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ERRATA ON "BANACH-SAKS PROPERTIES OF C^*-ALGEBRAS AND HILBERT C^*-MODULES"

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Abstract. Due to an example indicated to us in September 2009 we have to add one more restriction to the suppositions on the imprimitivity bimodules treated in Proposition 4.1, Theorem 5.1, Theorem 6.2 and Proposition 6.3. In the situation when the Banach-Saks property holds for the imprimitivity bimodule we can describe all possible additional examples violating the newly invented supposition. So the classification of Hilbert C^*-modules with the Banach-Saks property is complete. Beyond that, there is still an open problem for a certain class of imprimitivity bimodules with the weak or uniform weak Banach-Saks property which might violate the additional condition.

1. Introduction

In the end of September 2009 Lj. Arambašić and D. Bakić pointed out a counter-example to Proposition 4.1 of [7] to the authors, which will be described below. As a consequence, for full Hilbert C^*-modules E over non-unital C^*-algebras A the corresponding Hilbert A_1-module E_c need not be a *full* Hilbert A_1-module in certain situations (where $A_1 = A + \mathbb{C}1$). So this property of E has to be supposed additionally to keep the proofs of Proposition 4.1, Theorem 5.1, Theorem 6.2 and Proposition 6.3 correct, so far. As a result the problem of the general correctness of these statements has to be reconsidered. The presented

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proving technique does not work for the new particular examples. However, we would like to remark that the construction introduced in Section 3 of [7] is not affected by the nature of the newly found examples and is correct.

2. Main results

We start our work with some examples.

Example 2.1. Consider a separable Hilbert space H over the C^*-algebra of complex numbers $A = \mathbb{C}$. Both these Banach spaces H and A possess the Banach-Saks and the weak Banach-Saks properties. The C^*-algebra of 'compact' module operators $B = K_{\mathbb{C}}(H)$ on H has the weak Banach-Saks property. However, $K_{\mathbb{C}}(H)$ does not possess the Banach-Saks property. This is a counter-example to the formulation of Proposition 4.1 of [7]. The critical property of this example comes to light if one reverses the roles of A and B. Consider the (left) Hilbert B-module $E = K_{\mathbb{C}}(H)p$ for a minimal projection $p \in B$. Obviously, the C^*-algebra of 'compact' module operators $K_B(E)$ is $*$-isomorphic to $A = \mathbb{C}$. A careful analysis of E reveales the isometric C^*-module isomorphisms $E = E_c = E_d = \mathrm{End}_{\mathbb{C}}(H)p$. In other words, the Hilbert B_1-module E_c is not full as a Hilbert B_1-module as supposed in the proofs of Proposition 4.1 and Theorem 5.1.

Example 2.2. It happens that a certain A-B imprimitivity bimodule E has the Banach-Saks property, but both the C^*-algebras of coefficients A and B do not admit this property. To obtain an example we combine both the views on the example in the previous paragraph into one matrix-based example:

$$E = \begin{pmatrix} H & 0 \\ 0 & K_{\mathbb{C}}(H)p \end{pmatrix}, \; A = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & K_{\mathbb{C}}(H) \end{pmatrix}, \; B = \begin{pmatrix} K_{\mathbb{C}}(H) & 0 \\ 0 & \mathbb{C} \end{pmatrix}.$$

Here E has the Banach-Saks property because it is the direct sum of two Hilbert spaces in the Banach space sense, and Hilbert spaces admit the Banach-Saks property. However, both the C^*-algebras A and B contain a C^*-subalgebra $K_{\mathbb{C}}(H)$ for a separable infinite-dimensional Hilbert space H which does not possess the Banach-Saks property, and so do not A and B. Note, that the example splits using central projections of the C^*-algebra of bounded (adjointable) module operators on E. However, all three components possess the weak Banach-Saks property.

Now, we reformulate Proposition 4.1 of [7] with the additional condition of the existence of an identity in $\langle E_c, E_c\rangle$ necessary for the proof in the non-unital case (cf. [10, Thm. 3.6] for the unital case):

Proposition 2.3. *Let A be a (non-unital, in general) C^*-algebra and E be a full Hilbert A-module with the property that the C^*-algebra $\langle E_c, E_c\rangle$ is unital. Suppose, that E has the Banach-Saks property. Then A has to be finite-dimensional as a linear space, i.e. A is a finite direct sum of unital matrix algebras. In particular, any full Hilbert A-module over a non-trivial non-unital C^*-algebra A with the property that $\langle E_c, E_c\rangle$ is unital does not possess the Banach-Saks property, neither such non-unital C^*-algebras A themselves.*

The proof is the same as in the original paper since the additional supposition ensures the correctness of all arguments now.

To proceed we need information on the lattice of norm-closed two-sided ideals of type I W^*-factors $B(H)$ on separable and non-separable Hilbert spaces H. The lattice structure is described in [5, Prop. III.1.7.11]. If H has dimension $\aleph_\alpha$ as a Hilbert space then the non-trivial norm-closed two-sided ideals K_β of $B(H)$ are just the norm-closed linear spans of the orthogonal projections $p \in B(H)$ such that the dimension of the Hilbert spaces $p(H)$ are lower than $\aleph_\beta$, for $0 \leq \beta \leq \alpha$. In particular, these ideals are linearly ordered by inclusion. All these ideals admit $B(H)$ as their multiplier algebra. For proofs see [8, 12, 4].

Example 2.4. Let H be a non-separable Hilbert space and p be an orthogonal projection of H onto a fixed separable or non-separable Hilbert subspace H_0 with lower dimension than H. Set $A = K_{\mathbb{C}}(H)$, $B = K_{\mathbb{C}}(H_0) = pK_{\mathbb{C}}(H)p$ and $E = K_{\mathbb{C}}(H)p$. Then the construction of E_c and E_d from E according to §3 of [7] can be done separately for the left A-module E and for the right B-module E. Considering E as a left Hilbert A-module one arrives at $E_d^l = B_{\mathbb{C}}(H)p \neq K_{\mathbb{C}}(H)p$ since p is not similar to the identity operator. So $\langle E_d^l, E_d^l \rangle = B_{\mathbb{C}}(H)pB_{\mathbb{C}}(H)$ and this C^*-algebra does not contain an identity. Furthermore, $E_c^l = K_{\mathbb{C}}(H)p$ and $\langle E_c^l, E_c^l \rangle = K_{\mathbb{C}}(H)$ since every compact operator has separable range and separable support. The picture for the right Hilbert B-module is different. We obtain $E_d^r = B_{\mathbb{C}}(H)p$ and $E_c^r = K_{\mathbb{C}}(H)p \oplus \mathbb{C}p$. So the assumption that E would have the Banach-Saks property would lead to the statement that the unital C^*-algebra $K_{\mathbb{C}}(H_0) \oplus \mathbb{C}p$ would have the Banach-Saks property by Proposition 4.1 in its corrected version, a contradiction, because this C^*-algebra is not finite-dimensional.

The next step is the structural description of all examples of Hilbert C^*-modules E with the Banach-Saks property such that for E either the left completion E_c^l or the right completion E_c^r admit respective non-unital C^*-algebras $\langle E_c^l, E_c^l \rangle$ and $\langle E_c^r, E_c^r \rangle$, or both. Consequently, at least one of the minimal C^*-algebras of coefficients of the imprimitivity bimodule E has to be non-unital, too. We obtain a general structure similar to that one described in Example 2.2.

Proposition 2.5. *Let A and B be C^*-algebras, at least one of them non-unital, and let E be an A-B imprimitivity bimodule that has the Banach-Saks property. Then the centers of the multiplier C^*-algebras $M(A)$ and $M(B)$, which can be identified canonically, contain two positive projections p and q such that pA and qB are finite-dimensional C^*-algebras, at least one of the projections $p - pq$ and $q - pq$ is non-trivial and the identities of $M(A)$ and of $M(B)$ equal to $p + q - pq$. Consequently, E decomposes into a direct sum of the left Hilbert $(p-pq)A$-module $(p - pq)E$ for which $K_{(p-pq)A}((p - pq)E)$ is non-unital, of the right $(q - pq)B$-module $(q - pq)E$ for which $K_{(q-pq)B}((q - pq)E)$ is non-unital and of the finitely generated projective pqA-pqB imprimitivity bimodule pqE. The latter projective part and one of the other two parts can be trivial.*

To describe the picture clearly the matrix-notation is useful:

$$A = \begin{pmatrix} (p-pq)A & 0 & 0 \\ 0 & (q-pq)A & 0 \\ 0 & 0 & pqA \end{pmatrix}, \; B = \begin{pmatrix} (p-pq)B & 0 & 0 \\ 0 & (q-pq)B & 0 \\ 0 & 0 & pqB \end{pmatrix}$$
$$E = \begin{pmatrix} (p-pq)E & 0 & 0 \\ 0 & (q-pq)E & 0 \\ 0 & 0 & pqE \end{pmatrix}$$

with pA and qB unital and finite-dimensional, $(q-pq)A$ and $(p-pq)B$ non-unital.

Proof. Since E has the Banach-Saks property it is reflexive as a Banach space by [6, p. 85] and, hence, admits a predual Banach space. Since both the actions of A and of B on E are weakly continuous by the reflexivity of E, the Hilbert C^*-module E has to be a Hilbert W^*-module such that both the C^*-algebras of adjointable bounded module operators $End^*_A(E)$ and $End^*_B(E)$ have to be W^*-algebras ([15, Thm. 2.6]). Moreover, E is self-dual both as a left Hilbert A-module and a right Hilbert B-module. The centers of $End^*_A(E)$ and $End^*_B(E)$ can be isometrically identified, it is a commutative W^*-algebra C that slices the Hilbert A-B bimodule E. By [11] there exist isometric isomorphisms of $End^*_A(E)$ to the multiplier algebra $M(B)$ and of $End^*_B(E)$ to the multiplier algebra $M(A)$. The Hilbert W^*-module E has properties very similar to Hilbert spaces by [13]. In particular, the left Hilbert $M(A)$-module E is isometrically isomorphic to a certain w*-closed direct orthogonal sum of a collection of Hilbert $M(A)$-modules of type $M(A)r_\alpha$ for some orthogonal projections $r_\alpha \in M(A)$, [13, Thm. 3.12]. Consequently, the left Hilbert $M(A)$-modules $M(A)r_\alpha$ inherite the Banach-Saks property from E as norm-closed subspaces, and so do the unital C^*-algebras $r_\alpha M(A) r_\alpha$ of all 'compact' $M(A)$-linear operators on $M(A)r_\alpha$. The latter have to be finite-dimensional C^*-algebras by Proposition 2.3. So the projections $r_\alpha \in M(A)$ are atomic finite range projections in $M(A)$. Since all atomic finite range projections of a von Neumann algebra have the same central carrier projection which supports the atomic type I part of the W^*-algebra by [1, p. 278] and [2, p. I], the W^*-algebra $M(A)$ has to be atomic type I, as well as the W^*-algebra $M(B)$ by analogous considerations (and by Morita equivalence of W^*-algebras, cf. [14, Prop. 2.8, §8]). Consequently the isometrically isomorphic centers C of $M(A)$ and of $M(B)$ are an atomic commutative W^*-algebra.

Define

$$\begin{aligned} p &= sup\{r = r^2 \geq 0 : r \in C,\, r \in \langle rE, rE\rangle_A\}\,, \\ q &= sup\{r = r^2 \geq 0 : r \in C,\, r \in \langle Er, Er\rangle_B\}\,. \end{aligned}$$

By Proposition 2.3 the Hilbert C^*-modules pE and Eq which admit the Banach-Saks property as subspaces of E pass the Banach-Saks property to the unital C^*-algebras pA and qB. So both these C^*-algebras are finite-dimensional C^*-algebras. Note, that the Hilbert C^*-module pEq is finitely generated and projective since it is a pqA-pqB imprimitivity bimodule of two unital C^*-algebras.

Consider the complement $(1_C - (p + q - pq))E$. By construction both the C^*-algebras $(1_C - (p+q-pq))A$ and $(1_C - (p+q-pq))B$ are non-unital. Moreover, for any non-trivial subprojection $r \leq 1_C - (p+q-pq)$ in the center C of $M(A)$ and of $M(B)$ both the C^*-algebras rA and rB are non-unital. Fix a non-trivial minimal projection $r \in C$. Then both $rM(A)$ and $rM(B)$ are atomic type I W^*-algebras with trivial center, i.e. they are C^*-isomorphic to W^*-algebras of all bounded linear operators on certain infinite-dimensional Hilbert spaces. Since both rA and rB are non-unital and two-sided ideals in $rM(A)$ and $rM(B)$, respectively, by construction, they can only be either C^*-isomorphic to the respective C^*-algebras of all compact linear operators on these found Hilbert spaces or, in the non-separable case, at least contain the norm-closed two-sided ideals of all compact operators as two-sided closed strictly dense ideals. C^*-algebras of all compact operators on Hilbert spaces have a trivial Picard group by [3]. Therefore, the imprimitivity bimodule between them is unique up to unitary isomorphism. So the Hilbert rA-rB bimodule rE has to contain an isometric copy F of the unique imprimitivity bimodule interrelating both the C^*-algebras of compact operators on the respective Hilbert spaces. The space F is isometrically isomorphic to the set of all compact linear operators from one of these Hilbert spaces into the other. As in Example 2.4, $(rE)_c$ has to produce a unital C^*-algebra of 'compact' operators either for its left or for its right version, or for both of them, in dependency on the dimensions of the Hilbert spaces related to $rM(A)$ and to $rM(B)$, respectively. The identity arises from elements of $F \subseteq rE$. However, the self-duality of rE as a Hilbert W^*-module forces $rE \equiv (rE)_c$ in both situations, so at least one of the C^*-algebras rA or rB has to be unital and finite-dimensional by Proposition 2.3. This is a contradiction to our supposition that both the C^*-algebras rA and rB are set to be non-unital and infinite-dimensional, and hence, to the supposition on E to admit the Banach-Saks property. Finally, we arrive at the fact that the projection $(p + q - pq)$ is the carrier projection of A, B and E. □

While the classification of Hilbert C^*-modules with the Banach-Saks property is finally completed, the classification of Hilbert C^*-modules with the (uniform) weak Banach-Saks property has still the open problem with such A-B imprimitivity bimodules E for which both the C^*-algebras A and B are non-unital and neither the left Hilbert A_1-module E_c might be full nor the analogously built right Hilbert B_1-module E_c might be full. We do not know neither counter-examples to the statements made in [7] nor a theoretical classification of this remaining admissible situation. But mainly, we have to give a correct formulation of Theorem 5.1 of [7]. As in [7], we rely on a key partial result by M. Kusuda [10, Thm. 2.2]:

Theorem 2.6. *Let A and B be two strongly Morita equivalent C^*-algebras and E be an A-B imprimitivity bimodule. Suppose for the case of two non-unital C^*-algebras A and B that either the left Hilbert A_1-module E_c is full or the right Hilbert B_1-module E_c is full (, or both). Then the following four conditions are equivalent:*

(i) A has the weak Banach-Saks property.

(ii) B has the weak Banach-Saks property.
(iii) E has the weak Banach-Saks property.
(iv) L has the weak Banach-Saks property.

The proof can be made as presented in [7]. The gap in the arguments is filled by the additional assumption on E. Similarly, we give a correct version of Theorem 6.2 and Proposition 6.3 of [7]. As previously, we rely on the earlier results [9, Thm. 2.3] and [10, Thm. 2.2] by M. Kusuda. The proofs remain unchanged because of the additional assumption on E.

Theorem 2.7. *Let A and B be two strongly Morita equivalent C^*-algebras and E be an A-B imprimitivity bimodule. Suppose for the case of two non-unital C^*-algebras A and B that either the left Hilbert A_1-module E_c is full or the right Hilbert B_1-module E_c is full (, or both). The following four conditions are equivalent:*

(i) A has the uniform weak Banach-Saks property.
(ii) B has the uniform weak Banach-Saks property.
(iii) E has the uniform weak Banach-Saks property.
(iv) L has the uniform weak Banach-Saks property.

In particular, under the supposions made the conditions (i)-(iv) hold in case either A or B or E or L have the weak Banach-Saks property. Conversely, under the supposions made either of conditions (i)-(iv) implies A, B, E and L to have the weak Banach-Saks property.

Proposition 2.8. *Let A be a C^*-algebra and E be a full Hilbert A-module with the weak or uniform weak Banach-Saks property. Suppose for the case of two non-unital C^*-algebras A and B that either the left Hilbert A_1-module E_c is full or the right Hilbert B_1-module E_c is full (, or both). Then there exist a finite sequence $\{E_i : i = 0, ..., l\}$ of norm-closed A-submodules of E and a sequence $\{I_i : i = 0, ..., l\}$ of two-sided norm-closed ideals of A such that*

(i) $I_l = A$, $I_{i-1} \subset I_i$ and I_{i-1} is a two-sided ideal of I_i for any $i = 1, ..., l$.
(ii) The C^*-algebra I_0 and the factor C^*-algebras $\{I_i/I_{i-1} : i = 1, ..., l\}$ are dual C^*-algebras.
(iii) $E_l = E$, $E_{i-1} \subset E_i$ and the Hilbert A-modules E_i are full Hilbert I_i-modules for any $i = 0, ..., l$. In particular, the values $\langle x, y\rangle$ belong to I_i for any $x \in E_i$ and any $y \in E_j$ with $j \geq i$, $i, j = 0, ..., l$. The factor modules E_i/E_{i-1} are Hilbert C^*-modules over the dual C^*-algebras I_i/I_{i-1}.

So there is still a partial problem open to complete the classification. The technique used by the authors to clarify further situations left out by M. Kusuda does not help any more. New ideas and techniques are necessary.

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