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A GREGUS TYPE COMMON FIXED POINT THEOREM IN NORMED SPACES WITH APPLICATION

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Abstract. In this paper, we introduce the notion of ϕ-weakly compatible mapping for a pair of mappings. A fixed point theorem for two pairs of ϕ-weakly compatible mappings satisfying a rational type contraction in a normed space is also established. Subsequently we use our result to find existence of solutions of variational inequalities.

1. Introduction

Let $(X, \|\cdot\|)$ denote a normed linear space and $\mathbb{N}$ the set of positive integers. For self mappings T and I on X, we recall the following :

Sessa [10] defined T and I to be *weakly commuting* if $\|TIx - ITx\| \leq \|Tx - Ix\|$ for any $x \in X$.

Jungck [6] defined T and I to be *compatible mappings* if $\lim_{n\to\infty} \|TIx_n - ITx_n\| = 0$, whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = t$, for some $t \in X$.

Diviccaro et al. [4] established the a Gregus type common fixed point theorem for pair of weakly commuting mappings. Pathak and George [7] relaxed certain conditions on one of the pair of mapping and replaced the weakly commuting mappings with compatible mappings and presented the following theorem.

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Theorem 1.1. *Let T and I be compatible mappings of a closed convex bounded subset C of a normed linear space X satisfying the following:*

$$\begin{aligned} \|Tx - Ty\|^p &\leq a\|Ix - Iy\|^p + (1-a)\max\{\|Tx - Ix\|^p, \|Ty - Iy\|^p\} \\ I(C) &\supseteq (1-k)I(C) + kT(C) \end{aligned}$$

for all $x, y \in C$, where $0 < a < 1, p > 0$ and $0 < k < 1$. If, for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined by

for all $x, y \in C$, where $0 < a < 1, p > 0$ and $0 < k < 1$. If, for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined by

$$Ix_{n+1} = (1-k)Ix_n + kTx_n, n \in \mathbb{N} \cup \{0\}$$

converges to a point $z \in C$, and if I is continuous at z, then T and I have a unique common fixed point in C. Further, if I is continuous at Tz, then T and I have a unique common fixed point at which T is continuous.

Pathak et al. [8] introduced the concept of compatible mappings of type (T) (type (I)) in normed spaces and showed that these mappings are equivalent to compatible mappings under some conditions. They also proved a common fixed point theorem of Gregus type and applied this theorem to prove the existence of solution of variational inequalities.

Recently, Pathak et al. [9] proved the following:

Theorem 1.2. *Let $\{S, I\}$ and $\{T, J\}$ be two pairs of coincidentally commuting mappings of a normed space X into itself such that there exists a closed, convex subset C of X that is invariant under I, J, S and T, where I and J are one-to-one and the following conditions hold:*

$$\|Sx - Ty\|^p \leq a\|Ix - Jy\|^p + (1-a)\max\{\|Sx - Ix\|^p, \|Ty - Jy\|^p\}$$

for all $x, y \in C$, where $0 < a < 1, p > 0$ and

$$I(C) \supseteq (1-k)I(C) + kS(C), \qquad J(C) \supseteq (1-k^*)I(C) + k^*S(C)$$

for all $k, k^* \in (0,1)$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined inductively by

$$\begin{aligned} Ix_{2n+1} &= (1-a_{2n})Ix_{2n} + a_{2n}Sx_{2n}, \\ Jx_{2n+2} &= (1-a_{2n+1})Jx_{2n} + a_{2n+1}Tx_{2n+1}, n \in \mathbb{N} \cup \{0\} \end{aligned}$$

with $a_0 = 1, 0 < a_n \leq 1$ for all $n > 0$ and $\lim inf a_n > 0$, converges to a point z of C, then I, J, S and T have a unique common fixed point Tz in C. Further, if I and J are continuous at Tz, then I, J, S and T have a unique common fixed point at which S and T are continuous.

The main object of this paper is to introduce ϕ-weakly compatible mappings for pair of mappings and prove a Gregus type common fixed point theorem for two pairs of such mappings. Subsequently we use our result to find iterative solution of certain variational inequalities.

2. Main Result

In this section, we introduce the notion of ϕ-weakly compatible mappings of type (I, T). Further we prove a common fixed point theorem for ϕ-weakly compatible mappings of type (I, T) for the pair of mappings $\{I, T\}$ in a normed space.

Definition 2.1. Let I and T be mappings from a normed space X into itself. The pair of mappings $\{I, T\}$ is said to be ϕ- *weakly compatible of type (I,T)* at x in X, if for every $p > 0$, $Ix = Tx$ implies

$$\phi\Big(\frac{\|ITx - Ix\|^{p+1} + \|ITx - TIx\|^{p+1}}{\|ITx - Ix\| + \|TIx - ITx\|}\Big) \;\; \leq \;\; \|TIx - Tx\|^p,$$

where $\phi : [0, \infty) \to [0, \infty)$ is upper semi-continuous, non-decreasing and $\phi(t) < t$ for all $t > 0$, and $\|ITx - Ix\| + \|TIx - Tx\| \neq 0$.

If $\phi(t) = h\ t$, where $0 < h < 1$, then the pair of mappings $\{I, T\}$ is said to be $h -$ *weakly compatible of type (I,T).*

Example 2.2. Let $X = [0, \infty)$ with the Euclidean norm $\|\cdot\|$ and $\phi(t) = \frac{1}{2}t$. Define mappings I and T on X by $Ix = 1+x$ and $Tx = 1+2x$. Here $\|I0 - T0\| = 0$ and for this value of x, we have $\|IT0 - I0\| = 1$, $\|IT0 - TI0\| = 1$ and $\|TI0 - T0\| = 2$. Thus for all $p > 0$,

$$\phi\Big(\frac{\|IT0 - I0\|^{p+1} + \|IT0 - TI0\|^{p+1}}{\|IT0 - I0\| + \|TI0 - IT0\|}\Big) \;\; \leq \;\; \|TI0 - T0\|^p,$$

or

$$\frac{1}{2}\frac{1^{p+1} + 1^{p+1}}{1+1} \;\; \leq \;\; 2^p,$$

or

$$1 \;\; \leq \;\; 2^{p+1}.$$

Thus the pair mappings $\{I, T\}$ is ϕ-weakly compatible of type (I, T) at 0. On the other hand

$$\phi\Big(\frac{\|TI0 - T0\|^{p+1} + \|IT0 - TI0\|^{p+1}}{\|IT0 - I0\| + \|TI0 - IT0\|}\Big) \;\; \leq \;\; \|IT0 - I0\|^p,$$

implies

$$\frac{1}{2}\frac{2^{p+1} + 1^{p+1}}{1+1} \;\; \leq \;\; 1^p,$$

or

$$\frac{2^{p+1} + 1}{2^2} \;\; \leq \;\; 1,$$

which does not hold for $p > 1$. This shows that the pair of ϕ-weakly compatible type (I, T) is not necessarily ϕ-weakly compatible mappings of type (T, I).

Also, here for $I0 = T0 = 1$, $IT0 = I1 = 2$, $TI0 = S1 = 3$, so it is not satisfied that $ITx = TIx$ for all $x \in X$. Thus the pair of mappings $\{I, T\}$ is neither weakly compatible nor commutative nor weakly commuting.

Now we prove a Gregus type common fixed point theorem for pair of ϕ-weakly compatible mappings of type (I,S) and (J,T) in normed space.

Theorem 2.3. *Let I, J, S and T mappings from a normed linear space X into itself and C be a closed, convex bounded subset of X that is invariant under I, J, S and T, where I and J are one-to-one and the following conditions hold:*

$$\begin{aligned} &\|Sx-Ty\|^p \\ \leq\ & \phi\Big(\frac{a\|Ix-Jy\|^{p+1}+(1-a)\max\{\|Ix-Sx\|^{p+1},\|Jy-Ty\|^{p+1}\}}{\|Ix-Sx\|+\|Ix-Jy\|+\|Jy-Ty\|}\Big), \end{aligned} \tag{2.1}$$

where $\|Ix-Sx\|+\|Ix-Jy\|+\|Jy-Ty\| \neq 0$.

$$I(C) \supseteq (1-k)I(C)+kS(C), \qquad J(C) \supseteq (1-k')J(C)+k'T(C)$$

for all $x, y \in C$, where $0<a<1, p>0$ and $k, k' \in (0,1)$. Suppose that for some $x_0 \in C$ and $y_0 \in C$, the sequences $\{x_n\}$ and $\{y_n\}$ in C defined inductively by

$$Ix_{2n+1} = (1-a_n)Ix_n + a_nSx_n, n \in \mathbb{N}\cup\{0\} \tag{2.2}$$

$$Jy_{n+1} = (1-a_n)Jy_n + a_nTy_n, n \in \mathbb{N}\cup\{0\} \tag{2.3}$$

where $a_0 = 1, 0 < a_n \leq 1$ for all $n > 0$ and $\lim inf a_n > 0$, converge to a point $z \in C$. If $\{I,S\}$ and $\{J,T\}$ are ϕ-weakly compatible mappings of type (I,S) and type (J,T) at z, respectively and I and J are continuous at z, then S, T, I and J have a unique common fixed point Tz in C. Further, if I and J are continuous at Tz, then S, T, I and J have a unique common fixed point at which S and T are continuous.

Proof. Since the mapping I and J are one-to-one so, the sequence $\{x_n\}$ defined in (2.2) and (2.3) are well defined. Indeed it follows from (2.2) that

$$a_n(Sx_n - Ix_n) = Ix_{n+1} - Ix_n.$$

Define $\alpha = \liminf a_n$. Then there exists a positive integer N such that $n \geq N$ implies that $a_n > \frac{\alpha}{2}$. Thus from (2.2), for $n \geq N$

$$\|Sx_n - Ix_n\| \leq |\frac{2}{\alpha}|\|Ix_{2n+1} - Ix_n\|,$$

since I is continuous at z, we have $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ix_n = Iz$.
Now using (2.3), we have

$$a_nTy_n = Jy_{n+1} - (1-a_n)Jy_n$$

Again, since J is continuous at z, so taking $n \to \infty$, we have

$$\lim_{n\to\infty} Jy_n = \lim_{n\to\infty} Ty_n = Jz.$$

Let $Iz \neq Jz$. Putting $x = x_n$ and $y = y_n$ $(n \in \mathbb{N}\cup\{0\})$ in (2.1), we have

$$\begin{aligned} &\|Sx_n - Ty_n\|^p \\ &\leq \phi\Big(\frac{a\|Ix_n-Jy_n\|^{p+1}+(1-a)\max\{\|Ix_n-Sx_n\|^{p+1},\|Jy_n-Ty_n\|^{p+1}\}}{\|Ix_n-Sx_n\|+\|Ix_n-Jy_n\|+\|Jy_n-Ty_n\|}\Big) \end{aligned}$$

or

$$\begin{aligned}&\|Sx_n - Ty_n\|^p\\ &< \frac{a\|Ix_n - Jy_n\|^{p+1} + (1-a)\max\{\|Ix_n - Sx_n\|^{p+1}, \|Jy_n - Ty_n\|^{p+1}\}}{\|Ix_n - Sx_n\| + \|Ix_n - Jy_n\| + \|Jy_n - Ty_n\|}\end{aligned}$$

or

$$\begin{aligned}&\|Sx_n - Ty_n\|^p\big[\|Ix_n - Sx_n\| + \|Ix_n - Jy_n\| + \|Jy_n - Ty_n\|\big]\\ &< a\|Ix_n - Jy_n\|^{p+1} + (1-a)\max\{\|Ix_n - Sx_n\|^{p+1}, \|Jy_n - Ty_n\|^{p+1}\}.\end{aligned}$$

We obtain, on taking $\liminf$ on left side and $\limsup$ on the other, in above

$$\|Iz - Jz\|^p \leq \frac{a\|Iz - Jz\|^{p+1}}{\|Iz - Jz\|} = a\|Iz - Jz\|^p < a\|Iz - Jz\|^p,$$

which is a contradiction. Hence $Iz = Jz$.

Let $Iz \neq Tz$, putting $x = x_n$ and $y = z$ in (2.1), we have

$$\begin{aligned}&\|Sx_n - Tz\|^p\\ &\leq \phi\Big(\frac{a\|Ix_n - Jz\|^{p+1} + (1-a)\max\{\|Ix_n - Sx_n\|^{p+1}, \|Jz - Tz\|^{p+1}\}}{\|Ix_n - Sx_n\| + \|Ix_n - Jz\| + \|Jz - Tz\|}\Big)\end{aligned}$$

or

$$\begin{aligned}&\|Sx_n - Tz\|^p\\ &< \frac{a\|Ix_n - Jz\|^{p+1} + (1-a)\max\{\|Ix_n - Sx_n\|^{p+1}, \|Jz - Tz\|^{p+1}\}}{\|Ix_n - Sx_n\| + \|Ix_n - Jz\| + \|Jz - Tz\|}\end{aligned}$$

or

$$\begin{aligned}\|Sx_n - Tz\|^p\big[\|Ix_n - Sx_n\| + \|Ix_n - Jz\| + \|Jz - Tz\|\big] &< a\|Ix_n - Jz\|^{p+1}\\ &+(1-a)\max\{\|Ix_n - Sx_n\|^{p+1}, \|Jz - Tz\|^{p+1}\}.\end{aligned}$$

We obtain, on taking $\liminf$ on left side and $\limsup$ on the other, in above

$$\begin{aligned}\|Iz - Tz\|^p &\leq \frac{(1-a)\|Iz - Tz\|^{p+1}}{\|Iz - Tz\|}\\ &< (1-a)\|Iz - Tz\|^p,\end{aligned}$$

which is a contradiction again. Hence $Iz = Tz = Jz$.

Similarly, on putting $x = z$, $y = y_n$ and taking lim in similar manner, we have

$$\begin{aligned}\|Iz - Sz\|^p &\leq \frac{(1-a)\|Sz - Iz\|^{p+1}}{\|Sz - Iz\|}\\ &< (1-a)\|Iz - Sz\|^p,\end{aligned}$$

a contradiction, so $Iz = Sz$. Thus we have

$$Sz = Tz = Iz = Jz. \tag{2.4}$$

Now from (2.1) and (2.4), we have

$$\begin{aligned}&\|SSz - Tz\|^p\\ &\leq \phi\Big(\frac{a\|ISz - Jz\|^{p+1} + (1-a)\max\{\|SSz - ISz\|^{p+1}, \|Tz - Jz\|^{p+1}\}}{\|ISz - SSz\| + \|ISz - Jz\| + \|Jz - Tz\|}\Big)\end{aligned}$$

or

$$\|SIz - Sz\|^p \le \phi\Big(\frac{b\,\big[\|ISz - Iz\|^{p+1} + \|SIz - ISz\|^{p+1}\big]}{\|ISz - SIz\| + \|ISz - Iz\|}\Big),$$

where $b = \max\{a, 1-a\} < 1$. Since I and S are ϕ-weakly compatible of type (I, S) at z, it follows that

$$\|SIz - Sz\|^p < \|SIz - Sz\|^p$$

or

$$\|STz - Tz\|^p < \|STz - Tz\|^p,$$

which implies that $\|STz - Tz\|^p = 0$ and so $STz = Tz$; i.e. $SIz = Iz$. Again using ϕ-weakly compatibility of type (I, S) at z,

$$\phi\Big(\frac{\|ISz - Iz\|^{p+1} + \|SIz - ISz\|^{p+1}}{\|ISz - SIz\| + \|ISz - Iz\|}\Big) \le \|SIz - Sz\|^p = 0,$$

which implies that

$$SIz = ISz = ITz = STz = Tz$$

and so Tz is a common fixed point of I and S.
Similarly interchanging the roles of pairs I, S and J, T, we get

$$TTz = JTz = Tz$$

and thus Tz is also a common fixed point of J and T. Therefore, Tz is a common fixed point of I, J, S and T.

Next, let $\{v_n\}$ be an arbitrary sequence in C with the limit $Tz = z_1$. Then, using (2.1), we have

$$\begin{aligned}&\|Sv_n - Tz_1\|^p\\ &\le \phi\Big(\frac{a\|Iv_n - Jz_1\|^{p+1} + (1-a)\max\{\|Sv_n - Iv_n\|^{p+1}, \|Tz_1 - Jz_1\|^{p+1}\}}{\|Iv_n - Jz_1\| + \|Sv_n - Iv_n\| + \|Tz_1 - Jz_1\|}\Big).\end{aligned}$$

Since I and J are continuous at $Tz = z_1$, we have

$$\|Sv_n - Sz_1\|^p = \|Sv_n - Tz_1\|^p \le (1-a)\|Sv_n - Iz_1\|^p + \epsilon$$

for sufficiently large n and $\epsilon > 0$. Thus $\lim_{n\to\infty} Sv_n = Sz_1$, implies that S is continuous at Tz. Similarly, we have

$$\|Tz_1 - Tv_n\|^p = \|Sz_1 - Tv_n\|^p \le (1-a)\|Tv_n - Jz_1\|^p + \epsilon$$

for sufficiently large n and $\epsilon > 0$. Thus $\lim_{n\to\infty} Tv_n = Tz_1$, and so T is also continuous at Tz.

For uniqueness of fixed point, let u and w be distinct fixed points of S, T, I and J. Then using (2.1), we have

$$\begin{aligned}&\|Su - Tw\|^p\\ \le\ &\phi\Big(\frac{a\|Iu - Jw\|^{p+1} + (1-a)\max\{\|Su - Iu\|^{p+1}, \|Tw - Jw\|^{p+1}\}}{\|Iu - Su\| + \|Iu - Jw\| + \|Jw - Tw\|}\Big)\\ =\ &\phi\Big(\frac{a\|Iu - Jw\|^{p+1}}{\|Iu - Jw\|}\Big),\end{aligned}$$

or

$$\|u-w\|^p < a\|u-w\|^p,$$

a contradiction. Hence $u = w$ is the unique common fixed point of S, T, I and J. This completes the proof. $\square$

If we put $\phi(t) = h\ t\ (0 < h < 1)$ and $p = 1$ in Theorem 2.3, we have the following:

Corollary 2.4. *Let I, J, S and T mappings from a normed linear space X into itself and C be a closed, convex bounded subset of X that is invariant under I, J, S and T, where I and J are one-to-one and the following conditions hold:*

$$\|Sx-Ty\| \leq h\frac{a\|Ix-Jy\|^2 + (1-a)\max\{\|Ix-Sx\|^2, \|Jy-Ty\|^2\}}{\|Ix-Sx\| + \|Ix-Jy\| + \|Jy-Ty\|}\Big)$$

whenever $\|Ix-Sx\| + \|Ix-Jy\| + \|Jy-Ty\| \neq 0$.

$$I(C) \supseteq (1-k)I(C) + kS(C), J(C) \supseteq (1-k')J(C) + k'T(C)$$

for all $x, y \in C$, where $0 < a < 1, p > 0$ and $k, k' \in (0,1)$. Suppose that for some $x_0 \in C$ and $y_0 \in C$, the sequences $\{x_n\}$ and $\{y_n\}$ in C defined inductively by

$$\begin{aligned} Ix_{n+1} &= (1-a_n)Ix_n + a_nSx_n, n \in \mathbb{N}\cup\{0\} \\ Jy_{n+1} &= (1-a_n)Jy_n + a_nTyx_n, n \in \mathbb{N}\cup\{0\} \end{aligned}$$

where $a_0 = 1, 0 < a_n \leq 1$ for all $n > 0$ and $\lim inf\ a_n > 0$, converge to a point $z \in C$. If $\{I, S\}$ and $\{J, T\}$ are ϕ-weakly compatible mappings of type (I, S) and type (J, T) at z, respectively and I and J are continuous at z, then S, T, I and J have a unique common fixed point Tz in C. Further, if I and J are continuous at Tz, then S, T, I and J have a unique common fixed point at which S and T are continuous.

If we put $S = T$ and $I = J$ in Theorem 2.3, we have the following:

Corollary 2.5. *Let I and T be mappings from a normed linear space X into itself and C be a closed, convex bounded subset of X that is invariant under I and T, where the following conditions are satisfied :*

$$\|Tx-Ty\|^p \leq \phi\Big(\frac{a\|Ix-Iy\|^{p+1} + (1-a)\max\{\|Ix-Tx\|^{p+1}, \|Iy-Ty\|^{p+1}\}}{\|Ix-Tx\| + \|Ix-Iy\| + \|Iy-Ty\|}\Big)$$

whenever $\|Ix-Tx\| + \|Ix-Iy\| + \|Iy-Ty\| \neq 0$.

$$I(C) \supseteq (1-k)I(C) + kT(C)$$

for all $x, y \in C$, where $0 < a < 1, p > 0$ and $k \in (0,1)$. Suppose that for some $x_0 \in C$, the sequence $\{x_n\}$ in C defined by

$$Ix_{n+1} = (1-a_n)Ix_n + a_nTx_n, n \in \mathbb{N}\cup\{0\}$$

where $a_0 = 1, 0 < a_n \leq 1$ for all $n > 0$ and $\lim inf a_n > 0$, converge to a point $z \in C$. If I and T are ϕ-weakly compatible mappings of type (I, T) at z and I is continuous at z, then T and I have a unique common fixed point in C. Further, if I is continuous at Tz, then T and I have a unique common fixed point at which T is continuous.

For $I = I_X$ (the identity mapping on X) in above corollary, we have the following:

Corollary 2.6. *Let (T) be a mapping from normed linear space X into itself and C be a closed, convex bounded subset of X that is invariant under T and satisfies the following conditions:*

$$\|Tx - Ty\|^p \leq \phi\Big(\frac{a\|x-y\|^{p+1} + (1-a)\max\{\|x-Tx\|^{p+1}, \|y-Ty\|^{p+1}\}}{\|x-Tx\| + \|x-y\| + \|y-Ty\|}\Big)$$

whenever $\|x - Tx\| + \|x - y\| + \|y - Ty\| \neq 0$.

$$C \supseteq (1-k)C + kT(C)$$

for all $x, y \in C$, where $0 < a < 1, p > 0$ and $k \in (0,1)$. Suppose that for some $x_0 \in C$, the sequence $\{x_n\}$ in C defined by

$$x_{n+1} = (1-a_n)x_n + a_n T x_n, n \in \mathbb{N} \cup \{0\}$$

where $a_0 = 1, 0 < a_n \leq 1$ for all $n > 0$ and $\lim inf a_n > 0$, converge to a point $z \in C$. Then T has a unique fixed point at which T is continuous.

The following theorem is an immediate consequence of Theorem 2.3.

Theorem 2.7. *Let I, J and $T_n (n = 1, 2, \ldots)$ be mappings from a normed linear space X into itself and C be a closed, convex bounded subset of X that is invariant under I, J and $T_n (n = 1, 2, \ldots)$. Suppose that the pairs (I, T_{2n-1}) and (J, T_{2n}) are ϕ-weakly compatible mappings of type (I, T_{2n-1}) and type (J, T_{2n}), respectively satisfying the following :*

$$\begin{aligned}
\|T_{2n-1}x - T_{2n}y\|^p &\leq \phi\Big(a\|Ix - Jy\|^{p+1} + (1-a)\max\{\|Ix - T_{2n-1}x\|^{p+1}, \\
&\quad \|Jy - T_{2n}y\|^{p+1}\}[\|Ix - T_{2n-1}x\| + \|Ix - Jy\| \\
&\quad + \|Jy - T_{2n}y\|]^{-1}\Big) \\
I(C) &\supseteq (1-k)I(C) + kT_{2n-1}(C), J(C) \supseteq (1-k')J(C) + k'T_{2n}(C)
\end{aligned}$$

for all $x, y \in C$, $n \in \mathbb{N}$, $0 < a < 1, p > 0$ and $k, k' \in (0,1)$. Suppose that for some $x_0 \in C$ and $y_0 \in C$, the sequences $\{x_{m,n}\}$ and $\{y_{m,n}\}$ in C defined inductively by

$$\begin{aligned}
Ix_{m+1,n} &= (1-a_{m,n})Ix_{m,n} + a_{m,n}T_{2n-1}x_{m,n}, m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N} \\
Jy_{m+1,n} &= (1-a_{m,n})Jy_{m,n} + a_{m,n}T_{2n}y_{m,n}, m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}
\end{aligned}$$

where $a_{0,n} = 1, 0 < a_{m,n} \leq 1$ for each $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and $\lim inf a_{m,n} > 0$ for each $n \in \mathbb{N}$, converge to a point $z \in C$. If I and J are continuous at z, then I, J and $T_n (n = 1, 2, \ldots)$ have a unique common fixed point Iz in C.

3. Application

Variational inequalities arise in optimal stochastic control [2] as well as in other problems in mathematical physics, for example, deformation of elastic bodies steched over solid obstacles, elasto-plastic torsion etc. [5]. The iterative method for solutions of discrete variational inequalities is very suitable for implementation on parallel computers with single instruction, multiple-data architecture, particularly on massively parallel processors. In this section, we apply Corollary 2.4 to show the existence of solution of variational inequalities as in the recent work of Belbas and Mayergoyz [1].

The variational inequality problem is to find a function u such that

$$\begin{cases} \max\{Lu - f, u - \phi\} = 0 \ on \ \Omega, \\ u = 0 \ on \ \Omega. \end{cases}$$

Where Ω is a bounded open convex subset of $\mathbb{R}^N$ with smooth boundary, L is an elliptic operator defined on Ω by

$$L = -a_{ij}(x)\partial^2/\partial x_i \partial x_j + b_i(x)\partial/\partial x_i + c(x)I_N,$$

where summation with respect to repeated indices is implied, $c(x) \geq 0, [a_{ij}(x)]$ is a strictly positive definite matrix, uniformly in x, for $x \in \Omega$, f and ϕ are smooth functions defined in Ω and ϕ satisfies the condition : $\phi(x) \geq 0$ for $x \in \Omega$.
A problem related to above is the two-obstacle variational inequality. Given two functions ϕ and μ defined on Ω such that $\phi \leq \mu$ in Ω, $\phi \leq 0 \leq \mu$ on $\partial\Omega$, the corresponding variational inequality is as follows :

$$\begin{cases} \max\{\min[(Lu - f, u - \phi), u - \mu]\} = 0 \ on \ \Omega, \\ u = 0 \ on \ \Omega. \end{cases}$$

The above problem arises in stochastic game theory.

Let A be an $N \times N$ matrix corresponding to the finite difference discretizations of the operator L. We make the following assumption of the matrix A:

$$A_{ii} = 1, \sum_{j:j \neq i} A_{ij} > -1, A_{ij} < 0 \text{ for } i \neq j. \tag{3.1}$$

These assumptions are related to the definitions of "$M - matrices$"; matrices arising from the finite difference discretizations of continuous elliptic operators will have the property (3.1) under some appropriate conditions and Q denotes the set of all discretized vectors (see[3],[11]).
Let $B = I_N - A$. Then the corresponding property for the matrix B will be

$$B_{ii} = 0, \sum_{j:j \neq i} B_{ij} < 1, B_{ij} > 0 \text{ for } i \neq j. \tag{3.2}$$

Let $q = \max_i \sum_j B_{ij}$ and A^* be an $N \times N$ matrix such that $A^*_{ii} = 1 - q$ and $A^*_{ij} = -q$ for $i \neq j$. Then we have $B^* = I_N - A^*$.

Now, we are ready to show the existence of iterative solutions of variational inequalities :
Consider the following simultaneous discrete variational inequalities mentioned above :

$$\begin{aligned} \max[\min\{A(x - A^* \cdot \|Ix - Sx\| - f, x - A^* \cdot \|Ix - Sx\| - \phi\}, \\ x - A^* \cdot \|Ix - Sx\| - \mu] = 0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \max[\min\{A(x - A^* \cdot \|Jx - Tx\| - f, x - A^* \cdot \|Jx - Tx\| - \phi\}, \\ x - A^* \cdot \|Jx - Tx\| - \mu] = 0, \end{aligned} \tag{3.4}$$

where I and J are one-to-one and S, I and T, J are the pairs of weakly compatible operators of type (S, I) and (T, J), respectively with index 1 at x, where x is implicitly defined in R^N by

$$\begin{gathered}Sx = \min[\max\{BIx + A(1 - B^*) \cdot \|Ix - Sx\| + f,\\ (1 - B^*) \cdot \|Ix - Sx\| + \phi\}, (1 - B^*) \cdot \|Ix - Sx\| + \mu],\end{gathered}$$

$$\begin{gathered}Tx = \min[\max\{BJx + A(1 - B^*) \cdot \|Jx - Tx\| + f,\\ (1 - B^*) \cdot \|Jx - Tx\| + \phi\}, (1 - B^*) \cdot \|Jx - Tx\| + \mu],\end{gathered}$$

for all $x \in Q$. Then (3.3) and (3.4) are equivalent to the common fixed point problem :

$$x = Sx = Tx = Ix = Jx. \tag{3.5}$$

Now assume that $\overline{Q}$ is invariant under I, J, S and T and

$$I(\overline{Q}) \supseteq (1 - k)I(\overline{Q}) + kS(\overline{Q}), J(\overline{Q}) \supseteq (1 - k')J(\overline{Q}) + k'T(\overline{Q}), \tag{3.6}$$

where $0 < k, k' < 1$. Suppose there exists $x^{(0)} \in \Omega$ and $y^{(0)} \in \Omega$ such that the sequences $\{x^{(n)}\}$ and $\{y^{(n)}\}$ in R^N defined inductively as given below :

$$Ix^{(n+1)} = (1 - a_n)Ix^{(n)} + a_nSx^{(n)}, n \in \mathbb{N}\cup\{0\}, \tag{3.7}$$

$$Jy^{(n+1)} = (1 - a_n)Jy^{(n)} + a_nTy^{(n)}, n \in \mathbb{N}\cup\{0\}, \tag{3.8}$$

where $a_0 = 1, 0 < a_n \leq 1$ for all $n > 0$, $\liminf a_n > 0$ converge to a $z \in \overline{\Omega}$ and I and J are continuous at z.

Theorem 3.1. *Under the assumptions (3.1), (3.2), (3.6), (3.7) and (3.8), a solution of (3.5) exists.*

Proof. Let $(Ty)_i = h\ M^{-1}[(1 - B^*_{ij}) \cdot \|Jy_i - Ty_i\|^2 + \mu_i]$ for any $y \in \overline{Q}$ and any $i, j = 1, 2, \ldots, \mathbb{N}$. Now, since $(Sx)_i \leq h\ M^{-1}[(1 - B^*_{ij}) \cdot \|Ix_j - Sx_j\|^2 + \mu_i]$, for any $x \in \overline{Q}$ where

$$M = \sup_{x,y \in \overline{Q}} \Big\{\|Ix - Sx\| + \|Ix - Jy\| + \|Jy - Ty\|\Big\} \neq 0,$$

we have

$$(Sx)_i - (Ty)_i \leq h\ M^{-1}[(1 - B^*_{ij}) \cdot [\|Ix_j - Sx_j\|^2 - \|Jy_j - Ty_j\|^2],$$

or

$$\begin{aligned}(Sx)_i - (Ty)_i \ \ &\leq \ \ h\ M^{-1}(1 - B^*_{ij}) \cdot \\ &\quad\ \max\{\|Jy_i - Ty_j\|^2, \|Jy_j - Ty_j\|^2\}.\end{aligned} \tag{3.9}$$

If

$$\begin{aligned}(Ty)_i = \max\Big\{&h\ [B_{ij}Jy_j + (1 - B^*_{ij}) \cdot \|Jy_j - Ty_j\|^2 + f_i]M^{-1},\\ &h\ [(1 - B^*_{ij}) \cdot \|Jy_j - Ty_j\|^2 + \phi_i]M^{-1}\Big\},\end{aligned}$$

then we introduce the one sided operators:

$$\begin{aligned}
T^{+}x &= h \ \max\Big\{[BJx + A(1-B^{*})\cdot\|Jx-Tx\|^{2}+f]M^{-1}, \\
&\quad [(1-B^{*})\cdot\|Jx-Tx\|^{2}+\phi]M^{-1}\Big\},
\end{aligned}$$

$$\begin{aligned}
S^{+}x &= h \ \max\Big\{[BIx + A(1-B^{*})\cdot\|Ix-Sx\|^{2}+f]M^{-1}, \\
&\quad [(1-B^{*})\cdot\|Ix-Sx\|^{2}+\phi]M^{-1}\Big\}.
\end{aligned}$$

Therefore, we have $(Ty)_i = (T^{+}y)_i$.
Now, since $(Sx)_i = (S^{+}x)_i$, we have

$$(Sx)_i - (Ty)_i \leq (S^{+}x)_i - (T^{+}y)_i. \tag{3.10}$$

Now, if $(Sx)_i = h \ M^{-1}[B_{ij}Ix_j + A_{ij}(1-B^{*}_{ij})\cdot\|Ix_j - Sx_j\|^{2} + f_i]$, then since $(Ty)_i \geq h \ M^{-1}[B_{ij}Jy_j + A_{ij}(1-B^{*}_{ij})\cdot\|Jy_j - Ty_j\|^{2} + f_i]$, we have from (3.10),

$$\begin{aligned}
(S^{+}x)_i - (T^{+}y)_i &\leq h \ M^{-1}[B_{ij}\|Ix_i - Jy_i\| + (1-B^{*}_{ij})\cdot \\
&\quad \max\{\|Ix_j - Sx_j\|^{2}, \|Jy_j - Ty_j\|^{2}\}].
\end{aligned} \tag{3.11}$$

If $(Sx)_i = h \ M^{-1}[(1-B^{*}_{ij})\cdot\|Ix_j - Sx_j\|^{2} + \phi_i]$, then since $(Ty)_i \geq h \ M^{-1}[(1-B^{*}_{ij})\cdot\|Jy_j - Ty_j\|^{2} + \phi_i]$, we have

$$\begin{aligned}
(Sx)_i - (Ty)_i &\leq h \ M^{-1}[(1-B^{*}_{ij})\cdot \\
&\quad \max\{\|Ix - Sx\|^{2}, \|Jy - Ty\|^{2}\}].
\end{aligned} \tag{3.12}$$

Hence, from (3.9)-(3.12), we have

$$\begin{aligned}
(Sx)_i - (Ty)_i &\leq h \ M^{-1}[q\cdot\|Ix - Jy\| \\
&\quad +(1-q)\cdot\max\{\|Ix - Sx\|^{2}, \|Jy - Ty\|^{2}\}].
\end{aligned} \tag{3.13}$$

By interchanging the roles of (I, S) and (J, T) considering x and y be arbitrarily chosen, we have

$$\begin{aligned}
(Ty)_i - (Sx)_i &\leq h \ M^{-1}[q\cdot\|Ix - Jy\| \\
&\quad +(1-q)\cdot\max\{\|Ix - Sx\|^{2}, \|Jy - Ty\|^{2}\}].
\end{aligned} \tag{3.14}$$

Hence, from (3.13) and (3.14), it follows that

$$\|Sx - Ty\| \leq h \ M^{-1}[q\cdot\|Ix - Jy\| + (1-q)\cdot\max\{\|Ix - Sx\|^{2}, \|Jy - Ty\|^{2}\}],$$

which further implies

$$\|Sx - Ty\| \leq h \ \frac{q\cdot\|Ix - Jy\| + (1-q)\cdot\max\{\|Ix - Sx\|^{2}, \|Jy - Ty\|^{2}\}}{\|Ix - Sx\| + \|Ix - Jy\| + \|Jy - Ty\|}.$$

Thus the condition (2.1) is satisfied for $p = 1$. Therefore, Corollary 2.4 ensures the existence of a solution of (3.5). Hence the result. $\square$

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