\mathrm{Im}\,\Theta_A = \mathrm{E}(A)$ are in fact equivalent. It would be also interesting to know whether the SFT condition together with (2.4) implies the existence of elements $x_1, \ldots, x_m$ for which (2.3) holds.

We now give an example of a unital separable 2-SFT algebra which contains 2-primal ideals of arbitrarily large codimensions. First recall that if A is a unital C^*-algebra and $J \in \mathrm{Max}(Z(A))$ (the maximal spectrum of the center $Z(A)$) then the *Glimm ideal* of A generated by J is the proper (closed two-sided) ideal JA (which is indeed closed by Cohen's factorization theorem). Since $JA \cap Z(A) = J$, the mapping $J \mapsto JA$ defines a bijection from $\mathrm{Max}(Z(A))$ onto the set $\mathrm{Glimm}(A)$ of all Glimm ideals of A.

Example 2.6. Let (x_k) be a strictly increasing convergent sequence in $\mathbb{R}$ with limit x_0, and define

$$X := \bigsqcup_{k=1}^{\infty} [x_{2k-1}, x_{2k}] \cup \{x_0\},$$

which is a compact subset of $\mathbb{R}$. For every $k \in \mathbb{N}$ let $\mathbb{N}_k := \{1, \ldots, k\}$, and define $m(k) := \binom{k}{2}$. Let ϕ_k be some bijection from $\mathbb{N}_{m(k)}$ onto the set of all 2-element subsets of $\mathbb{N}_k$, and let $\phi_k(i) = \{\phi_{1,k}(i), \phi_{2,k}(i)\}$, where $\phi_{1,k}(i) < \phi_{2,k}(i)$ $(1 \le i \le m(k))$. For every $k \in \mathbb{N}$ let us fix some distinct points $s_{1,k}, \ldots, s_{m(k),k}$ from the interval (x_{2k-1}, x_{2k}). We define A to be a C^*-subalgebra of $B := C(X, \mathrm{M}_2(\mathbb{C}))$ consisting of all functions $a \in B$ for which there exist complex numbers $\{\lambda_{i,k}(a)\}$ $(k \in \mathbb{N}, 1 \le i \le k)$ and $\lambda(a)$ such that

$$a(s_{i,k}) = \begin{bmatrix} \lambda_{\phi_{1,k}(i)}(a) & 0 \\ 0 & \lambda_{\phi_{2,k}(i)}(a) \end{bmatrix} \quad (k \in \mathbb{N},\ 1 \le i \le m(k)),$$

and

$$a(x_0) = \begin{bmatrix} \lambda(a) & 0 \\ 0 & \lambda(a) \end{bmatrix}.$$

Then A is a (unital separable) 2-SFT algebra such that

$$\sup\{\dim(A/R) \ : \ R \in \mathrm{Primal}_2(A)\} = \infty.$$

Hence, by Theorem 2.3, $\ell(\mathrm{E}(A)) = \infty$ and $\mathrm{E}(A) \subsetneq \mathrm{Im}\,\Theta_A$. Moreover, $\mathrm{Im}\,\Theta_A$ is not even cb-closed.

Proof. Note that the 2-homogeneous ideal J of A is of the form

$$J = \{a \in A \ : \ a(s) = 0, \text{ for all } s \in X \setminus U\} = C_0(U, \mathrm{M}_2(\mathbb{C})),$$

where

$$U := X \setminus (\{s_{i,k} \ : \ k \in \mathbb{N}, 1 \le i \le m(k)\} \cup \{x_0\}).$$

Since A/J is commutative, A is a 2-SFT algebra. Since U is dense in X, the center $Z(A)$ of A consists of all elements $a \in A$ where $a(s)$ is a multiple of the identity (and hence $\lambda_{i,k}(a)$ does not depend on i). Let F be a quotient space obtained from X under the equivalence relation $x \sim y$ if and only if $x = y$ or $(x = s_{i,k}$ and $y = s_{j,k})$, for some $k \in \mathbb{N}$, $1 \leq i, j \leq m(k)$. Then $Z(A)$ is canonically isomorphic to the C^*-algebra $C(F)$ of all continuous complex-valued functions on F, and the space $\mathrm{Glimm}(A)$ can be identified with F. In particular, every ideal $G_k := \bigcap_{1 \leq i \leq k} \ker \lambda_{i,k}$ is Glimm ideal of A, and we have the following description of $\mathrm{Glimm}(A)$:

$$\mathrm{Glimm}(A) = \{\ker \pi_s : \ s \in U\} \cup \{G_k : \ k \in \mathbb{N}\} \cup \{\ker \lambda\},$$

where $\lambda_{i,k} : A \to \mathbb{C}$ $(k \in \mathbb{N},\ 1 \leq i \leq k)$, $\lambda : A \to \mathbb{C}$ and $\pi_s : A \to \mathrm{M}_2(\mathbb{C})$ $(s \in U)$ are irreducible representations of A defined respectively by $\lambda_{i,k} : a \mapsto \lambda_{i,k}(a)$, $\lambda : a \mapsto \lambda(a)$, and $\pi_s : a \mapsto a(s)$. Since A/J is commutative, it is also easy to check that every irreducible representation of A is (up to the equivalence) in one of this form, hence

$$\mathrm{Prim}(A) = \{\ker \pi_s : \ s \in U\} \cup \{\ker \lambda_{i,k} : \ k \in \mathbb{N}, 1 \leq i \leq k\} \cup \{\ker \lambda\}.$$

We claim that every Glimm ideal G_k is 2-primal. Let $k \in \mathbb{N}$ be arbitrary. We have to show that for every $P, Q \in \mathrm{Prim}(A/G_k)$ there exists a net in $\mathrm{Prim}(A)$ which converges simultaneously to P and Q. Since all primitive ideals of A/G_k are of the form $\ker \lambda_{i,k}$ $(1 \leq i \leq k)$, there are $1 \leq p, q \leq k$ such that $P = \ker \lambda_{p,k}$ and $Q = \ker \lambda_{q,k}$. We may assume that $p < q$ and let $1 \leq i \leq m(k)$ such that $\phi(i) = \{p, q\}$. If (s_α) is an arbitrary net in $[x_{2k-1}, x_{2k}] \setminus \{s_{i,k} : 1 \leq i \leq m(k)\}$ which converges to $s_{i,k}$, then it is not difficult to see that $(\ker \pi_{s_\alpha})$ is a net in $\mathrm{Prim}(J) \subseteq \mathrm{Prim}(A)$ which converges simultaneously to $\ker \lambda_{p,k} = P$ and $\ker \lambda_{q,k} = Q$. Thus, $G_k \in \mathrm{Primal}_2(A)$, for all $k \in \mathbb{N}$. Since $\dim(A/G_k) = k$, we have $\sup\{\dim(A/G_k) \ : \ k \in \mathbb{N}\} = \infty$, so $\ell(\mathrm{E}(A)) = \infty$ and $\mathrm{E}(A) \subsetneq \mathrm{Im}\,\Theta_A$, by Theorem 2.3. Moreover, using [15, 10.1] (see also [4]) we also conclude that $\mathrm{Im}\,\Theta_A$ is not even closed in $\mathrm{ICB}(A)$. □

3. C^*-algebras of finite central Haagerup rank

Recall from the introduction that for a unital C^*-algebra A we defined the central Haagerup rank (denoted by $\mathrm{rank}_{Z,h}(A)$) of A by (1.3). As already observed, $\mathrm{rank}_{Z,h}(A) < \infty$ obviously implies $\ell(\mathrm{E}(A)) < \infty$ and $\mathrm{Im}\,\Theta_A = \mathrm{E}(A)$. To show that the converse does not hold in general, we start with the following fact:

Proposition 3.1. *Let A be a unital* C**-algebra. If* $\mathrm{rank}_{Z,h}(A) < \infty$ *then*

$$\sup\{\dim(A/G) \ : \ G \in \mathrm{Glimm}(A)\} < \infty. \tag{3.1}$$

Proof. Let J_A be the closed two-sided ideal of $A \otimes_h A$ generated by tensors of the form $az \otimes b - a \otimes zb$, where $a, b \in A$ and $z \in Z(A)$. By definition, $A \otimes_{Z,h} A = (A \otimes_h A)/J_A$. Note that A satisfies $\mathrm{rank}_{Z,h}(A) < \infty$ if and only if

$$A \otimes_h A + J_A = A \overset{m}{\otimes} A + J_A$$

for some $m \in \mathbb{N}$. By [14, Theorem 1] we have

$$J_A = \bigcap \{G \otimes_h A + A \otimes_h G \ : \ G \in \mathrm{Glimm}(A)\} = \Lambda(\mathrm{Glimm}(A))$$

(using the same notation as in Lemma 2.2). Hence, by Lemma 2.2, (3.1) holds. $\square$

Example 3.2. Let A be the C^*-algebra from [9, 6.1] which consists of all elements $a \in C([0,\infty], \mathrm{M}_2(\mathbb{C}))$ such that

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \ (n \in \mathbb{N}),$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then $\mathrm{rank}_{Z,h}(A) = \infty$, but $\mathrm{Im}\,\Theta_A = \mathrm{E}(A)$ and $\ell(\mathrm{E}(A)) < \infty$.

Proof. It is easy to check that

$$Z(A) = \left\{ z = \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} \ : \ f \in C([0,\infty]), \ f|_{\mathbb{N}} \text{ is constant} \right\}.$$

If J is the maximal ideal of $Z(A)$ consisting of all $z \in Z(A)$ with $z|_{\mathbb{N}} = 0$, then the corresponding Glimm ideal $G := JA$ is of the form

$$G = \{a \in A \ : \ a|_{\mathbb{N}} = 0\}.$$

Obviously, $\dim(A/G) = \infty$, and hence by Proposition 3.1 $\mathrm{rank}_{Z,h}(A) = \infty$. On the other hand, it was shown in [9, 6.6] that $\mathrm{Im}\,\Theta_A = \mathrm{E}(A)$ and $\ell(\mathrm{E}(A)) < \infty$. $\square$

We now show that every unital SFT algebra with Hausdorff primitive spectrum has finite central Haagerup rank.

To show this, we introduce the following (rather standard) notation. For an operator space $X \subseteq \mathrm{B}(\mathcal{H})$ we define

$$\mathrm{R}_\infty(X) = \left\{ \mathbf{x} := [x_1 \, x_2 \, \ldots] \ : x_i \in X \text{ and } \sum_{i=1}^{\infty} x_i x_i^* \text{ converges in norm} \right\},$$

$$\mathrm{C}_\infty(X) = \left\{ \mathbf{y} := [y_1 \, y_2 \, \ldots]^\tau \ : y_i \in X \text{ and } \sum_{i=1}^{\infty} y_i^* y_i \text{ converges in norm} \right\}.$$

If $n \in \mathbb{N}$ we identify $\mathrm{R}_n(X)$ with the subspace of $\mathrm{R}_\infty(X)$ which consists of all $[x_1 \, x_2 \, \ldots]$ such that $x_i = 0$, for all $i > n$. Similarly, we identify $\mathrm{C}_n(X)$ with the corresponding subspace of $\mathrm{C}_\infty(X)$. For $\mathbf{x} := [x_1 \, x_2 \, \ldots] \in \mathrm{R}_\infty(X)$ and $\mathbf{y} := [y_1 \, y_2 \, \ldots]^\tau \in \mathrm{C}_\infty(X)$ we put

$$\|\mathbf{x}\| := \left\| \sum_{i=1}^{\infty} x_i x_i^* \right\|^{\frac{1}{2}}, \qquad \|\mathbf{y}\| := \left\| \sum_{i=1}^{\infty} y_i^* y_i \right\|^{\frac{1}{2}}$$

and

$$\mathbf{x} \odot \mathbf{y} := \sum_{i=1}^{\infty} x_i \otimes y_i \in X \otimes_h X.$$

There is also a natural operator space structure on $\mathrm{R}_\infty(X)$ and $\mathrm{C}_\infty(X)$, but we shall not need it. For $n \in \mathbb{N} \cup \{\infty\}$ we also put $\mathrm{M}_{\infty,n}(X) := \mathrm{C}_\infty(\mathrm{R}_n(X))$ and $\mathrm{M}_{n,\infty}(X) := \mathrm{R}_\infty(\mathrm{C}_n(X))$.

Lemma 3.3. *Let A be a* C^**-algebra. Suppose that there exists $m \in \mathbb{N}$ such that for every $\mathbf{y} := [y_1\, y_2\, \ldots]^\tau \in \mathrm{C}_\infty(A)$ there exists $\mathbf{x} := [x_1\, \ldots\, x_m]^\tau \in \mathrm{C}_m(A)$ and a matrix of central elements $\mathbf{Z} := [z_{i,j}] \in \mathrm{M}_{\infty,m}(Z(A))$ such that $\mathbf{Zx} = \mathbf{y}$. If A is unital then* $\mathrm{rank}_{Z,h}(A) \le m$.

Proof. Let $t \in A \otimes_h A$. By [7, 1.5.6], there exist $\mathbf{a} := [a_1\, a_2\, \ldots] \in \mathrm{R}_\infty(A)$ and $\mathbf{b} := [b_1\, b_2\, \ldots]^\tau \in \mathrm{C}_\infty(A)$ such that $t = \mathbf{a} \odot \mathbf{b}$. By assumption, we can find $\mathbf{x} := [x_1\, x_2\, \ldots\, x_m]^\tau \in \mathrm{C}_m(A)$ and $\mathbf{Z} := [z_{i,j}] \in \mathrm{M}_{\infty,m}(Z(A))$ such that $\mathbf{Zx} = \mathbf{b}$. For $1 \le j \le m$ let $\mathbf{z}_j$ be the j-th column of the matrix $\mathbf{Z}$. Then $\mathbf{z}_j \in \mathrm{C}_\infty(Z(A))$, so the series $\sum_{i=1}^\infty a_i z_{i,j} = \mathbf{a}\mathbf{z}_j$ is norm convergent. Then in $A \otimes_{Z,h} A$ we have

$$\begin{aligned} t_Z &= \lim_{k\to\infty} \sum_{i=1}^k a_i \otimes_Z b_i = \lim_{k\to\infty} \left(\sum_{i=1}^k a_i \otimes_Z \left(\sum_{j=1}^m z_{i,j} x_j \right) \right) \\ &= \lim_{k\to\infty} \left(\sum_{j=1}^m \sum_{i=1}^k z_{i,j} a_i \otimes_Z x_j \right) = \sum_{j=1}^m \lim_{k\to\infty} \left(\sum_{i=1}^k z_{i,j} a_i \right) \otimes_Z x_j \\ &= \sum_{j=1}^m \mathbf{a}\mathbf{z}_j \otimes_Z x_j. \end{aligned}$$

Thus, $\mathrm{rank}_{Z,h}(A) \le m$. □

Let us introduce the following auxiliary definition:

Definition 3.4. We say that a C^*-algebra A has property (P) if A satisfies the condition of Lemma 3.3.

Remark 3.5. Note that every C^*-algebra which satisfies (P) is quasicentral, that is A as a Banach module over its center $Z(A)$ is nondegenerate (see [2] of [9] for another descriptions). Furthermore, if A is quasicentral and $m \in \mathbb{N}$, then it is easy to see that $\mathrm{M}_{\infty,m}(A)$ is also a nondegenerate Banach $Z(A)$-module, under the natural action

$$z \cdot [a_{i,j}] := [z a_{i,j}] \quad (z \in Z(A),\ [a_{i,j}] \in \mathrm{M}_{\infty,m}(A)).$$

In particular, using Cohen's factorization theorem, we see that every matrix $\mathbf{a} \in \mathrm{M}_{\infty,m}(A)$ can be factorized in the form $\mathbf{a} = z \cdot \mathbf{b}$, for some $z \in Z(A)$ and $\mathbf{b} \in \mathrm{M}_{\infty,m}(A)$.

Lemma 3.6. *Let A be a* C^**-algebra which satisfies (P). Then every quasicentral ideal J of A satisfies (P).*

Proof. Let $\mathbf{a} \in \mathrm{C}_\infty(J)$. By Remark 3.5 there exist $z \in Z(J)$ and $\mathbf{b} \in \mathrm{C}_\infty(J)$ such that $\mathbf{a} = z \cdot \mathbf{b}$. Since A satisfies (P), there exists $m \in \mathbb{N}$ and the matrices $\mathbf{Z} \in \mathrm{M}_{\infty,m}(Z(A))$, $\mathbf{x} \in \mathrm{C}_m(A)$ such that $\mathbf{Zx} = \mathbf{b}$. Choose any factorization $z = z_1 z_2$, where $z_1, z_2 \in Z(J)$. Since $Z(J)$ is an ideal of $Z(A)$, we have

$$\mathbf{a} = z \cdot \mathbf{b} = z \cdot (\mathbf{Zx}) = (z_1 \cdot \mathbf{Z})(z_2 \cdot \mathbf{x}),$$

where $z_1 \cdot \mathbf{Z} \in \mathrm{M}_{\infty,m}(Z(J))$ and $z_2 \cdot \mathbf{x} \in \mathrm{C}_m(J)$. Hence, J satisfies (P). □

Proposition 3.7. *Let A be an n-homogeneous* C^**-algebra of finite type. Then A satisfies (P).*

Proof. By [8, 3.2], there exists a locally trivial C^*-bundle E over the (locally compact Hasudorff) space $\Delta := \mathrm{Prim}(A)$ whose fibers are isomorphic to $\mathrm{M}_n(\mathbb{C})$ such that $A \cong \Gamma_0(E)$ ($\Gamma_0(E)$ stands for the C^*-algebra of all continuous sections of E which vanish at ∞). Using the local triviality of E it is easy to see that A is quasicentral. Since by [11, 3.3] $M(A)$ is also n-homogeneous, Lemma 3.6 implies that it is sufficient to prove the claim when A is already unital. In this case Δ is compact and we identify A with $\Gamma(E)$. Choose a finite open covering $\{U_j\}_{1\le j\le m}$ of Δ such that every restriction bundle $E|_{\overline{U_j}}$ is trivial. Using a finite partition of unity argument, it is sufficient to prove the claim when E is already trivial. Then $A = C(\Delta, \mathrm{M}_n(\mathbb{C}))$, and let $(E_{i,j})$ be the standard matrix units of $\mathrm{M}_n(\mathbb{C})$ considered as constant elements of $C(\Delta, \mathrm{M}_n(\mathbb{C}))$. Let $[a_1\, a_2\, \ldots]^\tau \in \mathrm{C}_\infty(A)$. Then

$$a_k = \sum_{i,j=1}^{n} f_{k,i,j} E_{i,j},$$

for some functions $f_{k,i,j} \in C(\Delta) \cong Z(A)$. To show that A satisfies (P), it is sufficient to check that the series of functions $\sum_{k=1}^{\infty} |f_{k,i,j}|^2$ converge uniformly on Δ, for all $1 \le i, j \le n$. Indeed, since

$$a_k^* a_k = \sum_{i,j=1}^{n} \sum_{p=1}^{n} \overline{f_{k,p,j}} f_{k,p,i} E_{i,j},$$

and since the series $\sum_{k=1}^{\infty} a_k^* a_k$ is norm convergent if and only if its matrix values converge uniformly on Δ, we conclude that the series $\sum_{k=1}^{\infty} \sum_{p=1}^{n} |f_{k,p,j}|^2$ converge uniformly on Δ, for all $1 \le j \le n$. Since $|f_{k,i,j}|^2 \le \sum_{p=1}^{n} |f_{k,p,j}|^2$, it follows that the series $\sum_{k=1}^{\infty} |f_{k,i,j}|^2$ also converge uniformly on Δ. □

Lemma 3.8. *Let A and B be* C**-algebras and let $\phi : A \to B$ be a surjective $*$-homomorphism. If $m \in \mathbb{N}$ then the induced map*

$$\phi_{\infty,m} : \mathrm{M}_{\infty,m}(A) \to \mathrm{M}_{\infty,m}(B), \quad \phi_{\infty,m}([a_{i,j}]) = [\phi(a_{i,j})]$$

is also surjective

Proof. It is sufficient to prove this when $m = 1$. In this case $\mathrm{M}_{\infty,1}(A) = \mathrm{C}_\infty(A)$ and $\mathrm{M}_{\infty,1}(B) = \mathrm{C}_\infty(B)$ can be considered as the standard Hilbert C^*-modules $\mathcal{H}_A$ and $\mathcal{H}_B$ (see [17, Section 15]). Using the same notation as in [6, Section 2], $\phi_\infty := \phi_{\infty,1}$ is a ϕ-morphism between $\mathcal{H}_A$ and $\mathcal{H}_B$. Since ϕ is surjective, note that the image of ϕ_∞ is dense in $\mathcal{H}_B$. But [6, 2.5] implies that the image of ϕ_∞ is closed, so that ϕ_∞ is indeed surjective. □

Theorem 3.9. *Let A be a unital SFT algebra with Hausdorff primitive spectrum. Then A has a finite central Haagerup rank. In particular,* $\mathrm{Im}\,\Theta_A = \mathrm{E}(A)$, $\ell(\mathrm{E}(A)) < \infty$ *and* $\mathrm{E}(A)$ *is closed in the operator norm.*

Proof. Suppose that A is n-subhomogeneous and let (1.1) the standard composition series of A of length p. If $p = 1$ then A is n-homogeneous (of finite type), so the claim follows from Proposition 3.7. Let $p > 2$ and suppose that the result holds for all unital SFT algebras with Hausdorff primitive spectrum whose length of the corresponding composition series equals $p-1$. Let J be the n-homogeneous ideal of A and let $q_J : A \to A/J$ be the quotient map. Choose an arbitrary element $\mathbf{a} \in \mathrm{C}_\infty(A)$. Then obviously $\dot{\mathbf{a}} = (q_J)_{\infty,1}(\mathbf{a}) \in \mathrm{C}_\infty(A/J)$. By induction hypothesis (applied to A/J), there exists $m \in \mathbb{N}$ (which depends only on A/J) and matrices $\dot{\mathbf{Z}}_1 \in \mathrm{M}_{\infty,m}(Z(A/J))$, $\dot{\mathbf{x}}_1 \in \mathrm{C}_m(A/J)$ such that $\dot{\mathbf{Z}}_1\dot{\mathbf{x}}_1 = \dot{\mathbf{a}}$. Since $\mathrm{Prim}(A)$ is Hausdorff, [16, Corollary 1] implies that q_J maps $Z(A)$ surjectively onto $Z(A/J)$. By Lemma 3.8, matrices $\dot{\mathbf{Z}}_1$ and $\dot{\mathbf{x}}_1$ can be respectively lifted to the matrices $\mathbf{Z}_1 \in \mathrm{M}_{\infty,m}(Z(A))$ and $\mathbf{x}_1 \in \mathrm{C}_m(A)$ such that

$$\mathbf{b} := \mathbf{a} - \mathbf{Z}_1\mathbf{x}_1 \in \mathrm{C}_\infty(J). \tag{3.2}$$

Similarly, since J is homogeneous C^*-algebra of finite type, by invoking Proposition 3.7 we find $k \in \mathbb{N}$ (which depends only on J) and the matrices $\mathbf{Z}_2 \in \mathrm{M}_{\infty,k}(Z(J))$, $\mathbf{x}_2 \in \mathrm{C}_k(J)$ such that $\mathbf{Z}_2\mathbf{x}_2 = \mathbf{b}$. Since $Z(J) \subseteq Z(A)$, we have

$$\mathbf{Z} := \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \end{bmatrix} \in \mathrm{M}_{\infty,m+k}(Z(A)) \quad \text{and} \quad \mathbf{x} := \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \in \mathrm{C}_{m+k}(A),$$

and (3.2) implies that $\mathbf{a} = \mathbf{Z}\mathbf{x}$. By Lemma 3.3, $\mathrm{rank}_{Z,h}(A) \le m + k < \infty$. In particular, $\mathrm{Im}\,\Theta_A = \mathrm{E}(A)$ and $\ell(\mathrm{E}(A)) < \infty$. The claim that $\mathrm{E}(A)$ is closed in the operator norm follows from [9, 6.2] and [15, 10.1] (or [14, Theorem 4]). □

Remark 3.10. We also note that $\mathrm{rank}_{Z,h}(A) < \infty$ does not imply that $\mathrm{Prim}(A)$ is Hausdorff. For example, let A be a C^*-subalgebra of $B := C([0,1], \mathrm{M}_2(\mathbb{C}))$ consisting of all $a \in B$ such that $a(0)$ is diagonal. Then it is easy to see that $\mathrm{rank}_{Z,h}(A) < \infty$, even though $\mathrm{Prim}(A)$ is not Hausdorff.

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