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ON THE EQUIVALENCE OF HERMITIAN INNER PRODUCTS ON TOPOLOGICAL $*$-ALGEBRAS

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ABSTRACT. Sufficient conditions for a topological $*$-algebra under which several Hermitian inner products are mutually equivalent are given.

1. INTRODUCTION

For constructing a Hermitian K-theory for topological $*$-algebras, one usually supposes that the algebra under consideration is unital and locally m-convex (see [7]). In this paper we obtain similar results as in [7] for the case of both unital and non-unital topological $*$-algebras without assuming locally m-convexity.

1.1. **Preliminary definitions.** Throughout this paper $\mathbb{K}$ denotes either the set $\mathbb{R}$ of all real numbers or the set $\mathbb{C}$ of all complex numbers. Let A be a *-algebra over $\mathbb{K}$ and M a left A-module. $\mathrm{Hom}_A(M, A)$ stands for the set of all A-linear maps $f : M \to A$. Under the operations:

$$(f+g)(m) := f(m) + g(m), \quad (af)(m) := f(m)a^*, \quad (\lambda f)(m) := \overline{\lambda}[f(m)] \quad (1.1)$$

for all $f, g \in \mathrm{Hom}_A(M, A)$, $m \in M$ and $\lambda \in \mathbb{K}$, $\mathrm{Hom}_A(M, A)$ becomes a left A-module.

An *A-valued Hermitian inner product on M* is a map $\alpha : M \times M \to A$ which satisfies the following properties:

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(1) α is $\mathbb{K}$-linear on first component, i.e., $\alpha(\lambda x + \mu y, z) = \lambda\alpha(x, z) + \mu\alpha(y, z)$ for all $\lambda, \mu \in \mathbb{K}$ and $x, y, z \in M$;
(2) α is A-homogeneous in the first argument, i.e., $\alpha(ax, y) = a\alpha(x, y)$ for every $a \in A$ and $x, y \in M$;
(3) α is Hermitian, i.e., $\alpha(x, y) = \alpha(y, x)^*$ for every $x, y \in M$;
(4) the map $\phi : M \to \mathrm{Hom}_A(M, A)$, $x \mapsto \phi(x)$, defined by $[\phi(x)](y) := \alpha(y, x)$, is an isomorphism of A-modules.

Notice, that the conditions (1) and (3) together imply

(1') $\alpha(x, \lambda y + \mu z) = \bar{\lambda}\alpha(x, y) + \bar{\mu}\alpha(x, z)$ for every $\lambda, \mu \in \mathbb{K}$ and $x, y, z \in M$.

Similarly, (2) and (3) imply

(2') $\alpha(x, ay) = \alpha(x, y)a^*$ for every $a \in A$ and $x, y \in M$.

Hence, α is also $\mathbb{K}$-sesquilinear and A-homogeneous on both arguments.

Moreover, the condition

(4a) $\alpha(x, x) = \theta_A$ if and only if $x = \theta_M$

implies that the map ϕ defined in (4) is one-to-one. (Indeed, suppose that $\phi(x) = \phi(y)$ for some $x, y \in M$. Then

$$\alpha(x - y, x) = [\phi(x)](x - y) = [\phi(y)](x - y) = \alpha(x - y, y)$$

implies $\alpha(x - y, x - y) = \theta_A$. Hence, $x - y = \theta_M$ by 4a) and $x = y$.)

A Hermitian inner product α on M is said to be *spectrally positive definite* (for short *positive definite*)[1], if $\mathrm{Sp}_A(\alpha(x, x)) \subset [0, \infty)$ for every $x \in M$.

Let B be a non-unital algebra. The set $\{e_1, \ldots, e_m\}$ of elements of a B-module M is said to be a *basis* of M if for every $m \in M$ there exist unique elements $b_1, \ldots, b_m \in B$ and unique numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ such that

$$m = \sum_{i=1}^{m} b_i e_i + \sum_{i=1}^{m} \lambda_i e_i.$$

In case A is a unital algebra, the set $\{e_1, \ldots, e_m\}$ of elements of an A-module M is said to be a *basis* of M if for every $m \in M$ there exist unique elements $a_1, \ldots, a_m \in A$ such that

$$m = \sum_{i=1}^{m} a_i e_i.$$

2. On the existence of a Hermitian inner product

First, we show that under some conditions, which are automatically fulfilled for Hausdorff locally C^*-algebras (see [3, Remark 1.2, p. 184]), every topological $*$-algebra admits a positive definite Hermitian inner product.

Lemma 2.1. *Let A be a unital $*$-algebra for which the following conditions are fulfilled:*

(a) *If $a \in A$, then $aa^* = \theta_A$ if and only if $a = \theta_A$.*

[1]Another kind of a positive element in a $*$-algebra is given in [3, p.183]. This notion of positiveness agrees with spectrally positiveness for Hausdorff locally C^*-algebras (viz. complete locally m-convex C^*-algebras) [3, p. 184] (see also [4, Theorem 2.5, p. 205]).

(b) *If $n \in \mathbb{N}$ and $a_1, \dots, a_n \in A$, then* $\mathrm{Sp}_A(\sum_{i=1}^n a_i a_i^*) \subset [0, \infty)$.

(c) *The only self-adjoint element $a \in A$ with* $\mathrm{Sp}_A(a) = \{0\}$ *is the zero element θ_A of A.*

Moreover, let M be an A-module with basis $\{e_1, \dots, e_m\}$. Then the map $\alpha : M \times M \to A$, defined by

$$\alpha(x, y) = \alpha\Big(\sum_{i=1}^m x_i e_i, \sum_{i=1}^m y_i e_i\Big) := \sum_{i=1}^m x_i y_i^*$$

for every $x, y \in M$, defines an A-valued positive definite Hermitian inner product on M.

Proof. Let $x = \sum_{i=1}^m x_i e_i,\ y = \sum_{i=1}^m y_i e_i,\ z = \sum_{i=1}^m z_i e_i$ be elements of M, $a \in A$ and $\lambda, \mu \in \mathbb{K}$. Then

$$\alpha(\lambda x + \mu y, z) = \alpha\Big(\sum_{i=1}^m (\lambda x_i + \mu y_i) e_i, \sum_{i=1}^m z_i e_i\Big) = \sum_{i=1}^m (\lambda x_i + \mu y_i) z_i^* =$$

$$= \lambda \sum_{i=1}^m x_i z_i^* + \mu \sum_{i=1}^m y_i z_i^* = \lambda\alpha(x, z) + \mu\alpha(y, z),$$

$$\alpha(ax, y) = \alpha\Big(\sum_{i=1}^m (a x_i) e_i, \sum_{i=1}^m y_i e_i\Big) = \sum_{i=1}^m (a x_i) y_i^* = a \sum_{i=1}^m x_i y_i^* = a\alpha(x, y),$$

and

$$\alpha(x, y) = \sum_{i=1}^m x_i y_i^* = \sum_{i=1}^m (y_i x_i^*)^* = \Big(\sum_{i=1}^m y_i x_i^*\Big)^* = \alpha(y, x)^*.$$

Hence, the first 3 conditions of an A-valued Hermitian inner product are fulfilled. This implies that the conditions (1') and (2') are also fulfilled.

Clearly $\alpha(\theta_M, \theta_M) = \theta_A$. Suppose that $\alpha(x, x) = \theta_A$ for some $x \in M$. Then

$$\sum_{i=1}^m x_i x_i^* = \theta_A.$$

Hence,

$$\sum_{i=1}^{m-1} x_i x_i^* = -x_m x_m^*$$

and

$$\mathrm{Sp}_A\Big(\sum_{i=1}^{m-1} x_i x_i^*\Big) = \mathrm{Sp}_A(-x_m x_m^*) = -\mathrm{Sp}_A(x_m x_m^*).$$

By the condition (b), we get that

$$\mathrm{Sp}_A\Big(\sum_{i=1}^{m-1} x_i x_i^*\Big) \subset [0, \infty) \quad \text{and} \quad \mathrm{Sp}_A(x_m x_m^*) \subset [0, \infty).$$

Thus,

$$\mathrm{Sp}_A\Big(\sum_{i=1}^{m-1} x_i x_i^*\Big) = \{0\} = \mathrm{Sp}_A(x_m x_m^*).$$

Condition (c) implies that $x_m x_m^* = \theta_A$ from which by condition (a) follows that $x_m = \theta_A$. Similarly, we get that $x_{m-1} = \theta_A, ..., x_1 = \theta_A$. Hence, from $\alpha(x,x) = \theta_A$ it follows that $x = \theta_M$. Consequently, $\phi : M \to \mathrm{Hom}_A(M,A)$, defined by $[\phi(x)](y) = \alpha(y,x)$ is one-to-one.

Take now any $\psi \in \mathrm{Hom}_A(M,A)$, define $x_i := \psi(e_i)^*$ for every $i \in \{1, \dots m\}$ and $x := \sum_{i=1}^m x_i e_i$. Then $x \in M$ and

$$\psi(y) = \sum_{i=1}^m y_i \psi(e_i) = \sum_{i=1}^m y_i (\psi(e_i)^*)^* = \sum_{i=1}^m y_i x_i^* = \alpha(y,x) = [\phi(x)](y)$$

for every $y \in M$. Hence, ϕ is also onto.

Notice, that by the properties (1), (2), (3), (1') and (2') of α and the condition (1.1) of the operations on $\mathrm{Hom}_A(M,A)$, we have

$$[\phi(ax)](y) = \alpha(y, ax) = \alpha(y,x)a^* = [a\phi(x)](y),$$

$$[\phi(x+z)](y) = \alpha(y, x+z) = \alpha(y,x) + \alpha(y,z) = [\phi(x)](y) + [\phi(z)](y),$$

and

$$[\phi(\lambda x)](y) = \alpha(y, \lambda x) = \overline{\lambda}\alpha(y,x) = [\lambda\phi(x)](y)$$

for every $a \in A$, $x, y, z \in M$ and $\lambda \in \mathbb{K}$. Hence, $\phi(ax) = a\phi(x)$, $\phi(x+z) = \phi(x) + \phi(z)$ and $\phi(\lambda x) = \lambda\phi(x)$ for every $a \in A$, $\lambda \in \mathbb{K}$ and $x, z \in M$. Therefore, ϕ is an isomorphism of A-modules. Thus, α is an A-valued Hermitian inner product on M. Condition (b) implies that α is also positive definite. □

Corollary 2.2. *Let B be a non-unital $*$-algebra for which the following conditions are fulfilled:*

(a) *If $b \in B$, then $bb^* = \theta_B$ if and only if $b = \theta_B$.*

(b) *If $n \in \mathbb{N}$, $b_1, \dots, b_n \in B$ and $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, then*

$$\mathrm{Sp}_{B\times\mathbb{K}}\Big(\sum_{i=1}^n (b_i, \lambda_i)(b_i, \lambda_i)^*\Big) \subset [0, \infty).$$

(c) *The only self-adjoint element $(b, \lambda) \in B \times \mathbb{K}$ with $\mathrm{Sp}_{B\times\mathbb{K}}((b,\lambda)) = \{0\}$ is the zero element $(\theta_B, 0)$ of $B \times \mathbb{K}$.*

Moreover, let M be a B-module with basis $\{e_1, \dots, e_m\}$. Then the map $\alpha : M \times M \to B \times \mathbb{K}$, defined by

$$\alpha(x,y) = \alpha\Big(\sum_{i=1}^m (x_i e_i + \lambda_i e_i), \sum_{i=1}^m (y_i e_i + \mu_i e_i)\Big) := \sum_{i=1}^m (x_i, \lambda_i)(y_i, \mu_i)^*$$

for every $x, y \in M$, defines a $(B \times \mathbb{K})$-valued positive definite Hermitian inner product on M.

Proof. Remind, that every B-module with basis $\{e_1, \dots, e_m\}$ is also a $(B \times \mathbb{K})$-module with the same basis and that every B-linear map is also $(B \times \mathbb{K})$-linear (see [2, Proof of Corollary 3, p. 162]). Moreover, suppose that $(b,\lambda)(b,\lambda)^* = (\theta_B, 0)$. Then we have $(bb^* + \lambda^* b + \lambda b^*, \lambda\lambda^*) = (\theta_B, 0)$ (λ^* stands for the conjugate of $\lambda \in \mathbb{K}$). Since $\lambda\lambda^* =| \lambda |^2$, we get $\lambda = 0$. Hence, $(b,\lambda)(b,\lambda)^* = (\theta_B, 0)$ if and only if $bb^* = \theta_B$. By condition (a) we see that

$(b,\lambda)(b,\lambda)^* = (\theta_B, 0)$ if and only if $(b,\lambda) = (\theta_B, 0)$. Thus, taking $A := B \times \mathbb{K}$, we are in the situation of Lemma 2.1. Hence, the claim follows from Lemma 2.1. $\square$

3. On the matrix associated with a Hermitian inner product

Suppose again that A is a unital algebra. With every Hermitian inner product α on an A-module M with basis $\{e_1, \ldots, e_m\}$ (i.e., M is a free A-module of rank m), we can associate its matrix M_α as follows:

$$M_\alpha := (m_{i,j}), \text{ where } m_{i,j} = \alpha(e_i, e_j) \text{ for every } i, j \in \{1, \ldots, m\}.$$

It is known that for a $*$-algebra A and A-valued square matrix $M = (m_{i,j})$, one defines $M^* = (n_{i,j})$, where $n_{i,j} = m^*_{j,i}$ for every $i \in \{1, \ldots, m\}$ and every $j \in \{1, \ldots, m\}$. Since for a Hermitian inner product α we have $\alpha(e_i, e_j) = \alpha(e_j, e_i)^*$ for every $i \in \{1, \ldots, m\}$ and every $j \in \{1, \ldots, m\}$, then it is clear that $M^*_\alpha = M_\alpha$, i.e., M_α is Hermitian (alias, self-adjoint). From the condition (4) of a Hermitian inner product, it follows by [5, Proposition 12, p. 385] (see also [6, Proposition 6.1, p. 465 together with Proposition 4.16, p. 456]), that M_α is invertible. Moreover, for any

$$x = \sum_{i=1}^m x_i e_i \quad \text{and} \quad y = \sum_{i=1}^m y_i e_i$$

we have $\alpha(x, y) = (x_1\ x_2\ \ldots\ x_m) M_\alpha (y_1^*\ y_2^*\ \ldots\ y_m^*)^T$, where $(z_1\ z_2\ \ldots\ z_m)^T$ denotes the transpose matrix of the matrix $(z_1\ z_2\ \ldots\ z_m)$ with one row and m columns, i.e., $(z_1\ z_2\ \ldots\ z_m)^T$ is a matrix with m rows and 1 column.

Take any Hermitian invertible $(m \times m)$-matrix $H = (h_{i,j})$ and define a map $\beta : M \times M \to A$ by setting

$$\beta\left(\sum_{i=1}^m a_i e_i, \sum_{i=1}^m b_i e_i\right) := (a_1\ a_2\ \ldots\ a_m) H (b_1^*\ b_2^*\ \ldots\ b_m^*)^T.$$

Then it is clear that β is A-homogeneous and $\mathbb{K}$-sesquilinear. Next we show that the map $\phi : M \to \mathrm{Hom}_A(M, A)$, defined by

$$\left[\phi\left(\sum_{i=1}^m a_i e_i\right)\right]\left(\sum_{i=1}^m b_i e_i\right) := \beta\left(\sum_{i=1}^m b_i e_i, \sum_{i=1}^m a_i e_i\right)$$

is a bijection.

Suppose that

$$\phi(m_a) = \phi\left(\sum_{i=1}^m a_i e_i\right) = \phi\left(\sum_{i=1}^m b_i e_i\right) = \phi(m_b)$$

for some $m_a, m_b \in M$. Then

$$\sum_{i=1}^m h_{1,i} a_i^* = [\phi(m_a)](e_1) = [\phi(m_b)](e_1) = \sum_{i=1}^m h_{1,i} b_i^*,$$

$$\sum_{i=1}^m h_{2,i} a_i^* = [\phi(m_a)](e_2) = [\phi(m_b)](e_2) = \sum_{i=1}^m h_{2,i} b_i^*,$$

$$\ldots,$$

$$\sum_{i=1}^{m} h_{m,i} a_i^* = [\phi(m_a)](e_m) = [\phi(m_b)](e_m) = \sum_{i=1}^{m} h_{m,i} b_i^*.$$

Hence,

$$\sum_{i=1}^{m} h_{ji}(a_i^* - b_i^*) = \theta_A$$

for every $j \in \{1, \ldots, m\}$. If we denote by H_i the i-th column of the matrix H, then we get

$$\sum_{i=1}^{m} H_i(a_i^* - b_i^*) = (\theta_A \; \theta_A \; \ldots \; \theta_A)^T.$$

If $a_i^* - b_i^* \neq \theta_A$ for at least one value of i, then the columns of H are linearly dependent and H can not be invertible. Since H was assumed to be invertible, we must have $a_i^* - b_i^* = \theta_A$ for every $i \in \{1, \ldots, m\}$ from which $m_a = m_b$ and ϕ is one-to-one.

Take any $\psi \in \mathrm{Hom}_A(M, A)$. Since H is invertible, H^{-1} exists. Take

$$x := \sum_{i=1}^{m} x_i e_i,$$

where $(x_1 \; x_2 \; ... \; x_m)^T := H^{-1}(\psi(e_1)^* \; \psi(e_2)^* \; \ldots \; \psi(e_m)^*)^T$. Then $[\phi(x)](y) = \psi(y)$ for every $y \in M$. Hence, ϕ is onto. Consequently, ϕ is a bijection.

Thus, β, defined above, is a Hermitian inner product. Moreover, the matrix of β is actually H, i. e., $M_\beta = H$.

By the facts we just obtained, we have the following result.

Lemma 3.1. *Let A be a unital $*$-algebra and M a free A-module of rank m. Then there exists a bijection between the sets of Hermitian inner products on M and A-valued Hermitian invertible $(m \times m)$-matrices.*

By Lemma 3.1, we have the following result.

Corollary 3.2. *Let B be a non-unital $*$-algebra and M a free B-module of rank m. Then there exists a bijection between the sets of Hermitian inner products on M and $(B \times \mathbb{K})$-valued Hermitian invertible $(m \times m)$-matrices.*

Proof. Since every B-module is also a $(B \times \mathbb{K})$-module with the same basis, then taking $A := B \times \mathbb{K}$, we are in the situation of Lemma 3.1. □

Notice, that for the Hermitian inner product α, defined in Lemma 2.1 or Corollary 2.2, the matrix M_α, associated with α, is an identity matrix.

Definition 3.3. Let A be a unital $*$-algebra and M a free A-module of rank m. We say that two Hermitian inner products, α and β on M, are equivalent, if there exists an invertible $(m \times m)$-matrix N such that $M_\alpha = N^* M_\beta N$.

Notice, that if for any Hermitian inner product β there exists a Hermitian invertible matrix N such that $M_\beta = NN = N^2$, then β is equivalent to α defined in Lemma 2.1.

4. On the equivalence of Hermitian inner products

Let A be a topological algebra. A sequence $(x_n)_{n\in\mathbb{N}}$ in A is a *Mackey–Cauchy sequence* if there exists a bounded and balanced set U in A such that for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $x_n - x_m \in \epsilon U$ whenever $n, m > N_\epsilon$.

The algebra A is *sequentially Mackey complete* (one could also use the term *Mackey σ-complete*) if every Mackey–Cauchy sequence in A converges in A.

Proposition 4.1. *Let $m \in \mathbb{N}$ and A be a sequentially Mackey complete topological algebra. Then the algebra $M_m(A)$ of all $(m \times m)$-matrices with elements from A is also sequentially Mackey complete*[2].

Proof. The topology in the algebra $M_m(A)$ of all A-valued $(m \times m)$-matrices is induced by a product topology, i.e., a basis of this topology consists of sets

$$U_{O_1,\ldots,O_{m^2}} = \{M = (m_{ij}) \in M_m(A) : m_{ij} \in O_{(i-1)m+j}\},$$

where $O_1, \ldots, O_{m^2}$ vary in a basis of the topology of A.

Take any Mackey–Cauchy sequence $(M_n)_{n\in\mathbb{N}} = ((m_{ij}^n))_{n\in\mathbb{N}}$ in $M_m(A)$. Then the sequence $(m_{ij}^n)_{n\in\mathbb{N}}$ is a Mackey–Cauchy sequence in A for each fixed $i, j \in \{1, \ldots, m\}$. Indeed, let U be a bounded and balanced set in $M_m(A)$ such that for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ with $M_k - M_l \in \epsilon U$ whenever $k, l > N_\epsilon$. For each $i, j \in \{1, \ldots, m\}$ take $V_{i,j} := \{m_{ij} \in A : (m_{ij}) \in U\}$. Then all sets $V_{i,j}$ are balanced and bounded in A because U is balanced and bounded in $M_m(A)$. Now it is clear that $m_{ij}^k - m_{ij}^l \in \epsilon V_{i,j}$ whenever $k, l > N_\epsilon$. Hence, there exists balanced and bounded sets $V_{i,j}$ and numbers N_ϵ for every $\epsilon > 0$ such that the conditions of Mackey–Cauchy sequence are fulfilled.

Since $(m_{ij}^n)_{n\in\mathbb{N}}$ is a Mackey–Cauchy sequence in A for each $i, j \in \{1, \ldots, m\}$ and A is sequentially Mackey complete, then $(m_{ij}^n)_{n\in\mathbb{N}}$ converges in A to some element $s_{ij} \in A$ for each $i, j \in \{1, \ldots, m\}$. Take $S := (s_{ij}) \in M_m(A)$. Then $(M_n)_{n\in\mathbb{N}}$ converges to S in $M_m(A)$. Hence, $M_m(A)$ is sequentially Mackey complete as well. $\square$

Let us recall, that for an element a in a topological algebra A its *radius of boundedness* is defined as

$$\beta(a) := \inf\left\{\lambda > 0 : \left\{\left(\frac{a}{\lambda}\right)^n : n \in \mathbb{N}\right\} \text{ is bounded in } A\right\}.$$

We recall also that the terms "a is Hermitian" and "a is self-adjoint" are synonyms. In [1, Corollary 2.8], it was proved the following.

Theorem 4.2. *Let A be a unital sequentially Mackey complete topological algebra. If $a \in A$ satisfies the condition $\beta(a - e_A) < 1$, then there exists an element $b \in A$ such that $b^2 = a$. In particular, when A is a unital sequentially Mackey complete topological $*$-algebra with continuous involution and a is self-adjoint, then b is also self-adjoint.*

[2]It is clear that if A is unital, then also $M_m(A)$ is unital because the unit element in $M_m(A)$ is the identity matrix.

Let A be a topological algebra and $m \in \mathbb{N}$. For every $i, j \in \{1, \ldots, m\}$ define the projections $p_{i,j} : M_m(A) \to A$ by $p_{i,j}(M) = m_{ij}$ for every $M = (m_{ij}) \in M_m(A)$. A map $f : M_m(A) \to M_m(A)$ is continuous if and only if all of its projections are continuous, i.e., f is continuous if and only if $p_{i,j} \circ f$ is continuous for every $i, j \in \{1, \ldots, m\}$.

For the next result, see also [3, Lemma 5.3, p. 196], where the continuity of the involution of a locally m-convex $*$-algebra is inherited to the algebra of all infinite matrices with finite support and entries from A.

Lemma 4.3. *Let A be a topological $*$-algebra and $m \in \mathbb{N}$. The involution on $M_m(A)$ is continuous if and only if the involution is continuous on A.*

Proof. Suppose, that the involution $i_A : A \to A$, defined by $i_A(a) = a^*$ for every $a \in A$, is continuous. Consider the involution $i_m : M_m(A) \to M_m(A)$ defined by $i_m(M) = M^*$ for every $M \in M_m(A)$. Then $(p_{i,j} \circ i_m)(M) = m^*_{j,i}$ for every $M = (m_{i,j}) \in M_m(A)$. Let $T : M_m(A) \to M_m(A)$ be the transpose function, i.e., $T(M) = T((m_{i,j})) = (m_{j,i}) = M^T$ for every $M \in M_m(A)$. Then $(i_A \circ p_{i,j} \circ T)(M) = m^*_{j,i}$ for every $M = (m_{i,j}) \in M_m(A)$. Hence, $p_{i,j} \circ i_m = i_A \circ p_{i,j} \circ T$.

The tranpose function is continuous because for any neighbourhoods of zero $O_{i,j}$ in A there exist neighbourhoods $U_{i,j} = O_{j,i}$ of zero in A such that if $M \in U_{U_{1,1}, U_{1,2} \ldots, U_{i,m}, U_{2,1}, \ldots U_{m,m}}$ we get $T(M) \in U_{O_{1,1}, O_{1,2} \ldots, O_{i,m}, O_{2,1}, \ldots O_{m,m}}$. The projections $p_{i,j}$ are also continuous. Hence, $i_A \circ p_{i,j} \circ T$ is continuous for every $i, j \in \{1, \ldots, m\}$ as a composition of continuous maps. Therefore, $p_{i,j} \circ i_m$ is continuous for every $i, j \in \{1, \ldots, m\}$. It means that i_m is continuous.

Suppose that i_m is continuous. Take any neighbourhood O of zero in A. Then $P = U_{O_1, \ldots, O_{m^2}}$ with $O_1 = O_2 = \cdots = O_{m^2} = O$ is a neighbourhood of zero in $M_m(A)$. Since the involution is continuous in $M_m(A)$, then there exists a neighbourhood $V = U_{V_1, \ldots, V_{m^2}}$ of zero in $M_m(A)$ such that $i_m(M) \in P$ for every $M \in V$. Take

$$W := \bigcap_{1 \leq i \leq m^2} V_i$$

and $Z = U_{Z_1, \ldots, Z_{m^2}}$ with $Z_1 = Z_2 = \cdots = Z_{m^2} = W$. Then $i_m(M) \in P$ also for every $M \in Z$. Now, it is clear that $i_A(a) \in O$ for every $a \in W$ because $i_A(a) = p_{1,1} \circ i_m(M_a)$, where M_a is a matrix having all its elements equal to a. Hence, i_A is continuous as well. □

For $m \in \mathbb{N}$, $I_m \equiv I$ denotes the identity matrix in $M_m(A)$. Using Theorem 4.2, we get the following result.

Theorem 4.4. *Let A be a unital sequentially Mackey complete topological $*$-algebra with continuous involution, M a free A-module of rank m and $\alpha : M \times M \to A$ a Hermitian inner product on M. If the matrix $M_\alpha \in M_m(A)$ associated with α fulfils the condition $\beta(M_\alpha - I) < 1$, then there exists a Hermitian inner product $\gamma : M \times M \to A$ such that $M_\alpha = {M_\gamma}^2$.*

Proof. By assumption, m is a free A-module of rank m. Consider the $*$-algebra $M_m(A)$. By Proposition 4.1, $M_m(A)$ is a unital sequentially Mackey complete topological algebra. The involution in $M_m(A)$ is continuous by Lemma 4.3.

Let $\alpha : M \times M \to A$ be a Hermitian inner product on M and let its matrix M_α fulfil the condition $\beta(M_\alpha - I) < 1$. Then, by the first part of Theorem 4.2, there exists a matrix $N \in M_m(A)$ such that $N^2 = M_\alpha$.

Since the involution on $M_m(A)$ is continuous and M_α is a Hermitian matrix, N is Hermitian, by the second part of Theorem 4.2. Moreover, since M_α is invertible, N must be also invertible (its inverse is $N^{-1} = M_\alpha^{-1}N$). Now, by Lemma 3.1, we get that N is actually a matrix of some Hermitian inner product $\gamma : M \times M \to A$, i.e., $N = M_\gamma$. Hence, $M_\alpha = {M_\gamma}^2$ for some Hermitian inner product γ. □

Using Lemma 2.1, we get the following result.

Theorem 4.5. *Let A be a unital sequentially Mackey complete topological $*$-algebra with continuous involution for which the following conditions are fulfilled:*

(a) *If $a \in A$, then $aa^* = \theta_A$ if and only if $a = \theta_A$.*
(b) *If $n \in \mathbb{N}$ and $a_1, \dots, a_n \in A$, then $\mathrm{Sp}_A(\sum_{i=1}^n a_i a_i^*) \subset [0, \infty)$.*
(c) *The only self-adjoint element $a \in A$ with $\mathrm{Sp}_A(a) = \{0\}$ is the zero element θ_A of A.*

Moreover, let M be a free A-module of rank m. Then all Hermitian inner products $\delta : M \times M \to A$, with matrices M_δ such that $\beta(M_\delta - I) < 1$, are mutually equivalent.

Proof. Let δ be a Hermitian inner product for which $\beta(M_\delta - I) < 1$. By Theorem 4.4, there exists a Hermitian inner product $\gamma : M \times M \to A$ such that ${M_\gamma}^2 = M_\delta$. By Lemma 2.1, we know that there exists an inner product $\alpha : M \times M \to A$ with $M_\alpha = I$. Since M_γ is Hermitian, then $M_\gamma^* = M_\gamma$. Therefore, $M_\delta = {M_\gamma}^2 = M_\gamma^* M_\gamma = M_\gamma^* I M_\gamma = M_\gamma^* M_\alpha M_\gamma$. Hence, the Hermitian inner products δ and α are equivalent.

Let $\kappa : M \times M \to A$ be another Hermitian inner product with $\beta(M_\kappa - I) < 1$. As before, we can now show that κ and α are equivalent. Hence, κ is equivalent to δ. Therefore, all such Hermitian inner products δ with $\beta(M_\delta - I) < 1$ are mutually equivalent. □

Let B be a non-unital algebra, $m \in \mathbb{N}$ and J denote the identity matrix in the algebra $M_m(B \times \mathbb{K})$. Suppose that the involution $i_B : B \to B$, defined by $i_B(b) := b^*$ for every $b \in B$, is continuous on B. Take any neighbourhood O of zero in $B \times \mathbb{K}$. Then there exist neighbourhoods of zero U in B and V in $\mathbb{K}$ such that $U \times V \subset O$. Since involution is continuous on B and $\mathbb{K}$, there exist neighbourhoods of zero W in B and Z in $\mathbb{K}$ such that $i_B(b) \in U$ for every $b \in W$ and $i_{\mathbb{K}}(\lambda) \in V$ for every $\lambda \in Z$ (here $i_{\mathbb{K}}$ denotes the involution on $\mathbb{K}$). Denote the involution in $B \times \mathbb{K}$ by $i_{B\times\mathbb{K}}$. Since $P := U \times V$ is a neighbourhood of zero in $B \times \mathbb{K}$ and since $i_{B\times\mathbb{K}}((b, \lambda)) \in O$ for every $(b, \lambda) \in P$, then the involution $i_{B\times\mathbb{K}}$ in $B \times \mathbb{K}$ is also continuous.

From the last two Theorems we can have the following results in nonunital case.

Corollary 4.6. *Let B be a non-unital sequentially Mackey complete topological $*$-algebra with continuous involution, M a free B-module of rank m and*

$\alpha : M \times M \to B \times \mathbb{K}$ a Hermitian inner product on M. If the matrix $M_\alpha \in M_m(B \times \mathbb{K})$, associated with α, fulfils the condition $\beta(M_\alpha - J) < 1$, then there exists a Hermitian inner product $\gamma : M \times M \to B \times \mathbb{K}$ such that $M_\alpha = {M_\gamma}^2$.

Proof. Since $\mathbb{K}$ is complete, it is also Mackey complete. By assumption, B is sequentially Mackey complete, so $B \times \mathbb{K}$, endowed with the product topology, turns to be Mackey complete. For the latter, one can argue as in the proof of Proposition 4.1, that $B \times \mathbb{K}$ is Mackey complete. Since every B-module with m elements in its basis is also a $(B \times \mathbb{K})$-module with the same basis, then we are in the context of Theorem 4.4, if we take $A := B \times \mathbb{K}$. Hence, the claim follows by Theorem 4.4. □

Corollary 4.7. *Let B be a non-unital sequentially Mackey complete topological $*$-algebra with continuous involution for which the following conditions are satisfied:*

(a) *If $b \in B$, then $bb^* = \theta_B$ if and only if $b = \theta_B$.*
(b) *If $n \in \mathbb{N}$, $b_1, \ldots, b_n \in B$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, then*

$$\mathrm{Sp}_{B\times\mathbb{K}}\Big(\sum_{i=1}^{n}(b_i,\lambda_i)(b_i,\lambda_i)^*\Big) \subset [0,\infty).$$

(c) *The only self-adjoint element $(b,\lambda) \in B \times \mathbb{K}$ with $\mathrm{Sp}_{B\times\mathbb{K}}((b,\lambda)) = \{0\}$ is the zero element $(\theta_B, 0)$ of $B \times \mathbb{K}$.*

Moreover, let M be a free B-module of rank m. Then all Hermitian inner products $\delta : M \times M \to B \times \mathbb{K}$ with matrices M_δ such that $\beta(M_\delta - J) < 1$ are mutually equivalent.

Proof. Using the same argumentation as in the proofs of Corollaries 2.2 and 4.6, we see that by taking $A := B \times \mathbb{K}$, we are in the situation of Theorem 4.5, thus the assertion follows. □

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