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INDEX COMPUTATION FOR AMALGAMATED PRODUCTS OF PRODUCT SYSTEMS

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Abstract. The notion of amalgamation of product systems has been introduced in [7] which generalizes the concept of Skeide product, introduced by Skeide, of two product systems via a pair of normalized units. In this paper we show that amalgamation leads to a setup where a product system is generated by two subsystems and conversely whenever a product system is generated by two subsystems, it could be realized as an amalgamated product. We parameterize all contractive morphism from a Type I product system to another Type I product system and compute index of amalgamated product through contractive morphisms.

1. Introduction

Arveson [1] associated to every E_0 semigroup, a product system of Hilbert spaces. He showed that this association classifies E_0 semigroups up to cocycle conjugacy. In the context of product system of Hilbert modules, Skeide [14] introduced spatial product of product systems as there is no natural tensor product operation on product system of Hilbert modules and index is additive under spatial product. At the 2002 AMS summer conference on 'Advances in Quantum Dynamics' held at Mount Holyoke, R.T. Powers posed the following problem : Let $\mathcal{B}(H)$ and $\mathcal{B}(K)$ be algebras of all bounded operators on two Hilbert spaces H and $\mathcal{K}$. Suppose $\phi = \{\phi_t : t \geq 0\}$ and $\psi = \{\psi_t : t \geq 0\}$ are two E_0 semigroups on $\mathcal{B}(H)$ and $\mathcal{B}(K)$ respectively and $U = \{U_t : t \geq 0\}$ and

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$V = \{V_t : t \geq 0\}$ are two strongly continuous semigroups of isometries which intertwine ϕ_t and ψ_t respectively. Consider the CP semigroup τ_t on $\mathcal{B}(H \oplus K)$ defined by $\tau_t \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} \phi_t(X) & U_t Y V_t^* \\ V_t Z U_t^* & \psi_t(W) \end{pmatrix}$. How is the minimal dilation (in the sense of [3],[4]) of τ related to ϕ and ψ? In fact, Powers was interested in a more specific question. It is the following. Since the minimal dilation is unique we might say that we are associating a product system to a given contractive CP semigroup [3]. (This can also be done more directly as in [8]). The question was 'What is the product system of the Powers' sum τ in terms of the product systems of ϕ and ψ. Is it the tensor product?' Still during the workshop, Skeide [13] identified the product system as a spatial product through normalized units. It turns out that though the index of product system of τ is sum of indices of ϕ and ψ, it is not the tensor product but the spatial product [10]. Definition of Powers' sum easily extends to CP semigroups and the product system of Powers' sum in that case also is the spatial product of the product systems of summands ([6],[14]). In [7], amalgamated product of two product systems of Hilbert space through general contractive morphism has been introduced which generalizes the spatial product for product system of Hilbert spaces. The spatial product of product system of Hilbert spaces may be viewed as amalgamated product through the contractive morphism defined through normalized units. This answers Powers' problem for the Powers' sum obtained from non necessarily isometric intertwining semigroups.

To begin with we briefly recapitulate the notion of inclusion systems introduced in [7]. They were also introduced by Shalit and Solel, under the name *subproduct systems* [12]. These are parameterized families of Hilbert spaces exactly like product systems except that now unitaries are replaced by isometries. Every inclusion system generates a product system via inductive limit procedure [8]. To every CP semigroup, an inclusion system can be associated and we have, the product system associated to the minimal dilation of the CP semigroup is isomorphic to product system generated by the inclusion system. Basic properties of the product systems such as existence/non-existence of units, structure of morphisms etc. can be defined at the level of inclusion systems and there is a bijective correspondence with those in generated product system. Given two inclusion systems and a contractive morphism between them, there is a natural way to amalgamate them to get a new inclusion system. On product systems level, this is called the amalgamated product of two product systems. All these constructions can be found [7]. Loosely speaking amalgamation is nothing but 'the construction of a product system which is generated by two given product subsystems.' This is a kind of universality result (See Theorem 2.7). We use this to show in Theorem 2.9, that under the assumption of separability, the amalgamated product of type I parts is the type I part of the amalgamated product.

In the last section, we parameterize all contractive morphisms from a type I product system to another type I product system. In [7], the index of the amalgamated product through contractive units has been calculated. Here in Theorem

3.8, we compute the index of amalgamated product of two spatial product systems via a general contractive morphism. This is our main result.

We recall some results from [7]. From now on our product systems are algebraic with no measurability conditions.

Definition 1.1. *An inclusion system (E,β) is a family of Hilbert spaces $E = \{E_t, t \in (0,\infty)\}$ together with isometries $\beta_{s,t}{:}E_{s+t} \to E_s \otimes E_t$, for $s,t \in (0,\infty)$, such that $\forall\ r,s,t \in (0,\infty)$, $(\beta_{r,s} \otimes 1_{E_t})\beta_{r+s,t} = (1_{E_r} \otimes \beta_{s,t})\beta_{r,s+t}$. It is said to be a product system if further every $\beta_{s,t}$ is a unitary.*

Consider an inclusion system (E,β). For $t \in \mathbb{R}_+$, let $J_t = \{(t_n, t_{n-1}, \dots, t_1) : t_i > 0, \sum_{i=1}^n t_i = t, n \geq 1\}$.
For $\mathbf{s} = (s_m, s_{m-1}, \dots, s_1) \in J_s$, and $\mathbf{t} = (t_n, t_{n-1}, \dots, t_1) \in J_t$ we define $\mathbf{s} \smile \mathbf{t} := (s_m, s_{m-1}, \dots, s_1, t_n, t_{n-1}, \dots, t_1) \in J_{s+t}$. Now fix $t \in \mathbb{R}_+$. On J_t define a partial order $\mathbf{t} \geq \mathbf{s} = (s_m, s_{m-1}, \dots, s_1)$ if for each i, $(1 \leq i \leq m)$ there exists (unique) $\mathbf{s}_i \in J_{s_i}$ such that $\mathbf{t} = \mathbf{s}_m \smile \mathbf{s}_{m-1} \smile \cdots \smile \mathbf{s}_1$. For $\mathbf{t} = (t_n, t_{n-1}, \dots t_1)$ in J_t define $E_\mathbf{t} = E_{t_n} \otimes E_{t_{n-1}} \otimes \cdots \otimes E_{t_1}$. For $\mathbf{s} = (s_m, \dots, s_1) \leq \mathbf{t} = (\mathbf{s}_m \smile \cdots \smile \mathbf{s}_1)$ in J_t, define $\beta_{\mathbf{t},\mathbf{s}} : E_\mathbf{s} \to E_\mathbf{t}$ by $\beta_{\mathbf{t},\mathbf{s}} = \beta_{\mathbf{s}_m, s_m} \otimes \beta_{\mathbf{s}_{m-1}, s_{m-1}} \otimes \cdots \otimes \beta_{\mathbf{s}_1, s_1}$ where we define $\beta_{\mathbf{s},s} : E_s \to E_\mathbf{s}$ inductively as follows: Set $\beta_{s,s} = id_{E_s}$. For $\mathbf{s} = (s_m, s_{m-1}, \dots, s_1)$, $\beta_{\mathbf{s},s}$ is the composition of maps:

$$(\beta_{s_m, s_{m-1}} \otimes I)(\beta_{s_m + s_{m-1}, s_{m-2}} \otimes I) \cdots (\beta_{s_m + \cdots + s_3, s_2} \otimes I)\beta_{s_m + \cdots + s_2, s_1}.$$

Theorem 1.2. *Suppose (E,β) is an inclusion system. Let $\mathcal{E}_t = indlim_{J_t} E_\mathbf{s}$ be the inductive limit of $E_\mathbf{s}$ over J_t for $t > 0$. Then $\mathcal{E} = \{\mathcal{E}_t : t > 0\}$ has the structure of a product system of Hilbert spaces.*

Definition 1.3. *Given an inclusion system (E,β), the product system $(\mathcal{E}, B)$ constructed as in the theorem above is called the product system* generated by *the inclusion system (E,β).*

Definition 1.4. *Let (E,β) and (F,γ) be two inclusion systems. Let $A = \{A_t : t > 0\}$ be a family of linear maps $A_t : E_t \to F_t$, satisfying $\|A_t\| \leq e^{tk}$ for some $k \in \mathbb{R}$. Then A is said to be a* morphism *or a* weak morphism *from (E,β) to (F,γ) if*

$$A_{s+t} = \gamma^*_{s,t}(A_s \otimes A_t)\beta_{s,t}\ \forall s,t > 0.$$

It is said to be a strong morphism *if*

$$\gamma_{s,t} A_{s,t} = (A_s \otimes A_t)\beta_{s,t}\ \forall s,t > 0.$$

Definition 1.5. *Let (E,β) be an inclusion system. Let $u = \{u_t : t > 0\}$ be a family of vectors with $u_t \in E_t$, for all $t > 0$, such that $\|u_t\| \leq e^{tk}$ for some $k \in \mathbb{R}$ and $u \not\equiv 0$. Then u is said to be a* unit *or a* weak unit *if*

$$u_{s+t} = \beta^*_{s,t}(u_s \otimes u_t)\ \forall s,t > 0.$$

It is said to be a strong unit *if*

$$\beta_{s,t} u_{s+t} = u_s \otimes u_t\ \forall s,t > 0.$$

Theorem 1.6. *Let (E,β) be an inclusion system and let $(\mathcal{E},B)$ be the product system generated by it. Then the canonical map $i_t : E_t \to \mathcal{E}_t$, $t>0$ is an isometric strong morphism of inclusion systems. Further i^* is an isomorphism between units of $(\mathcal{E},B)$ and units of (E,β).*

Theorem 1.7. *Let $(E,\beta),(F,\gamma)$ be two inclusion systems generating two product systems $(\mathcal{E},B),(\mathcal{F},C)$ respectively. Let i,j be the respective inclusion maps. Suppose $A:(E,\beta)\to(F,\gamma)$ is a weak morphism then there exists a unique morphism $\hat{A}:(\mathcal{E},B)\to(\mathcal{F},C)$ such that $A_s=j_s^*\hat{A}_s i_s$ for all s. This is a one to one correspondence of weak morphisms. Further more, $\hat{A}$ is isometric/unitary if A is isometric/unitary.*

2. Universality of amalgamation

Suppose H and K are two Hilbert spaces and $D:K\to H$ is a linear contraction. Define a semi inner product on $H\oplus K$ by

$$\begin{aligned}\Big\langle \begin{pmatrix} u_1\\ v_1\end{pmatrix}, \begin{pmatrix} u_2\\ v_2\end{pmatrix}\Big\rangle_D &= \langle u_1,u_2\rangle+\langle u_1,Dv_2\rangle+\langle Dv_1,u_2\rangle+\langle v_1,v_2\rangle\\ &= \Big\langle \begin{pmatrix} u_1\\ v_1\end{pmatrix}, \tilde{D}\begin{pmatrix} u_2\\ v_2\end{pmatrix}\Big\rangle,\end{aligned}$$

where $\tilde{D}:=\begin{bmatrix} I & D\\ D^* & I\end{bmatrix}$. Note that as D is contractive, $\tilde{D}$ is positive definite. Take

$$N=\{\begin{pmatrix} u\\ v\end{pmatrix} : \Big\langle \begin{pmatrix} u\\ v\end{pmatrix}, \begin{pmatrix} u\\ v\end{pmatrix}\Big\rangle_D=0\}.$$

Then N is the kernel of bounded operator $\tilde{D}$ and hence it is a closed subspace of $H\oplus K$. Set G as completion of $(H\oplus K)/N$) with respect to norm of $\langle .,.\rangle_D$. We denote G by $H\oplus_D K$ and further denote the image of vector $\begin{pmatrix} u\\ v\end{pmatrix}$ by $\begin{bmatrix} u\\ v\end{bmatrix}$ for $u\in H$ and $v\in K$. Now

$$\Big\langle \begin{bmatrix} u_1\\ 0\end{bmatrix}, \begin{bmatrix} u_2\\ 0\end{bmatrix}\Big\rangle_D=\langle u_1,u_2\rangle_H;\ \Big\langle \begin{bmatrix} 0\\ v_1\end{bmatrix}, \begin{bmatrix} 0\\ v_2\end{bmatrix}\Big\rangle_D=\langle v_1,v_2\rangle_K.$$

So H and K are naturally embedded in $H\oplus_D K$ and their closed liner span is $H\oplus_D K$ but they need not be orthogonal. We call $H\oplus_D K$ as the amalgamation of H and K through D. It is to be noted that if range $(\tilde{D})$ is closed, then no completion is needed in the construction, and every vector of G is of the form $\begin{bmatrix} u\\ v\end{bmatrix}$ for $u\in H$ and $v\in K$.

In the converse direction, if H and K are two closed subspaces of a Hilbert space G. Then by a simple application of Riesz representation theorem for Hilbert space, there exists unique contraction $D:K\to H$ such that for $u\in H$, $v\in K$

$$\langle u,v\rangle_G=\langle u,Dv\rangle.$$

Now we consider amalgamation at the level of inclusion systems. Let (E,β) and (F,γ) be two inclusion systems. Let $D=\{D_s : s>0\}$ be a weak contractive morphism from F to E. Define $G_s := E_s \oplus_{D_s} F_s$ and $\delta_{s,t} := i_{s,t}(\beta_{s,t} \oplus_D \gamma_{s,t})$ where $i_{s,t} : (E_s \otimes E_t) \oplus_{D_s \otimes D_t} (F_s \otimes F_t) \to G_s \otimes G_t$ is the map defined by

$$i_{s,t} \begin{bmatrix} u_1 \otimes u_2 \\ v_1 \otimes v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ v_2 \end{bmatrix},$$

and $(\beta_{s,t} \oplus_D \gamma_{s,t}) : E_{s+t} \oplus_{D_{s+t}} F_{s+t} \to E_s \otimes E_t \oplus_{D_s \otimes D_t} F_s \otimes F_t$ is the map defined by

$$(\beta_{s,t} \oplus_D \gamma_{s,t}) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \beta_{s,t}(u) \\ \gamma_{s,t}(v) \end{bmatrix}$$

Lemma 2.1. *The maps $i_{s,t} : (E_s \otimes E_t) \oplus_{D_s \otimes D_t} (F_s \otimes F_t) \to G_s \otimes G_t$ and $(\beta_{s,t} \oplus_D \gamma_{s,t}) : E_{s+t} \oplus_{D_{s+t}} F_{s+t} \to (E_s \otimes E_t) \oplus_{D_s \otimes D_t} (F_s \otimes F_t)$ are well defined isometries.*

Proof. Proof can be found in [7], Section 3. □

Proposition 2.2. *Let $(G, \delta) = \{G_s, \delta_{s,t} : s, t > 0\}$ be defined as above. Then $\{G, \delta\}$ forms an inclusion system*

Proof. Proof can be found in [7], Section 3. □

Definition 2.3. *The inclusion system (G, δ) constructed above is called the amalgamation of inclusion systems (E, β) and (F, γ) via the morphism D. If $(\mathcal{E}, B)$, $(\mathcal{F}, C)$, and $(\mathcal{G}, L)$ are product systems generated respectively by (E, β), (F, γ), and (G, δ), then $(\mathcal{G}, L)$ is said to be the amalgamated product of $(\mathcal{E}, B)$ and $(\mathcal{F}, C)$ via D and is denoted by $\mathcal{G} =: \mathcal{E} \otimes_D \mathcal{F}$.*

Remark 2.4. *In this terminology, we have defined amalgamation of inclusion system via contractive morphism to be an operation on the category of inclusion systems. On the other hand, amalgamated product is an operation on the category of product systems. More precisely, Given two product systems $(\mathcal{E}, W^{\mathcal{E}})$ and $(\mathcal{F}, W^{\mathcal{F}})$ and a contractive morphism D, first we form the inclusion system (G, δ) which is the amalgamation of inclusion systems $(\mathcal{E}, W^{\mathcal{E}})$ and $(\mathcal{F}, W^{\mathcal{F}})$ via the morphism D. Then the product system generated by the inclusion system (G, δ) is the amalgamated product $\mathcal{E} \otimes_D \mathcal{F}$ of $(\mathcal{E}, W^{\mathcal{E}})$ and $(\mathcal{F}, W^{\mathcal{F}})$.*

Definition 2.5. *Suppose (E, β) is an inclusion system. Then a family of Hilbert spaces $F = (F_t)_{t>0}$ is said to be an inclusion subsystem of (E, β) if $F_t \subset E_t$ is closed for every $t > 0$, and $\beta_{s,t}|_{F_{s+t}}(F_{s+t}) \subset F_s \otimes F_t$ for every $s, t > 0$.*

Definition 2.6. *Suppose $(\mathcal{E}, W)$ is a product system. A family of Hilbert spaces $\mathcal{F} = (\mathcal{F}_t)_{t>0}$ is said to be a product subsystem of $\mathcal{E}$ if $\mathcal{F}_t \subset \mathcal{E}_t$ is closed for every $t > 0$, and $W|_{\mathcal{F}_{s+t}}$ is a unitary from $\mathcal{F}_{s+t}$ onto $\mathcal{F}_s \otimes \mathcal{F}_t$ for every $s, t > 0$.*

Suppose H is a Hilbert space and let M and N be two closed subspaces of H. We denote by $M \vee N$, the smallest closed subspace of H containing M and N. i.e.

$$M \vee N = \overline{\text{span}}\{x + y : x \in M, y \in N\}$$

Suppose $\mathcal{G}$ is a product system and let $\mathcal{E}$ and $\mathcal{F}$ be two product subsystem of $\mathcal{G}$. Then we denote by $\mathcal{E} \bigvee \mathcal{F}$, the smallest product system containing $\mathcal{E}$ and $\mathcal{F}$. The product system $\mathcal{E}\bigvee\mathcal{F}$ having the fiber: for each $t>0$,

$$(\mathcal{E}\bigvee\mathcal{F})_t=\overline{\text{span}}\{x^1_{t_1}\otimes\cdots\otimes x^n_{t_n}:\ \sum_i^n t_i=t,\ x^i_{t_i}\in\mathcal{E}_{t_i}\text{ or }\mathcal{F}_{t_i}\ ,i=1,2,\cdots n,n\geq 1\}$$

Note that for $t>0$, $\mathcal{E}_t\vee\mathcal{F}_t\neq(\mathcal{E}\bigvee\mathcal{F})_t$.

Theorem 2.7. *Suppose $(\mathcal{E},W^{\mathcal{E}})$ and $(\mathcal{F},W^{\mathcal{F}})$ are two product systems and let $C:(\mathcal{F},W^{\mathcal{F}})\to(\mathcal{E},W^{\mathcal{E}})$ be a contractive morphism. Suppose $(\mathcal{G},W^{\mathcal{G}})$ is the amalgamated product of $(\mathcal{E},W^{\mathcal{E}})$ and $(\mathcal{F},W^{\mathcal{F}})$. i.e. $\mathcal{G}=\mathcal{E}\otimes_C\mathcal{F}$. Then there are isometric product system morphism $I:\mathcal{E}\to\mathcal{G}$ and $J:\mathcal{F}\to\mathcal{G}$ such that the following holds:*

(i) $\langle I_s(x),J_s(y)\rangle=\langle x,C_sy\rangle$ for all $x\in\mathcal{E}_s$ and $y\in\mathcal{F}_s$.

(ii) $\mathcal{G}=I(\mathcal{E})\bigvee J(\mathcal{F})$.

Conversely, suppose $\mathcal{E}$ and $\mathcal{F}$ are two product subsystems of a product system $(\mathcal{H},W)$. Then there is a contraction morphism $C:\mathcal{F}\to\mathcal{E}$ such that the amalgamated product $\mathcal{G}$ of $\mathcal{E}$ and $\mathcal{F}$ via C is isomorphic via ϕ to the product system generated by $\mathcal{E}$ and $\mathcal{F}$. i.e. $\mathcal{E}\otimes_C\mathcal{F}\sim\mathcal{E}\bigvee\mathcal{F}$ which is canonical in the sense that

$$\phi(\begin{bmatrix} a\\ b\end{bmatrix})=a+b\ ,a\in\mathcal{E},b\in\mathcal{F}\tag{2.1}$$

Proof. Let (G,δ) be the amalgamation as inclusion system of $(\mathcal{E},W^{\mathcal{E}})$ and $(F,W^{\mathcal{F}})$. Let $i_s:\mathcal{E}_s\to G_s$ and $j_s:\mathcal{F}_s\to G_s$ be the maps

$$i_s(a)=\begin{bmatrix} a\\ 0\end{bmatrix}\ ,\ a\in\mathcal{E}_s$$

and

$$j_s(b)=\begin{bmatrix} 0\\ b\end{bmatrix}\ ,\ b\in\mathcal{F}_s$$

Let $g_s:G_s\to\mathcal{G}_s$ be the canonical map. Set $I_s=g_si_s$ and $J_s=g_sj_s$. As i, j and g are isometries so are I and J. Suppose for $a\in\mathcal{E}_{s+t}$, $W^{\mathcal{E}}_{s,t}(a)=\sum_r x_r\otimes y_r$. Now

$$\begin{aligned}
W^{\mathcal{G}}_{s,t}I_{s+t}(a) &= W^{\mathcal{G}}_{s,t}g_{s+t}i_{s+t}(a)\\
&= (g_s\otimes g_t)\delta_{s,t}i_{s+t}(a)\\
&= (g_s\otimes g_t)i_{s,t}(W^{\mathcal{E}}_{s,t}\oplus_C W^{\mathcal{F}}_{s,t})\begin{bmatrix} a\\ 0\end{bmatrix}\\
&= (g_s\otimes g_t)\sum_r i_{s,t}\begin{bmatrix} x_r\otimes y_r\\ 0\end{bmatrix}\\
&= (g_s\otimes g_t)\sum_r \begin{bmatrix} x_r\\ 0\end{bmatrix}\otimes\begin{bmatrix} y_r\\ 0\end{bmatrix}\\
&= (g_s\otimes g_t)\sum_r i_s(x_r)\otimes i_t(y_r)\\
&= (g_s\otimes g_t)(i_s\otimes i_t)W^{\mathcal{E}}_{s,t}(a)\\
&= (I_s\otimes I_t)W^{\mathcal{E}}_{s,t}(a).
\end{aligned}$$

So I is an isometric morphism of product systems. Similarly $J_s : \mathcal{F}_s \to G_s$ is an isometric morphism of product systems. For $a \in \mathcal{E}_{s+t}$, Now for $x \in \mathcal{E}_s$ and $y \in \mathcal{F}_s$,

$$\begin{aligned}\langle I_s(x), J_s(y)\rangle &= \langle \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix}\rangle \\ &= \langle x, C_s y\rangle.\end{aligned}$$

So (i) is verified. As $I(\mathcal{E}) \subset \mathcal{G}$ and $J(\mathcal{F}) \subset \mathcal{G}$, it implies that $I(\mathcal{E}) \bigvee J(\mathcal{F}) \subset \mathcal{G}$. On the other hand, I and J maps $\mathcal{E}$ and $\mathcal{F}$ into $I(\mathcal{E}) \bigvee J(\mathcal{F})$ isometrically and (i) is satisfied. So from universal property of amalgamated product Proposition 18, [7], there is an isometric morphism $A_t : \mathcal{G}_t \to (I(\mathcal{E}) \bigvee J(\mathcal{F}))_t$ by $A_t g_t \begin{bmatrix} a \\ b \end{bmatrix} = I_t(a) + J_t(b)$. For $a \in \mathcal{E}_t$ and $b \in \mathcal{F}_t$, $I_t(a) = g_t i_t(a)$ and $J_t(b) = g_t j_t(b)$ implies $A_t g_t(i_t(a) + j_t(b)) = g_t(i_t(a) + j_t(b))$. We get that A_t is the inclusion morphism from $\mathcal{G}_t$ into $(I(\mathcal{E}) \bigvee J(\mathcal{F}))_t$, for every $t > 0$. So $\mathcal{G} \subset I(\mathcal{E}) \bigvee J(\mathcal{F})$.

Now for the converse, let for every $t > 0$, $P^{\mathcal{E}_t}$ and $P^{\mathcal{F}_t}$ be the projections onto $\mathcal{E}_t$ and $\mathcal{F}_t$ respectively in $\mathcal{H}_t$. Then $P^{\mathcal{E}} = \{P^{\mathcal{E}_t} : t > 0\}$ and $P^{\mathcal{F}} = \{P^{\mathcal{F}_t} : t > 0\}$ are projection morphisms on $(\mathcal{H}, W)$. Define for every $t > 0$, $C_t = P^{\mathcal{E}_t} P^{\mathcal{F}_t}|_{\mathcal{F}_t}$. Then clearly $C = \{C_t : t > 0\}$ is a contraction morphism from $\mathcal{F}$ to $\mathcal{E}$. Now set for every $t >, 0$ $G_t = \mathcal{E}_t \oplus_{C_t} \mathcal{F}_t$.

Claim 1 : $\mathcal{E} \vee \mathcal{F} := (\mathcal{E}_t \vee \mathcal{F}_t, W|_{\mathcal{E}_t \vee \mathcal{F}_t})$ is an inclusion subsystem of $(\mathcal{H}, W)$ which generates the product system $\mathcal{E} \bigvee \mathcal{F}$.

Proof of claim 1 : The first part of the claim is easy. Indeed for every $s, t > 0$, $\mathcal{E}_{s+t} \vee \mathcal{F}_{s+t} \subset \mathcal{E}_s \otimes \mathcal{E}_t \vee \mathcal{F}_s \otimes \mathcal{F}_t \subset (\mathcal{E}_s \vee \mathcal{F}_s) \otimes (\mathcal{E}_t \vee \mathcal{F}_t)$. For the second part, let $\mathcal{G}'$ be the generated product system. As $\mathcal{E} \subset \mathcal{G}'$ and $\mathcal{F} \subset \mathcal{G}'$, we have $\mathcal{E} \bigvee \mathcal{F} \subset \mathcal{G}'$. On the other hand, $\mathcal{E}_t \subset (\mathcal{E} \bigvee \mathcal{F})_t$ and $\mathcal{F}_t \subset (\mathcal{E} \bigvee \mathcal{F})_t$, implies the inclusion system $\mathcal{E} \vee \mathcal{F}$ is an inclusion subsystem of $\mathcal{E} \bigvee \mathcal{F}$. So $\mathcal{G}'$ is a product subsystem of $\mathcal{E} \bigvee \mathcal{F}$, proving the claim.

Define $l_t : G_t \to \mathcal{E}_t \vee \mathcal{F}_t$ by $l_t \begin{bmatrix} u \\ v \end{bmatrix} = u + v$.

Claim 2 : $l : G \to \mathcal{E} \vee \mathcal{F}$ is a unitary morphism of inclusion systems.

Proof of claim 2 : Clearly l_t is a unitary for each $t > 0$. To prove that it is a morphism, consider $\begin{bmatrix} a \\ b \end{bmatrix} \in G_{s+t}$. Decomposing $W_{s,t}(a) = \sum_k x_k \otimes y_k$ and $W_{s,t}(b) = \sum_r u_r \otimes v_r$, $(l_s \otimes l_t)\delta_{s,t} \begin{bmatrix} a \\ b \end{bmatrix} = (l_s \otimes l_t) i_{s,t} \begin{bmatrix} \sum_k x_k \otimes y_k \\ \sum_r u_r \otimes v_r \end{bmatrix} = \sum_k (l_s \otimes l_t) \begin{bmatrix} x_k \\ 0 \end{bmatrix} \otimes \begin{bmatrix} y_k \\ 0 \end{bmatrix} + \sum_r (l_s \otimes l_t) \begin{bmatrix} 0 \\ u_r \end{bmatrix} \otimes \begin{bmatrix} 0 \\ v_r \end{bmatrix} = \sum_k x_k \otimes y_k + \sum_r u_r \otimes v_r = W_{s,t}(a) + W_{s,t}(b) = W_{s,t} l_{s+t} \begin{bmatrix} a \\ b \end{bmatrix}$, proving claim 2. Let $\phi : \mathcal{G} \to \mathcal{E} \bigvee \mathcal{F}$ be the lift of l. Then (2.1) is satisfied. This proves the converse part. $\square$

The theorem means that the amalgamation of two product systems is nothing but a construction of a large product system containing the two product systems as subsystems such that it is generated by those product subsystems and the two subsystems sit inside the amalgamated product by the defining condition (i) of the theorem. So from now on we will always assume that $\mathcal{E}$ and $\mathcal{F}$ are subsystems of $\mathcal{E} \otimes_C \mathcal{F}$. This construction may lead to a non separable product system even if the individual components are separable. If we choose our contractive morphism to be a zero morphism, then the amalgamated product is always non separable product system as the following lemma shows.

Lemma 2.8. *Let $\mathcal{G}$ be a product system. Let $\mathcal{E}, \mathcal{F}$ be nonzero subsystems of $\mathcal{G}$ with the property that*

$$\langle x, y\rangle = 0 \ , \ \textit{for all } x \in \mathcal{E} \ , \ y \in \mathcal{F}$$

Then $\mathcal{G}$ is a non separable product system.

Proof. Choose $e_t \in \mathcal{E}_t$ and $f_t \in \mathcal{F}_t$ with $\|e_t\| = \|f_t\| = 1$ for all $t > 0$. Consider the uncountable set $\{g_s := e_s \otimes f_{t-s} : 0 < s < t\} \subset \mathcal{G}_t$. Now for $0 < s < s' < t$, we have $g_s \in \mathcal{E}_s \otimes \mathcal{F}_{t-s}$ and $g_{s'} \in \mathcal{E}_{s'} \otimes \mathcal{F}_{t-s'}$. Decomposing $\mathcal{E}_s \otimes \mathcal{F}_{t-s} = \mathcal{E}_s \otimes \mathcal{F}_{s'-s} \otimes \mathcal{F}_{t-s'}$ and $\mathcal{E}_{s'} \otimes \mathcal{F}_{t-s'} = \mathcal{E}_s \otimes \mathcal{E}_{s'-s} \otimes \mathcal{F}_{t-s'}$. Now it follows from the hypothesis that $\langle g_s, g_{s'}\rangle = 0$, for all $0 < s < s' < t$. Hence the lemma follows. □

Similar problem arises in dilation theory of CP semigroups. There are CP semigroups which are not strongly continuous at 0, for which the minimal E dilations act on non-separable Hilbert spaces. Though in this paper, we are not concerned about solving the problem of classifying all contractive morphisms for which the amalgamated product is a separable product system. As for most of our examples, amalgamated product is always separable.

For a product system $\mathcal{G}$, let $\mathcal{U}^{\mathcal{G}}$ and $\mathcal{G}^I$ denote the set of all units and the type I part of $\mathcal{G}$ respectively. i.e.

$$\mathcal{G}^I = \overline{\text{span}}\{u^1_{t_1} \otimes ... \otimes u^n_{t_n} : u^i \in \mathcal{U}^{\mathcal{G}}, 1 \leq i \leq n, \sum_i t_i = t, n \geq 1\}$$

Theorem 2.9. *Let $\mathcal{E}$ and $\mathcal{F}$ be two spatial product systems and $C : \mathcal{F} \to \mathcal{E}$ be a contractive morphism such that $\mathcal{E} \otimes_C \mathcal{F}$ is a separable product system. Then $(\mathcal{E} \otimes_C \mathcal{F})^I = \mathcal{E}^I \otimes_C \mathcal{F}^I$.*

Proof. Part of this theorem has been proved in [7],Theorem 24. Let $\mathcal{G} = (\mathcal{E} \otimes_C \mathcal{F})$. As units of $\mathcal{E}$ and of $\mathcal{F}$ are units of the amalgamated product, it follows that $\mathcal{E}^I$ and $\mathcal{F}^I$ are contained in $\mathcal{G}^I$. So the smallest product system generated by $\mathcal{E}^I$ and $\mathcal{F}^I$ is contained in $\mathcal{G}^I$. Hence right hand side is contained in left hand side. Now for the converse, Choose any unit $w \in \mathcal{G}$. Denoting $P^{\mathcal{E}}$ and $P^{\mathcal{F}}$ the projection morphism respectively on $\mathcal{E}$ and $\mathcal{F}$, separability assumption on $\mathcal{G}$ implies that $P^{\mathcal{E}}w$ and $P^{\mathcal{F}}w$ are nonzero hence they are units of $\mathcal{E}$ and $\mathcal{F}$, hence units of $\mathcal{G}$. From the Theorem 2.7, we get

$$I^{\mathcal{G}} = \bigvee\{P^{\epsilon_1}_{t_n} \otimes ... P^{\epsilon_n}_{t_1} : \mathbf{t} = (t_n, ... t_1) \in J_t, \epsilon_i = \{\mathcal{E}, \mathcal{F}\}, 1 \leq i \leq n, n \geq 1\}.$$

Now $w = I^{\mathcal{G}}w \in \overline{\text{span}}\{z_{t_1} \otimes .. z_{t_n} : \mathbf{t} = (t_1, ..., t_n) \in J_t, n \geq 1, z = P^{\mathcal{E}}w, P^{\mathcal{F}}w\} \in \mathcal{E}^I \bigvee \mathcal{F}^I = \mathcal{E}^I \otimes_C \mathcal{F}^I$. Hence the theorem is proved. □

Proposition 2.10. *Let $\mathcal{E}$ and $\mathcal{F}$ be two product systems and $C : \mathcal{F} \to \mathcal{E}$ be a contractive morphism. Then for every $t > 0$, $P^{\mathcal{E}_t}$ and $P^{\mathcal{F}_t}$ commute in $\mathcal{B}((\mathcal{E} \otimes_C \mathcal{F})_t)$ if and only if C is a morphism of partial isometries.*

Proof. From the Theorem 2.7, we know that for every $t > 0$, $C_t = P^{\mathcal{E}_t}P^{\mathcal{F}_t}|_{\mathcal{F}_t} : \mathcal{F}_t \to \mathcal{E}_t$. If $P^{\mathcal{E}_t}$ and $P^{\mathcal{F}_t}$ commute, then for every $t > 0$, $C_t^*C_t = P^{\mathcal{E}_t} \wedge P^{\mathcal{F}_t} = P^{\mathcal{E}_t \wedge \mathcal{F}_t}$ as a projection on $\mathcal{F}_t$ and $C_tC_t^* = P^{\mathcal{E}_t} \wedge P^{\mathcal{F}_t} = P^{\mathcal{E}_t \wedge \mathcal{F}_t}$ as a projection on $\mathcal{E}_t$, implying C to be a morphism of partial isometry between $\mathcal{F}$ to $\mathcal{E}$. Conversely, assume C is a morphism of partial isometry. Then for every $t > 0$, $P^{\mathcal{F}_t}P^{\mathcal{E}_t}P^{\mathcal{F}_t}|_{\mathcal{F}_t}$ and $P^{\mathcal{E}_t}P^{\mathcal{F}_t}P^{\mathcal{E}_t}|_{\mathcal{E}_t}$ are projections on $\mathcal{F}_t$ and $\mathcal{E}_t$ respectively. It implies easily that $P^{\mathcal{F}_t}P^{\mathcal{E}_t}P^{\mathcal{F}_t}$ and $P^{\mathcal{E}_t}P^{\mathcal{F}_t}P^{\mathcal{E}_t}$ are projections on $\mathcal{G}_t$. Now for every $t > 0$, writing $P^{\mathcal{E}_t}$ and $P^{\mathcal{F}_t}$ as a matrix decomposing it as $P^{\mathcal{F}_t}$ and $I^{(\mathcal{E} \otimes_C \mathcal{F})_t} - P^{\mathcal{F}_t}$,

$$P^{\mathcal{E}_t} = \begin{pmatrix} A_t & B_t \\ B_t^* & C_t \end{pmatrix} , \ P^{\mathcal{F}_t} = \begin{pmatrix} I_{\mathcal{F}_t} & 0 \\ 0 & 0 \end{pmatrix},$$

we get that $P^{\mathcal{F}_t}P^{\mathcal{E}_t}P^{\mathcal{F}_t} = \begin{pmatrix} A_t & 0 \\ 0 & 0 \end{pmatrix}$. It implies $A_t^2 = A_t$, for all $t > 0$. From the first matrix we get, for every $A_t^2 + B_tB_t^* = A_t$, for all $t > 0$, which imply $B_t = 0$, for all $t > 0$. Hence for ever $t > 0$, $P^{\mathcal{E}_t}$ and $P^{\mathcal{F}_t}$ commute. □

3. CONTRACTIVE MORPHISMS

In this Section we will mainly concentrate on the category of product systems as defined by Arveson [1],page 68. In particular, all Hilbert spaces are separable and the product system has a measurable structure. By a contractive morphism, we mean a contractive morphism of product systems in that category i.e. the family of maps is a measurable family. We will call this category as category of Arveson's product systems. Contractive morphisms on product systems have been studied since long time ago. Arveson [1], Theorem 8.8, characterized all unitary morphisms on CCR flows. All positive contractive morphisms and projection morphisms on CCR flows have been parameterized in Bhat [5], Theorem 7.5. A complete characterization of morphisms on time ordered Fock modules has been done in [2], Theorem 5.2.1. Here we give a complete parametrization of all contractive morphisms from a type I product system to another type I product system.

Let D_t be a contractive morphism from a type I product system $\mathcal{F}$ to a type I product system $\mathcal{E}$. Thanks to Arveson, without loss of generality we may assume $\mathcal{E}_t = \Gamma_{sym}(L^2[0,t], K_1)$ and $\mathcal{F}_t = \Gamma_{sym}(L^2[0,t], K_2)$ for some separable Hilbert spaces K_1 and K_2 with corresponding unitaries

$$W^{\mathcal{E}}_{s,t}e(f) \otimes e(g) = e(g + S^1_t f), \ f, g \in L^2([0,t], K_1)$$

and

$$W^{\mathcal{F}}_{s,t}e(f) \otimes e(g) = e(g + S^2_t f), \ f, g \in L^2([0,t], K_2)$$

where S^1 and S^2 are usual shift semigroup respectively on $L^2([0,\infty), K_1)$ and $L^2([0,\infty), K_2)$ defined below. For $f \in L^2([0,t], K_1)$ and $g \in L^2([0,t], K_2)$,

$$S^1_t f(x) = \begin{cases} f(x-t) & \text{if } t \leq x \\ 0 & \text{otherwise} \end{cases}$$

and

$$S_t^2 g(x) = \begin{cases} g(x-t) & \text{if } t \le x \\ 0 & \text{otherwise} \end{cases}$$

Any unit has the form $u(t) = e^{-pt} e(y\chi_{t]})$ for some $(p, y) \in \mathbb{C} \times K_i, i = 1, 2$. We know that the morphisms D and D^* sends units to units.

Hence for $y \in K_2$, and $z \in K_1$,

$$D_t e(y\chi_{t]}) = e^{-p_y t} e(B_y \chi_{t]})$$

$$D_t^* e(z\chi_{t]}) = e^{-q_z t} e(C_z \chi_{t]})$$

for some $(p_y, B_y) \in \mathbb{C} \times K_1$ and $(q_z, C_z) \in \mathbb{C} \times K_2$. We claim that $y \mapsto B_y - B_0$ and $z \mapsto C_z - C_0$ are linear. We have

$$\begin{aligned} \langle e(z\chi_{t]}), D_t e(y\chi_{t]})\rangle &= e^{-p_y t + \langle z, B_y\rangle t} \\ &= \langle D_t^* e(z\chi_{t]}), e(y\chi_{t]})\rangle \\ &= \langle e^{-q_z t} e(C_z \chi_{t]}), e(y\chi_{t]})\rangle \\ &= e^{-\overline{q_z} t} e^{\langle C_z, y\rangle} t. \end{aligned}$$

Hence

$$-p_y + \langle z, B_y\rangle = -\overline{q_z} + \langle C_z, y\rangle.$$

Put $y = z = 0$ to get $p_0 = \bar{q}_0$. Putting only $y = 0$ gives $-p_0 + \langle z, B_0\rangle = -\overline{q_z}$. Putting only $z = 0$ gives $-p_y = -p_0 + \langle C_0, y\rangle$. Hence we have the following

$$\langle z, B_y - B_0\rangle = \langle C_z - C_0, y\rangle. \tag{3.1}$$

Now for $y, w \in K_2$ and $a \in \mathbb{C}$,

$$\begin{aligned} \langle z, B_{ay+w} - B_0\rangle &= \langle C_z - C_0, ay + w\rangle \\ &= a\langle C_z - C_0, y\rangle + \langle C_z - C_0, w\rangle \\ &= a\langle z, B_y - B_0\rangle + \langle z, B_w - B_0\rangle \\ &= \langle z, a(B_y - B_0) + (B_w - B_0)\rangle. \end{aligned}$$

This shows that the map $y \mapsto B_y - B_0$ is linear from K_2 to K_1. Set $Ay = B_y - B_0$, for $y \in K_2$. Take $u = B_0, x = C_0, q = p_0$. So $\overline{q} = q_0$ and from (3.1), $A^* z = Cz - C_0$. Hence

$$D_t e(y\chi_{t]}) = e^{-qt + \langle x, y\rangle t} e(Ay\chi_{t]} + u\chi_{t]}) \tag{3.2}$$

$$D_t^* e(z\chi_{t]}) = e^{-\bar{q}t + \langle u, z\rangle t} e(A^* z\chi_{t]} + x\chi_{t]})$$

Now for $y \in K_2$, $\|D_t e(y\chi_{t]})\|^2 \le \|e(y\chi_{t]})\|^2$. This yields

$$e^{-(q+\bar{q})t + \langle y, x\rangle t + \langle x, y\rangle t} e^{\langle Ay+u, Ay+u\rangle t} \le e^{\langle y, y\rangle t},$$

which imply $q + \bar{q} - \|u\|^2 - \langle y, A^* u + x\rangle - \langle A^* u + x, y\rangle + \langle y, (I - A^* A)y\rangle \ge 0$. With little bit of calculation this imply the following matrix

$$\begin{pmatrix} q + \bar{q} - \|u\|^2 & -(A^* u + x)^* \\ -(A^* u + x) & I - A^* A \end{pmatrix} \ge 0 \tag{3.3}$$

Now from [5] this matrix is positive if and only if A is a contraction, $A^*u + x \in \text{Range}(I - A^*A)^{1/2}$, $q + \bar{q} \geq \|u\|^2 + q_0(A, x, u)$ where $q_0(A, x, u) = \inf\{\|a\|^2 : A^*x + u = (I - A^*A)^{1/2}a\}$.

Define $\mathcal{C}(K_2, K_1) \subset \mathbb{C} \times K_2 \times K_1 \times B(K_2, K_1)$ as the set of all tuples (q, x, u, A) such that (3.3) holds. For $(q, x, u, A) \in \mathcal{C}(K_2, K_1)$, define $[q, x, u, A]_t$ on the set $\{e(f) : f \in L^2([0, t], K_2)\}$ by

$$[q, x, u, A]_t e(f) = e^{-qt + \langle x\chi|_{t]}, f\rangle} e(Af + u\chi_{t]}),$$

where $Af \in L^2([0, t], K_1)$ is defined by $(Af)(s) = Af(s)$. From the fact that $\{e(f) : f \in L^2([0, t], K_2)\}$ is total in $\Gamma_{sym}(L^2[0, t], K_2)$, and are linearly independent. we extend linearly $[q, x, u, A]_t$ to all of $\Gamma_{sym}(L^2[0, t], K_2)$. Now we will show that it is contractive which in turn shows that the extension is contractive. For this we need the notion of conditionally positive definiteness.

Definition 3.1. *An $n \times n$ matrix $B = [b_{ij}]$ is said to be conditionally positive definite if it is self-adjoint and $\sum_{i,j=1}^n \overline{c_i} c_j b_{ij} \geq 0$ for all $c_1, c_2, \cdots, c_n \in \mathbb{C}$ with $\sum c_i = 0$.*

Lemma 3.2. *For $n \times n$ complex matrix B, the following are equivalent*

[1] $[e^{tb_{ij}}] \geq 0$, for all $t \geq 0$;

[2] B is conditionally positive definite;

[3] $b_{ij} = b + \overline{b_i} + b_j + c_{ij}$, for some $b \in \mathbb{R}$, $b_1, b_2, \cdots, b_n \in \mathbb{C}$ and $C = [c_{ij}] \geq 0$.

Suppose B is conditionally positive definite then for any matrix D, $[e^{tb_{ij}}] \leq [e^{td_{ij}}]$ for all t if and only if $B \leq D$.

Proof. See [9], [11] for equivalence of [1] to [3]. For the last statement, suppose $[e^{tb_{ij}}] \leq [e^{td_{ij}}]$ for all t. Let J be the matrix with all of its entries equal to 1. Then we get, for every $t \geq 0$,

$$\frac{1}{t}([e^{tb_{ij}}] - J) \leq \frac{1}{t}([e^{td_{ij}}] - J).$$

taking limit as $t \downarrow 0$, we get $B \leq D$. For the converse,

$$[e^{td_{ij}} - e^{tb_{ij}}] \quad = \quad [e^{tb_{ij}}] \cdot [e^{t(d_{ij} - b_{ij})} - J]$$

where $\cdot$ indicate the Schur product or entry wise product of two matrices. Letting $C = D - B \geq 0$. We get $[e^{t(d_{ij} - b_{ij})} - J] = tC + (t^2/2!)C \cdot C + (t^3/3!)C \cdot C \cdot C + \cdots \geq 0$. Now as B is conditionally positive, $[e^{tb_{ij}}] \geq 0$. The result now follows as Schur product of two positive matrices is positive. □

Proposition 3.3. *Suppose D is a contractive morphism from the product system $\Gamma(L^2[0, t], K_2)$ to the product system $\Gamma(L^2(0, t), K_1)$. Then there exists $(q, x, u, A) \in \mathcal{C}(K_2, K_1)$ such that $D_t = [q, x, u, A]_t$. Conversely, for any tuple $(q, x, u, A) \in \mathcal{C}(K_2, K_1)$, $[q, x, u, A]_t$ defines a contractive morphism from $\Gamma(L^2(0, t), K_2)$ to $\Gamma(L^2(0, t), K_1)$.*

Proof. We have shown in the previous section that a contractive morphism $D_t : \Gamma(L^2[0, t], K_2) \to \Gamma(L^2[0, t], K_1)$ is of the form $D_t = [q, x, u, A]_t$ for some $(q, x, u, A) \in \mathcal{C}(K_2, K_1)$. Conversely, for $\{q, x, u, A\} \in \mathcal{C}(K_2, K_1)$, consider $[q, x, u, A]_t$. We will show that it is contractive, indeed for $c_1, c_2, \cdots, c_n \in \mathbb{C},\ f_1, f_2, \cdots, f_n \in L^2([0,t], K_2)$,

$$\|[q,x,u,A]_t \sum_i c_i e(f_i)\|^2$$
$$= \sum_{i,j} \overline{c_i} c_j e^{-(q+\overline{q}-\|u\|^2)t + \langle (x+A^*u)\chi_{t]}, f_i\rangle + \langle f_j, (x+A^*u)\chi_{t]}\rangle + \langle f_i, A^*Af_j\rangle}$$

Now from Lemma 3.2,

$$\|[q,x,u,A]_t \sum_i c_i e(f_i)\|^2 \leq \|\sum_i c_i e(f_i)\|^2$$

is equivalent to $[a_{ij}(t)] \geq 0$, where

$a_{ij}(t) = (q+\overline{q}-\|u\|^2)t - \langle (x+A^*u)\chi_{t]}, f_i\rangle - \langle f_j, (x+A^*u)\chi_{t]}\rangle + \langle f_i, (I-A^*A)f_j\rangle$

Set

$b_{ij}(s) = (q+\overline{q}-\|u\|^2) - \langle (x+A^*u), f_i(s)\rangle - \langle f_j(s), (x+A^*u)\rangle + \langle f_i(s), (I-A^*A)f_j(s)\rangle$

From (3.3), we have $[b_{ij}(s)] \geq 0$, for every $0 \leq s \leq t$. Then

$$a_{ij}(t) = \int_0^t b_{ij}(s)ds.$$

Hence $[a_{ij}(t) \geq 0]$.

Now we prove that $[q,x,u,A]_t : \Gamma_{sym}(L^2[0,t], K_2) \to \Gamma_{sym}(L^2[0,t], K_1)$ is a morphism of product system, indeed, for $f \in L^2([0,s], K_2)$ and $g \in L^2([0,t], K_2)$,

$$\begin{aligned}
[q,x,u,A]_{s+t} \quad W^{\mathcal{F}}_{s,t} \quad & (e(f)\otimes e(g)) \\
= \quad & [q,x,u,A]_{s+t} e(g + S_t f) \\
= \quad & e^{-q(s+t)+\langle x\chi|_{s+t]}, (g+S_tf)\rangle} e(A(g+S_tf) + u\chi_{s+t]}) \\
= \quad & e^{-qt+\langle x\chi|_{t]}, g\rangle} e^{-qs\langle x\chi|_{s]}, f\rangle} e((Ag+u\chi_{t]}) + S_t(Af+u\chi_{s]})) \\
= \quad & W^{\mathcal{E}}_{s,t} e^{-qs+\langle x\chi|_{s]}, f\rangle} e(Af+u\chi_{s]}) \otimes e^{-qt+\langle x\chi|_{t]}, g\rangle} e(Ag+u\chi_{t]}) \\
= \quad & W^{\mathcal{E}}_{s,t}([q,x,u,A]_s e(f) \otimes [q,x,u,A]_t e(g)).
\end{aligned}$$

□

We wish to compute the index of the amalgamated products of product systems. As noted before, to define index, we need separability and measurability structure on the product system. A priori it is not clear whether the amalgamated product system has any measurable structure even if the components are Arveson's product systems. We will handle this technical problem by restricting ourselves into a subclass where the amalgamated product is an Arveson's product system. Recalling Theorem 2.7, this is equivalent to the following assumption that there is a big Arveson's product system $\mathcal{H}$ which contains the two systems $\mathcal{E}$ and $\mathcal{F}$ as subsystems. So now on we will always assume this setup.

Let $\mathcal{U}^{\mathcal{E}}$ denote the units of a product system $\mathcal{E}$. Then the measurability ensures the existence a function

$$\gamma : \mathcal{U}^{\mathcal{E}} \times \mathcal{U}^{\mathcal{E}} \to \mathbb{C}$$

called the covariance function satisfying:

$$\langle u_t, v_t \rangle = e^{t\gamma(u,v)} \ \forall t,$$

for units u, v. The function γ is a conditionally positive definite function [1], Proposition 4.5. If Z is a non-empty subset of $\mathcal{U}^{\mathcal{E}}$, we may do the usual GNS construction for the kernel γ restricted to $Z \times Z$ to obtain a Hilbert space H_Z, which we call as the Arveson Hilbert space associated to Z. Note that the index of the product system $\mathcal{E}$ is nothing but the dimension of $\mathcal{K} := H_{\mathcal{U}^{\mathcal{E}}}$ (Arveson Hilbert space of $\mathcal{U}^{\mathcal{E}}$). In [1], Theorem 4.7, it is shown that there exists a bijection $u \mapsto (\lambda(u), \mu(u)) \in \mathbb{C} \times \mathcal{K}$, between $\mathcal{U}^{\mathcal{E}}$ and $\mathbb{C} \times \mathcal{K}$, satisfying

$$\gamma(u, u') = \overline{\lambda(u)} + \lambda(u') + \langle \mu(u), \mu(u') \rangle.$$

In the following, for simplicity of notation, though we have different product systems, we will be using same λ and μ for the corresponding bijections. This shouldn't cause any confusion. We need couple of lemmas before we state our main theorem.

For an operator A on a Hilbert space H, we denote $N(A) = \dim \{x \in H : Ax = 0\}$.

Lemma 3.4. *Let X be a linear operator from a Hilbert space H into K. Then rank of the operator $Z := \begin{pmatrix} I_K & X \\ X^* & I_H \end{pmatrix} : (K \oplus H) \to (K \oplus H)$ is $dim(K) + dim(H) - N(I - X^*X)$.*

Proof. If dim H or dim K is infinite dimensional then clearly the rank of Z is infinite. So Assume dim H , dim $K < \infty$. First assume $H = K$ and dim $H = n$. Then by polar decomposition of $X = U|X|$, where U is a unitary we get,

$$Z = \begin{pmatrix} I_H & U|X| \\ |X|U^* & I_H \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & I_H \end{pmatrix} \begin{pmatrix} I_H & |X| \\ |X| & I_H \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & I_H \end{pmatrix}.$$

Which implies that the rank of Z is same as the rank of $\begin{pmatrix} I_H & |X| \\ |X| & I_H \end{pmatrix}$.

By spectral theorem we write $|X| = VDV^*$, where V is a unitary and $D = \mathrm{diag}\{x_1, x_2, \cdots, x_n\}$. Then

$$\begin{pmatrix} I_H & |X| \\ |X| & I_H \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} I_H & D \\ D & I_H \end{pmatrix} \begin{pmatrix} V^* & 0 \\ 0 & V^* \end{pmatrix}.$$

So the rank of Z is same as the rank of $\begin{pmatrix} I_H & D \\ D & I_H \end{pmatrix}$. Conjugating by sequence of permutation matrices we get that the rank of $\begin{pmatrix} I_H & D \\ D & I_H \end{pmatrix}$ is same as the rank of

$$\begin{pmatrix} B_1 & 0 & . & . & . & 0 \\ 0 & B_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & B_{n_1} & 0 \\ 0 & . & . & . & 0 & B_n \end{pmatrix},$$

where $B_i = \begin{pmatrix} 1 & x_i \\ x_i & 1 \end{pmatrix}$, for $i = 1, 2, \cdots, n$. Now rank $(B_i) = 2$ if and only if $x_i \neq 1$, for each $i = 1, 2, \cdots, n$ and $\{\sharp\ i : x_i = 1\} = N(I - |X|)$, As $(I + |X|)$ is invertible, we have $N(I - |X|) = N(I - X^*X)$.

So we have rank $(Z) = 2n - N(I - X^*X)$. Now for different Hilbert spaces H and K, consider

$$\tilde{Z} = \begin{pmatrix} I_K & 0 & 0 & X \\ 0 & I_H & 0 & 0 \\ 0 & 0 & I_K & 0 \\ X^* & 0 & 0 & I_H \end{pmatrix}.$$

So by what we have proved, rank $(\tilde{Z}) = 2$ dim $(H) + 2$ dim $(K) - N(I - X^*X)$. Clearly $\tilde{Z}$ is conjugate via permutation matrices to

$$\begin{pmatrix} Z & 0 & 0 \\ 0 & I_H & 0 \\ 0 & 0 & I_K \end{pmatrix}.$$

Hence rank $(Z) =$ dim $(H) +$ dim $(K) - N(I - X^*X)$. $\square$

Lemma 3.5. *Let X be a positive linear operator of rank k on a finite Hilbert space H. Let $x \in H$ and $\alpha \in \mathbb{C}$ be fixed. Define a linear operator $[\alpha, x, X] := \begin{pmatrix} \alpha & x^* \\ x & X \end{pmatrix}$ on $\mathbb{C} \oplus H$ via*

$$\begin{pmatrix} \alpha & x^* \\ x & X \end{pmatrix} \begin{pmatrix} \beta \\ h \end{pmatrix} = \begin{pmatrix} \alpha\beta + \langle x, h\rangle \\ \beta x + Xh \end{pmatrix}.$$

Then

$$\textit{rank}\ [\alpha, x, X] = \begin{cases} k & \textit{if and only if } x \in \textit{range}(X) \textit{ and } \alpha = \langle x, y\rangle, \\ & \qquad \textit{where } Xy = x. \\ k+1 & \qquad \textit{otherwise} \end{cases}$$

Proof. If $x \notin$ range(X) then clearly rank of $[\alpha, x, X]$ is greater than the rank of X. Suppose that rank of $[\alpha, x, X] = k$. It implies that $x = Xy$, for some $y \in H$. So the rank of the linear operator $(x, X) : \mathbb{C} \oplus H \to H$ is k. Now $\begin{pmatrix} \alpha \\ x \end{pmatrix}$ is in the range of $(x, X)^*$ means there exists $z \in H$ such that $(x, X)^*z = \begin{pmatrix} \alpha \\ x \end{pmatrix}$ which implies $\langle x, z\rangle = \alpha$ and $Xz = x$. Now if $y, z \in H$ such that $Xy = Xz = x$. Then $X(y-z) = 0\mathbb{R}ightarrow y - z \in$ range $(X)^\perp \mathbb{R}ightarrow \langle x, y - z\rangle = 0 \mathbb{R}ightarrow \langle x, y\rangle = \langle x, z\rangle$. Conversely suppose $x = Xy$ for some $y \in H$ and $\alpha = \langle x, y\rangle$. Consider the operators $\begin{pmatrix} (Xy)^* \\ X \end{pmatrix} : H \to \mathbb{C} \oplus H$ and $\begin{pmatrix} \langle Xy, y\rangle \\ Xy \end{pmatrix} : \mathbb{C} \to \mathbb{C} \oplus H$. As $\begin{pmatrix} (Xy)^* \\ X \end{pmatrix}(y) = \begin{pmatrix} \langle Xy, y\rangle \\ Xy \end{pmatrix}(1)$. We conclude the rank of $[\alpha, x, X]$ is equal to the rank of $\begin{pmatrix} (Xy)^* \\ X \end{pmatrix}$ which is equal to the rank of X as $x \in \text{range}(X)$. $\square$

Lemma 3.6. *Let $\mathcal{E}$ be a spatial product system and $Z \subset \mathcal{U}^{\mathcal{E}}$ be a subset of the set of all units in $\mathcal{E}$. Let H_Z be the Arveson Hilbert space associated to Z. Then $dimH_Z = ind\,\mathcal{E}$ if and only if $\overline{span}\{u^1_{t_1} \otimes u^2_{t_2} \otimes ... \otimes u^k_{t_k} : 1 \leq i \leq k, \sum_i t_i = t, u^i \in Z, k \geq 1\} = \mathcal{E}^I_t$.*

Proof. [7, Lemma 23] $\square$

Lemma 3.7. *Let γ be the covariance kernel on the set of all units in a spatial product system $\mathcal{E}$. Let $Z \subset \mathcal{U}^{\mathcal{E}}$ be such that $\overline{span}\{u^1_{t_1} \otimes u^2_{t_2} \otimes ... \otimes u^k_{t_k} : 1 \leq i \leq k, \sum_i t_i = t, u^i \in Z, k \geq 1\} = \mathcal{E}^I_t$. Suppose there is a function $a : Z \to \mathbb{C}$ such that the function $L : Z \times Z \to \mathbb{C}$ defined by $L(x, y) = \gamma(x, y) - \overline{a(x)} - a(y)$, $x, y \in \mathcal{U}^{\mathcal{E}}$ is positive definite, then*

$$rank\ L := maximum\ rank\{(L(x_i, x_j))_{n\times n} : x_i, x_j \in Z, n \geq 1\} = ind\ (\mathcal{E})$$

if and only if L is a non-constant function on $Z \times Z$. Further if L is a constant function on $Z \times Z$, then $ind\ (\mathcal{E}) = 0$.

Proof. Without loss of generality we may assume $\mathcal{E}$ to be type I as index $(\mathcal{E}^I) =$ index $(\mathcal{E})$. We then represent our product system on Fock space. Then Z can be identified with a subset of $\mathbb{C} \times H_{\mathcal{U}^{\mathcal{E}}}$ and $\gamma((\lambda, x), (\alpha, y)) = \overline{\lambda} + \alpha + \langle x, y\rangle$. Set $A = \{x \in H_{\mathcal{U}^{\mathcal{E}}} : (\lambda, x) \in Z\}$. Fix $x_0 \in A$. Now define $\phi : \mathbb{C}_0 Z := \{f : Z \to \mathbb{C} : f$ is zero all but finitely many points $, \sum_{u\in Z} f(u) = 0\} \to \overline{\text{span}}(A - x_0)$ by $\phi(f) = \sum_{\lambda,x} f(\lambda, x)(x - x_0)$. Then ϕ induces a unitary from H_Z onto $\overline{\text{span}}\,(A - x_0)$. This implies dim $H_Z =$ dim $\overline{\text{span}}(A - x_0)$. Now from Lemma 3.6, we get ind $(\mathcal{E}) =$ dim $\overline{\text{span}}\ (A - x_0)$. Hence dim $H_{\mathcal{U}^{\mathcal{E}}} =$ dim $\overline{\text{span}}\ (A - x_0)$. From positivity of the function L we get that $L((\lambda, x), (\alpha, y)) = b + \langle x, y\rangle$ for some $b \geq 0$. Now L is a constant function if and only if $A = \{y_0\}$ for some $y_0 \in H_{\mathcal{U}^{\mathcal{E}}}$. Consequently ind $(\mathcal{E}) =$ dim $\overline{\text{span}}\ (A - y_0) = 0$. On the other hand if L is not a constant function, then rank $(L) =$ dim $\overline{\text{span}}\ (A)$. Hence rank $(L) =$ ind $(\mathcal{E})$. $\square$

Theorem 3.8. *Suppose $\mathcal{E}$ and $\mathcal{F}$ are two spatial Arveson product systems of index k_1 and k_2 respectively. Let $D : \mathcal{F} \to \mathcal{E}$ be a contractive morphism such that $\mathcal{E} \otimes_D \mathcal{F}$ is an Arveson product system. Then $D|_{\mathcal{F}^I} : \mathcal{F}^I \to \mathcal{E}^I$ is a contractive morphism. So they can be represented as $\mathcal{E}^I_t = \Gamma_{sym}(L^2[0, t], K_1)$ and $\mathcal{F}^I_t = \Gamma_{sym}(L^2[0, t], K_2)$. Then $D_t|_{\mathcal{F}^I_t} = [q, x, y, A]_t$ for some $(q, x, y, A) \in \mathcal{C}(K_2, K_1)$ and*

$$ind\ (\mathcal{E} \otimes_D \mathcal{F}) = \begin{cases} \infty & \text{if } k_1 \text{ or } k_2 \text{ is } \infty \\ k_1 + k_2 - N(I - A^*A) & \text{if } q + \overline{q} - \|y\|^2 = \langle x + A^*y, a\rangle \\ & \text{where } (I - A^*A)a = x + A^*y \\ k_1 + k_2 - N(I - A^*A) + 1 & \text{otherwise} \end{cases}$$

Proof. From the universality Theorem 2.7, we may assume that $\mathcal{E}$ and $\mathcal{F}$ are product subsystems of $\mathcal{E} \otimes_D \mathcal{F}$. Now from Theorem 2.9, we know that every unit in the amalgamated product can be generated by units of $\mathcal{E}$ and of $\mathcal{F}$. So from Lemma 3.6, to compute the index of the amalgamated product, it is enough to compute the index on the set $\mathcal{U}^{\mathcal{E}} \bigcup \mathcal{U}^{\mathcal{F}}$. For $u, u' \in \mathcal{U}^{\mathcal{E}}$ and $v, v' \in \mathcal{U}^{\mathcal{F}}$,

$$\gamma(u, u') = \overline{\lambda(u)} + \lambda(u') + \langle \mu(u), \mu(u') \rangle$$

$$\gamma(v, v') = \overline{\lambda(v)} + \lambda(v') + \langle \mu(v), \mu(v') \rangle$$

Now

$$\begin{aligned} \langle u_t, v_t \rangle_{(\mathcal{E}\otimes_D \mathcal{F})_t} &= \langle u_t, D_t v_t \rangle_{\mathcal{E}_t} \\ &= \langle e^{\lambda(u)t} e(\mu(u)\chi|_{t]}), e^{-qt+\lambda(v)t+\langle x,\mu(v)\rangle t} e((y + A\mu(v))\chi|_{t]}) \rangle \\ &= e^{-qt+(\overline{\lambda(u)}+\lambda(v))t+\langle x,\mu(v)\rangle t+\langle \mu(u),y+A\mu(v)\rangle t} \end{aligned}$$

Hence

$$\gamma(u, v) = \overline{\lambda(u)} + \lambda(v) - q + \langle x, \mu(v) \rangle + \langle \mu(u), y + A\mu(v) \rangle$$

Now take $Y = \mathcal{U}^{\mathcal{E}} \cup \mathcal{U}^{\mathcal{F}}$ and define $a : Y \to \mathbb{C}$ by

$$a(u) = \lambda(u) + \langle y, \mu(u) - \frac{1}{2} y \rangle \text{ if } u \in \mathcal{U}^{\mathcal{E}}.$$

and

$$a(v) = \lambda(v) - q + \langle x, \mu(v) \rangle + \langle y, \frac{1}{2} y + A\mu(v) \rangle \text{ if } v \in \mathcal{U}^{\mathcal{F}}.$$

We will show that $a : Y \to \mathbb{C}$ is well defined. Fix $t > 0$. For $x \in \mathcal{E}_t$ and $y \in \mathcal{F}_t$, we have $\|x-y\|^2_{(\mathcal{E}\otimes_D\mathcal{F})_t} = \|x-D_t y\|^2 + \|y\|^2 - \|D_t y\|^2 \geq \|x-D_t y\|^2$. From this it follows that x and y are identified in $(\mathcal{E} \otimes_D \mathcal{F})_t$ if and only if $x = D_t y$ and $\|y\| = \|D_t y\|$. So $w \in \mathcal{U}^{\mathcal{E}} \cap \mathcal{U}^{\mathcal{F}}$ if and only if there are $u \in \mathcal{U}^{\mathcal{E}}$ and $v \in \mathcal{U}^{\mathcal{F}}$ such that for every $t > 0$, $u_t = D_t v_t$ and $\|v_t\| = \|D_t v_t\|$. Equating $\lambda(u) = \lambda(Dv)$ and $\mu(u) = \mu(Dv)$, we get $\lambda(u) = -q + \langle x, \mu(v) \rangle + \lambda(v)$ and $\mu(u) = y + A\mu(v)$. Plugging those values, we get $a(u) = \lambda(u) + \langle y, \mu(u) - \frac{1}{2} y \rangle = -q + \langle x, \mu(v) \rangle + \lambda(v) + \langle y, y + A\mu(v) - \frac{1}{2} y \rangle = \lambda(v) - q + \langle x, \mu(v) \rangle + \langle y, \frac{1}{2} y + A\mu(v) \rangle = a(v)$. This shows that $a : Y \to \mathbb{C}$ is well defined. Now define $L : Y \times Y \to \mathbb{C}$ by

$$L(z, w) = \gamma(z, w) - \overline{a(z)} - a(w).$$

Then by direct computation: For $u, u' \in \mathcal{U}^{\mathcal{E}}$ and $v, v' \in \mathcal{U}^{\mathcal{F}}$,

$$L(u, u') = \langle \mu(u) - y, \mu(u') - y \rangle.$$

$$L(u, v) = \langle \mu(u) - y, A\mu(v) \rangle.$$

$$L(v, v') = q + \bar{q} - \|y\|^2 - \langle x + A^* y, \mu(v') \rangle - \langle \mu(v), x + A^* y \rangle + \langle \mu(v), \mu(v') \rangle.$$

If one of the product system has infinite index then it easily follows that the amalgamated product has infinite index. So let us assume that ind $\mathcal{E} < \infty$, ind $\mathcal{F} < \infty$.

We claim that $L : Y \times Y \to \mathbb{C}$ is a positive kernel and rank L = rank C. Set $\tilde{K} = K_1 \oplus \mathbb{C} \oplus K_2$. Define $C : \tilde{K} \to \tilde{K}$ by

$$C = \begin{pmatrix} I_{K_1} & 0 & A \\ 0 & q+\bar{q}-\|y\|^2 & -(x+A^*y)^* \\ A^* & -(x+A^*y) & I_{K_2} \end{pmatrix}.$$

Now

$$C = \begin{pmatrix} I & 0 & A \\ 0 & 0 & 0 \\ A^* & 0 & A^*A \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & q+\bar{q}-\|y\|^2 & -(A^*y+x)^* \\ 0 & -(A^*y+x) & I-A^*A \end{pmatrix}.$$

It follows from Equation (3.3) that C is positive. For any finite choices of $\{u_1, u_2, \cdots, u_l\} \in \mathcal{U}^{\mathcal{E}}$ and $\{v_1, v_2, \cdots, v_m\} \in \mathcal{U}^{\mathcal{F}}$, set $a_i = \mu(u_i) - y \oplus 0 \oplus 0 \in \tilde{K}$ for $i = 1, 2, \cdots, l$ and $a_j = 0 \oplus 1 \oplus \mu(v_j) \in \tilde{K}$ for $j = l+1, l+2, \cdots, l+m$. Then

$$\begin{pmatrix} ((L(u_i,u_j)))_{l\times l} & ((L(u_i,v_j)))_{l\times m} \\ ((L(v_j,u_i)))_{m\times l} & ((L(v_i,v_j)))_{m\times m} \end{pmatrix}_{(l+m)\times(l+m)} = ((\langle a_i, Ca_j\rangle))_{(l+m)\times(l+m)}.$$

This implies L is a positive definite kernel and rank $(L) \leq$ rank (C). Choose $u_i \in \mathcal{U}^{\mathcal{E}}$, $1 \leq i \leq l$ and $v_j \in \mathcal{U}^{\mathcal{F}}$, $1 \leq j \leq m+1$ such that $\{\mu(u_1) - y, \mu(u_2) - y, \cdots, \mu(u_l) - y\}$ is a basis of K_1 and $\{\mu(v_2), \mu(v_3), \cdots, \mu(v_{m+1})\}$ is a basis of K_2 and choose v_1 such that $\mu(v_1) = 0$. Let a_i, $1 \leq i \leq l+m+1$ be defined as above. Then $\{a_1, a_2, \cdots, a_{l+m+1}\}$ is a basis of $\tilde{K}$. So we have

$$\begin{aligned} &\text{rank } (L) \geq \\ \text{rank} &\begin{pmatrix} ((L(u_i,u_j)))_{l\times l} & ((L(u_i,v_j)))_{l\times(m+1)} \\ ((L(v_j,u_i)))_{(m+1)\times l} & ((L(v_i,v_j)))_{(m+1)\times(m+1)} \end{pmatrix}_{(l+m+1)\times(l+m+1)} \\ &= \text{rank } (C) \end{aligned}.$$

So the claim is proved. Observe that L is constant implies $K_1 = \{0\}$, $K_2 = \{0\}, q+\bar{q} = 0$. So $x = 0$, $y = 0$, $A = 0$, $\mu(u) = 0$ for all $u \in \mathcal{U}^{\mathcal{E}}$, $\mu(v) = 0$ for all $v \in \mathcal{U}^{\mathcal{F}}$. Hence L is identically zero. So $\gamma(x, y) = \overline{a(x)} + a(y)$. Consequently, $H_{\mathcal{U}^{\mathcal{E}\otimes_D\mathcal{F}}} = \{0\}$. In this special case, $\mathcal{E}$ and $\mathcal{F}$ are two type I_0 product systems and $D = \{D_t = [q, 0, 0, 0]_t\}_{t>0}$ is a contractive morphism. Then ind $(\mathcal{E} \otimes_D \mathcal{F}) = 0$ if and only if $q + \bar{q} = 0$. Now assume L is a non-constant function. Then from Lemma 3.7, it is enough to calculate the rank of L. Now rank of L is same as the rank of the matrix C. Clearly C is conjugate via permutation matrix to the following matrix,

$$C' = \begin{pmatrix} q+\bar{q}-\|y\|^2 & 0 & -(x+A^*y)^* \\ 0 & I_{K_1} & A \\ -(x+A^*y) & A^* & I_{K_2} \end{pmatrix}$$

Invoking Lemma 3.5,

$$\text{rank } (C') = \begin{cases} \text{rank } (Z) & \text{if } 0 \oplus -(x+A^*y) \in \text{range } (Z) \text{ and} \\ & q+\bar{q}-\|y\|^2 = \langle 0 \oplus -(x+A^*y), e \oplus f\rangle, \\ & \text{where } Z(e \oplus f) = 0 \oplus -(x+A^*y) \\ \text{rank } (Z)+1 & \text{otherwise} \end{cases} \tag{3.4}$$

where $Z = \begin{pmatrix} I_{K_1} & A \\ A^* & I_{K_2} \end{pmatrix}$. From Lemma 3.4, we know

$$\text{rank } (Z) \;=\; k_1 + k_2 - N(I - A^*A). \tag{3.5}$$

Solving the equation, $\begin{pmatrix} I_{K_1} & A \\ A^* & I_{K_2} \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ x + A^*y \end{pmatrix}$, we get $Af = -e$ and $A^*e + f = -(x + A^*y)$, which imply $(I - A^*A)(-f) = x + A^*y$. Now this holds as $\{q, x, y, A\} \in \mathcal{C}(K_2, K_1)$ and range $(I - A^*A)^{1/2} =$ range $(I - A^*A)$. Now it follows from equations (3.4) and (3.5) that rank $(L) =$ rank $(C) =$ rank $(C') = k_1 + k_2 - N(I - A^*A)$ if $q + \overline{q} - \|y\|^2 = \langle x + A^*y, -f\rangle$ where $(I - A^*A)(-f) = x + A^*y$. Setting $a = -f$ our theorem follows. □

The following corollary has been proved in [7], Theorem 24, which answers that the index of the amalgamated product through strictly contractive units is one more than that of through normalized units.

Corollary 3.9. *Suppose $\mathcal{E}$ and $\mathcal{F}$ are two spatial product systems of index k_1 and k_2 respectively. Let u^0 and v^0 be two units of $\mathcal{E}$ and $\mathcal{F}$ respectively such that $\|u_t^0\|, \|v_t^0\| \le 1$ for all $t > 0$. Set $D_t = |u_t^0\rangle\langle v_t^0|$. Then $D_t : \mathcal{F}_t \to \mathcal{E}_t$ is a contractive morphism and*

$$\mathit{ind}\ (\mathcal{E} \otimes_D \mathcal{F}) = \begin{cases} k_1 + k_2 & \text{if } \|u_t\| = \|v_t\| = 1 \text{ for all } t > 0 \\ k_1 + k_2 + 1 & \text{otherwise} \end{cases}$$

Proof. Let $(\lambda, y) \in \mathbb{C} \times K_1$ and $(\mu, x) \in \mathbb{C} \times K_2$ be the parametrization of u^0 and v^0 respectively. Then D_t is given by $D_t = [-\lambda - \overline{\mu}, x, y, 0]$. Now $\|u_t^0\| = 1$ implies $\lambda + \bar{\lambda} + \|y\|^2 = 0$. Similarly $\|v_t^0\| = 1$ implies $\mu + \bar{\mu} + \|x\|^2 = 0$. So by the previous theorem, ind $(\mathcal{E} \otimes_D \mathcal{F}) = k_1 + k_2$. □

Corollary 3.10. *Suppose $\mathcal{E}$ and $\mathcal{F}$ are two spatial product systems of index k_1 and k_2 respectively. Suppose $D : \mathcal{F} \to \mathcal{E}$ is a morphism of partial isometries. Then*

$$\mathit{ind}\ (\mathcal{E} \otimes_D \mathcal{F}) = k_1 + k_2 - N(I - A^*A)$$

Proof. As DD^* and D^*D are projection morphisms, it follows that $(I - A^*A)$ is a projection and $q + \bar{q} - \|y\|^2 = \|x + A^*y\|^2$. So $(I - A^*A)(x + A^*y) = (x + A^*y)$. So the Corollary follows. □

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