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ON A HILBERT-TYPE INTEGRAL INEQUALITY IN THE SUBINTERVAL AND ITS OPERATOR EXPRESSION

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ABSTRACT. In this paper, by using the methods of real analysis and functional analysis, a Hilbert-type integral inequality in the subinterval (a, ∞) $(a > 0)$ with the homogeneous kernel of $-\lambda$-degree and a best constant factor and its operator expression are given. As applications, a few improved results, the equivalent forms and some new inequalities with the particular kernels are obtained.

1. INTRODUCTION

If $f, g \geq 0, f, g \in L^2(0, \infty), ||f|| = \{\int_0^\infty f^2(x)dx\}^{\frac{1}{2}}$ and $||g|| = \{\int_0^\infty g^2(x)dx\}^{\frac{1}{2}}$, then we have the following Hilbert's integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \pi ||f|| \cdot ||g||, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is important in analysis and its applications (cf. [2, 5]). Define an integral operator $T : L^2(0, \infty) \to L^2(0, \infty)$ as: for $f(\geq 0) \in L^2(0, \infty)$,

$$T(f)(y) := \int_0^\infty \frac{f(x)}{x+y} dx (y \in (0, \infty)). \tag{1.2}$$

Then inequality (1.1) is rewritten as: $(Tf, g) \leq \pi ||f|| \cdot ||g||$, where $(Tf, g) := \int_0^\infty (\int_0^\infty \frac{f(x)}{x+y} dx) g(y) dy$ is the inner product of Tf and g. We named of T Hilbert

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integral operator. By (1.1), we can prove the equivalent form that $||Tf|| \leq \pi||f||$, and conclude that $||T|| = \pi$; [1].

If we replace $\frac{1}{x+y}$ by a bilinear function $k(x,y)(\geq 0)$ in (1.1), then the problem is how to make sure the conditions of $k(x,y)$ for giving an integral operator T as (1.2) and the inequality with a best constant factor as (1.1). In recent years, Yang [7, 8] considered the case of $k(x,y)$ being continuous and symmetric in the function space $L^p(0,\infty)$, Yang [9, 10, 11] considered the same case of $k(x,y)$ in the disperse space l^p, and Zhong et al. [18] considered the case of $k(x,y)$ in $L^p(R^n_+)$. But their given conditions are not quite simple.

In 1998, by introducing $\lambda \in (0,1]$ and the Beta function $B(u,v)$ as[6]:

$$B(u,v) := \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{-u+1} dt (u,v>0), \tag{1.3}$$

Yang [12] gave an extension of (1.1) in the subinterval $(a,\infty)(a>0)$ as:

$$\int_a^\infty \int_a^\infty \frac{f(x)g(y)dxdy}{(x+y)^\lambda} \leq k_\lambda \{\int_a^\infty \sigma(x)x^{1-\lambda} f^2(x)dx \int_a^\infty \sigma(x)x^{1-\lambda} g^2(x)dx\}^{\frac{1}{2}}, \tag{1.4}$$

where $k_\lambda = B(\frac{\lambda}{2},\frac{\lambda}{2})$ and $\sigma(x) = 1 - \frac{1}{2}(\frac{a}{x})^{\frac{\lambda}{2}}$. When $\lambda = 1, a \to 0^+$, inequality (1.4) deduces to (1.1). In recent years, a number of papers studied some improvements and extensions of (1.4) (cf. [13, 14, 15, 17]).

In this paper, a simple condition of the homogeneous kernel with $-\lambda$-degree $(\lambda > 0)$ is considered. By using the methods of real analysis and functional analysis, a Hilbert-type integral inequality in the subinterval (a,∞) with the homogeneous kernel and a best constant factor and its operator expression are given. As applications, a few improved results, the equivalent forms and some new inequalities with the particular kernels are obtained.

2. Lemmas and main results

If $\lambda > 0$, the function $k_\lambda(x,y)$ is non-negative measurable in $(0,\infty)\times(0,\infty)$, satisfying $k_\lambda(ux,uy) = u^{-\lambda}k_\lambda(x,y)$ for any $u,x,y>0$, then we call $k_\lambda(x,y)$ the homogeneous function of $-\lambda-$degree; if for any $x,y>0, k_\lambda(x,y) = k_\lambda(y,x)$, then we call the homogeneous function $k_\lambda(x,y)$ is symmetric. Assume that $r > 1, \frac{1}{r}+\frac{1}{s} = 1$. Setting $k_\lambda(r)$ and $\widetilde{k}_\lambda(s)$ as

$$k_\lambda(r) := \int_0^\infty k_\lambda(u,1)u^{\frac{\lambda}{r}-1}du, \widetilde{k}_\lambda(s) := \int_0^\infty k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du,$$

then it follows $k_\lambda(r) = \widetilde{k}_\lambda(s)$. In fact, setting $v = \frac{1}{u}$, we obtain

$$\widetilde{k}_\lambda(s) = \int_0^\infty k_\lambda(1,\frac{1}{v})v^{\frac{-\lambda}{s}+1}\frac{dv}{v^2} = \int_0^\infty k_\lambda(v,1)v^{\frac{\lambda}{r}-1}dv = k_\lambda(r).$$

Suppose that $k_\lambda(r)$ is a positive number. For $a>0, x,y\in(a,\infty)$, define the weight functions $\omega_\lambda(r,y,a)$ and $\varpi_\lambda(s,x,a)$ as:

$$\omega_\lambda(r,y,a) := \int_a^\infty k_\lambda(x,y)\frac{y^{\frac{\lambda}{s}}}{x^{1-\frac{\lambda}{r}}}dx, \varpi_\lambda(s,x,a) := \int_a^\infty k_\lambda(x,y)\frac{x^{\frac{\lambda}{r}}}{y^{1-\frac{\lambda}{s}}}dy. \tag{2.1}$$

Setting $u = \frac{y}{x}$ in the integral $\omega_\lambda(r, y, a)$, for any $y \in (a, \infty)$, we find

$$\omega_\lambda(r, y, a) = \int_0^{\frac{y}{a}} k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du \leq \int_0^{\infty} k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du = \widetilde{k}_\lambda(s).$$

Similarly, $\varpi_\lambda(s, x, a) \leq k_\lambda(r)$ $(x \in (a, \infty))$. Setting $\theta_\lambda(r)$ and $\widetilde{\theta}_\lambda(s)$ as

$$\theta_\lambda(r) := \int_0^1 k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du, \widetilde{\theta}_\lambda(s) := \int_0^1 k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du,$$

if $\theta_\lambda(r), \widetilde{\theta}_\lambda(s) > 0$, then for any $y > a$, we find

$$\omega_\lambda(r, y, a) = \int_0^{\frac{y}{a}} k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du \geq \int_0^1 k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du = \widetilde{\theta}_\lambda(s) > 0.$$

Similarly, $\varpi_\lambda(s, x, a) \geq \theta_\lambda(r) > 0$ $(x > a)$. Hence by (2.1), for fixed $y > a, k_\lambda(x, y) > 0$ *a.e.* in (a, ∞), and for fixed $x > a, k_\lambda(x, y) > 0$ a.e. in (a, ∞).

Lemma 2.1. *If both $k_\lambda(1, u), k_\lambda(u, 1) \geq l_\lambda > 0, u \in (0, 1]$, then we have*

$$\omega_\lambda(r, y, a) \leq k_\lambda(r)[1 - \frac{r l_\lambda}{\lambda k_\lambda(r)} (\frac{a}{y})^{\frac{\lambda}{r}}] \ (y \in (a, \infty)); \tag{2.2}$$

$$\varpi_\lambda(s, x, a) \leq k_\lambda(r)[1 - \frac{s l_\lambda}{\lambda k_\lambda(r)} (\frac{a}{x})^{\frac{\lambda}{s}}] \ (x \in (a, \infty)). \tag{2.3}$$

Proof. Setting $u = \frac{x}{y}$, we find

$$\begin{aligned} \omega_\lambda(r, y, a) &= \int_{\frac{a}{y}}^{\infty} k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du = k_\lambda(r) - \int_0^{\frac{a}{y}} k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du \\ &\leq k_\lambda(r) - l_\lambda \int_0^{\frac{a}{y}} u^{\frac{\lambda}{r}-1} du = k_\lambda(r) - \frac{r l_\lambda}{\lambda} (\frac{a}{y})^{\frac{\lambda}{r}} \ (y \in (a, \infty)). \end{aligned}$$

Hence we have (2.2). Similarly we have (2.3). The lemma is proved. □

Lemma 2.2. *If both $k_\lambda(1, u)$ and $k_\lambda(u, 1)$ are derivable decreasing function in $(0, 1]$, then we have*

$$\omega_\lambda(r, y, a) \leq k_\lambda(r)[1 - \frac{\theta_\lambda(r)}{k_\lambda(r)} (\frac{a}{y})^{\frac{\lambda}{r}}] \ (y \in (a, \infty)); \tag{2.4}$$

$$\varpi_\lambda(s, x, a) \leq k_\lambda(r)[1 - \frac{\widetilde{\theta}_\lambda(s)}{k_\lambda(r)} (\frac{a}{x})^{\frac{\lambda}{s}}] \ (x \in (a, \infty)). \tag{2.5}$$

In particular, if $k_\lambda(x, y)$ is symmetric, setting $k_\lambda := k_\lambda(2)$, then

$$\omega_\lambda(2, y, a) \leq k_\lambda[1 - \frac{1}{2} (\frac{a}{y})^{\frac{\lambda}{2}}]; \varpi_\lambda(2, x, a) \leq k_\lambda[1 - \frac{1}{2} (\frac{a}{x})^{\frac{\lambda}{2}}] \ (x, y > a). \tag{2.6}$$

Proof. Since $k'_\lambda(u, 1) \leq 0, u \in (0, 1)$, for $y \in (0, 1)$, we obtain

$$\begin{aligned} \frac{d}{dy} [y^{\frac{-\lambda}{r}} \int_0^y k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du] &= \frac{-\lambda}{r} y^{\frac{-\lambda}{r}} \int_0^y k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du + k_\lambda(y, 1) y^{-1} \\ &= -y^{\frac{-\lambda}{r}} \int_0^y k_\lambda(u, 1) du^{\frac{\lambda}{r}} + k_\lambda(y, 1) y^{-1} = y^{\frac{-\lambda}{r}} \int_0^y k'_\lambda(u, 1) u^{\frac{\lambda}{r}} du \leq 0, \end{aligned}$$

and $y^{\frac{-\lambda}{r}}\int_0^y k_\lambda(u,1)u^{\frac{\lambda}{r}-1}du \geq \int_0^1 k_\lambda(u,1)u^{\frac{\lambda}{r}-1}du = \theta_\lambda(r)$. Hence we find

$$\begin{aligned}\omega_\lambda(r,y,a) &= k_\lambda(r) - [(\frac{a}{y})^{\frac{-\lambda}{r}}\int_0^{\frac{a}{y}} k_\lambda(u,1)u^{\frac{\lambda}{r}-1}du](\frac{a}{y})^{\frac{\lambda}{r}} \\ &\leq k_\lambda(r) - \theta_\lambda(r)(\frac{a}{y})^{\frac{\lambda}{r}}\ (y \in (a,\infty)).\end{aligned}$$

Then we obtain (2.4). Similarly, we obtain (2.5). If $k_\lambda(x,y)$ is symmetric, then we find $\theta_\lambda(2) = \widetilde{\theta}_\lambda(2)$ and

$$k_\lambda = \theta_\lambda(2) + \int_1^\infty k_\lambda(1,u)u^{\frac{\lambda}{2}-1}du = \theta_\lambda(2) + \int_0^1 k_\lambda(v,1)v^{\frac{\lambda}{2}-1}dv = 2\theta_\lambda(2).$$

Then by (2.4) and (2.5), we have (2.6). The lemma is proved. □

For the measurable function $\varphi(x) > 0$, set the function spaces as:

$$L_\varphi^\rho(a,\infty) := \{h \geq 0; ||h||_{\rho,\varphi} := \{\int_a^\infty \varphi(x)h^\rho(x)dx\}^{\frac{1}{\rho}} < \infty\}(\rho = p,q).$$

Theorem 2.3. *Assume that* $p,r > 1, \frac{1}{p}+\frac{1}{q} = 1, \frac{1}{r}+\frac{1}{s} = 1,\ \lambda > 0,\ k_\lambda(x,y)$ *is a homogeneous function of* $-\lambda-$*degree in* $(0,\infty)\times(0,\infty)$, *satisfying* $k_\lambda(r), \theta_\lambda(r)$ *and* $\widetilde{\theta}_\lambda(s)$ *are positive numbers. For* $a > 0$, *there exist measurable functions* $\kappa(y)$ *and* $\widetilde{\mu}(x)$, *such that* $0 < \kappa(y), \widetilde{\mu}(x) \leq 1$ *and*

$$\omega_\lambda(r,y,a) \leq k_\lambda(r)\kappa(y), \varpi_\lambda(s,x,a) \leq k_\lambda(r)\widetilde{\mu}(x)(x,y \in (a,\infty)). \tag{2.7}$$

If $\phi_r(x) := x^{p(1-\frac{\lambda}{r})-1}, \psi_s(x) := x^{q(1-\frac{\lambda}{s})-1}(x \in (a,\infty)), f \in L^p_{\phi_r}(a,\infty), g \in L^q_{\psi_s}(a,\infty)$, $||f||_{p,\phi_r}, ||g||_{q,\psi_s} > 0$, *then we have the equivalent inequalities as*

$$I_\lambda(a) := \int_a^\infty\int_a^\infty k_\lambda(x,y)f(x)g(y)dxdy < k_\lambda(r)||f||_{p,\widetilde{\mu}\cdot\phi_r}||g||_{q,\kappa\cdot\psi_s}; \tag{2.8}$$

$$J_\lambda(a) := \int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)}(\int_a^\infty k_\lambda(x,y)f(x)dx)^p dy < k_\lambda^p(r)||f||^p_{p,\widetilde{\mu}\cdot\phi_r}, \tag{2.9}$$

where the constant factors $k_\lambda(r)$ *and* $k_\lambda^p(r)$ *are the best possible.*

In particular, for $\kappa(y) = \widetilde{\mu}(x) = 1$, *we have the equivalent inequalities as:*

$$I_\lambda(a) < k_\lambda(r)\{\int_a^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy\}^{\frac{1}{q}}; \tag{2.10}$$

$$\int_a^\infty y^{\frac{p\lambda}{s}-1}(\int_a^\infty k_\lambda(x,y)f(x)dx)^p dy < k_\lambda^p(r)\int_a^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx. \tag{2.11}$$

Proof. Since $0 < \theta_\lambda(r)/k_\lambda(r) \leq \kappa(y) \leq k_\lambda(r), 0 < \widetilde{\theta}_\lambda(s)/k_\lambda(r) \leq \widetilde{\mu}(x) \leq k_\lambda(r)$ $(x,y \in (a,\infty))$, it is obvious that the condition $0 < ||f||_{p,\phi_r}, ||g||_{q,\psi_s} < \infty$ is equivalent to the condition $0 < ||f||_{p,\widetilde{\mu}\cdot\phi_r}, ||g||_{q,\kappa\cdot\psi_s} < \infty$.

By Hölder's inequality [3], in view of (2.1) and Fubini's theorem [4], we find

$$I_\lambda(a) = \int_a^\infty\int_a^\infty k_\lambda(x,y)[\frac{x^{(1-\frac{\lambda}{r})/q}}{y^{(1-\frac{\lambda}{s})/p}}f(x)][\frac{y^{(1-\frac{\lambda}{s})/p}}{x^{(1-\frac{\lambda}{r})/q}}g(y)]dxdy$$

$$\leq \{\int_a^\infty \int_a^\infty k_\lambda(x,y)\frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}}f^p(x)dxdy\}^{\frac{1}{p}} \times\{\int_a^\infty \int_a^\infty k_\lambda(x,y)\frac{y^{(1-\frac{\lambda}{s})(q-1)}}{x^{1-\frac{\lambda}{r}}}g^q(y)dxdy\}^{\frac{1}{q}}$$
$$= \{\int_a^\infty \varpi_\lambda(s,x,a)\phi_r(x)f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty \omega_\lambda(r,y,a)\psi_s(y)g^q(y)dy\}^{\frac{1}{q}}. \quad (2.12)$$

If inequality (2.12) keeps the form of equality, then [3] there exist constants A and B, such that they are not all zero and

$$A\frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}}f^p(x) = B\frac{y^{(1-\frac{\lambda}{s})(q-1)}}{x^{1-\frac{\lambda}{r}}}g^q(y) \ \ a.e. \text{ in } (a,\infty)\times(a,\infty).$$

It follows $Ax^{p(1-\frac{\lambda}{r})}f^p(x) = By^{q(1-\frac{\lambda}{s})}g^q(y)$ a.e. in $(a,\infty)\times(a,\infty)$. Assuming that $A \neq 0$, there exists $y > a$, $x^{p(1-\frac{\lambda}{r})-1}f^p(x) = [By^{q(1-\frac{\lambda}{s})}g^q(y)]\frac{1}{Ax}$ a.e. in $x \in (a,\infty)$. This contradicts the fact that $0 < ||f||_{p,\phi_r} < \infty$. Then inequality (2.12) keeps the strict form and inequality (2.12) is valid by using (2.7).

For $x \in (a,\infty)$, setting a bounded measurable function $[f(x)]_n$ as

$$[f(x)]_n := \min\{f(x),n\} = \begin{cases} f(x), & \text{for } f(x) \leq n \\ n, & \text{for } f(x) > n, \end{cases}$$

since $||f||_{p,\phi_r} > 0$, there exists $n_0 \in N$, such that $\int_a^n \phi_r(x)[f(x)]_n^p dx > 0 (n \geq n_0)$, and then $\int_a^n \widetilde{\mu}(x)\phi_r(x)[f(x)]_n^p dx > 0$. Setting $\widetilde{g}_n(y)$ as

$$\widetilde{g}_n(y) := \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)}(\int_a^n k_\lambda(x,y)[f(x)]_n dx)^{p-1} \, (y \in (a,n); n \geq n_0),$$

then by (2.8), we have

$$0 < \int_a^n \kappa(y)\psi_s(y)\widetilde{g}_n^q(y)dy = \int_a^n \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)}(\int_a^n k_\lambda(x,y)[f(x)]_n dx)^p dy$$
$$= \int_a^n \int_a^n k_\lambda(x,y)[f(x)]_n \widetilde{g}_n(y)dxdy$$
$$< k_\lambda(r)\{\int_a^n \widetilde{\mu}(x)\phi_r(x)[f(x)]_n^p dx\}^{\frac{1}{p}}\{\int_a^n \kappa(y)\psi_s(y)\widetilde{g}_n^q(y)dy\}^{\frac{1}{q}} < \infty; \quad (2.13)$$

$$0 < \int_a^n \kappa(y)\psi_s(y)\widetilde{g}_n^q(y)dy < k_\lambda^p(r)\int_a^\infty \widetilde{\mu}(x)\phi_r(x)f^p(x)dx < \infty. \quad (2.14)$$

It follows $0 < ||g||_{q,\kappa\cdot\psi_s} < \infty$ and $0 < ||g||_{q,\psi_s} < \infty$. For $n \to \infty$, by (2.8), both (2.13) and (2.14) still keep the forms of strict inequality. Hence we have (2.9).

On the other-hand, suppose that (2.9) is valid. By Hölder's inequality,

$$J_\lambda(a) = \int_a^\infty [y^{\frac{-1}{p}+\frac{\lambda}{s}}\kappa^{\frac{-1}{q}}(y)\int_a^\infty k_\lambda(x,y)f(x)dx][\kappa^{\frac{1}{q}}(y)y^{\frac{1}{p}-\frac{\lambda}{s}}g(y)]dy$$

$$\leq \{\int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)}(\int_a^\infty k_\lambda(x,y)f(x)dx)^p dy\}^{\frac{1}{p}}\{\int_a^\infty \kappa(y)\psi_s(y)g^q(y)dy\}^{\frac{1}{q}}. \quad (2.15)$$

In view of (2.9), we have (2.8). Hence (2.8) is equivalent to (2.9).

For $n \in N, n > \max\{\frac{\lambda}{r}, \frac{\lambda}{s}\}$, setting f_n, g_n as: $f_n(x) = x^{\frac{\lambda}{r}-\frac{1}{np}-1}, g_n(x) = x^{\frac{\lambda}{s}-\frac{1}{nq}-1}$, for $x \in (a,\infty)$, if there exists $0 < K \leq k_\lambda(r) = \widetilde{k}_\lambda(s)$, such that (2.8) is still valid if we replace $k_\lambda(r)$ by K, then we have

$$I_\lambda^{(n)}(a) := \int_a^\infty \int_a^\infty k_\lambda(x,y) f_n(x) g_n(y) dxdy < K||f_n||_{p,\cdot\phi_r}||g_n||_{q,\psi_s} = \frac{nK}{a^{\frac{1}{n}}}; \quad (2.16)$$

$$I_\lambda^{(n)}(a) = \int_a^\infty [\int_a^\infty k_\lambda(x,y) x^{\frac{\lambda}{r}-\frac{1}{np}-1} y^{\frac{\lambda}{s}-\frac{1}{nq}-1} dx] dy$$

$$\begin{aligned} &\overset{u=y/x}{=} \int_a^\infty y^{-1-\frac{1}{n}} [\int_0^{\frac{y}{a}} k_\lambda(1,u) u^{\frac{\lambda}{s}+\frac{1}{np}-1} du] dy \\ = &\frac{n}{a^{\frac{1}{n}}} \int_0^1 k_\lambda(1,u) u^{\frac{\lambda}{s}+\frac{1}{np}-1} du + \int_a^\infty y^{-1-\frac{1}{n}} \int_1^{\frac{y}{a}} k_\lambda(1,u) u^{\frac{\lambda}{s}+\frac{1}{np}-1} dudy \\ = &\frac{n}{a^{\frac{1}{n}}} \int_0^1 k_\lambda(1,u) u^{\frac{\lambda}{s}+\frac{1}{np}-1} du + \int_1^\infty (\int_{au}^\infty y^{-1-\frac{1}{n}} dy) k_\lambda(1,u) u^{\frac{\lambda}{s}+\frac{1}{np}-1} du \\ = &\frac{n}{a^{\frac{1}{n}}} [\int_0^1 k_\lambda(1,u) u^{\frac{\lambda}{s}+\frac{1}{np}-1} du + \int_1^\infty k_\lambda(1,u) u^{\frac{\lambda}{s}-\frac{1}{nq}-1} du]. \end{aligned} \quad (2.17)$$

Hence by (2.16) and (2.17), we have

$$\int_0^1 k_\lambda(1,u) u^{\frac{\lambda}{s}+\frac{1}{np}-1} du + \int_1^\infty k_\lambda(1,u) u^{\frac{\lambda}{s}-\frac{1}{nq}-1} du < K,$$

and by Fatou's lemma [4], it follows

$$\begin{aligned} \widetilde{k}_\lambda(s) &= \int_0^1 \lim_{n\to\infty} k_\lambda(1,u) u^{\frac{\lambda}{s}+\frac{1}{np}-1} du + \int_1^\infty \lim_{n\to\infty} k_\lambda(1,u) u^{\frac{\lambda}{s}-\frac{1}{nq}-1} du \\ &\leq \underline{\lim}_{n\to\infty} [\int_0^1 k_\lambda(1,u) u^{\frac{\lambda}{s}+\frac{1}{np}-1} du + \int_1^\infty k_\lambda(1,u) u^{\frac{\lambda}{s}-\frac{1}{nq}-1} du] \leq K. \end{aligned}$$

Therefore $K = k_\lambda(r)$ is the best constant factor of (2.8). If the constant factor in (2.9) is not the best possible, then by (2.15), we can get a contradiction that the constant factor in (2.8) is not the best possible. The theorem is proved. □

Define an operator $T_a : L^p_{\phi_r}(a,\infty) \to L^p_{\psi_s^{1-p}}(a,\infty)$ as: for $f \in L^p_{\phi_r}(a,\infty)$,

$$(T_a f)(y) := \int_a^\infty k_\lambda(x,y) f(x) dx (y \in (a,\infty)).$$

In view of (2.11), it follows $T_a f \in L^p_{\psi_s^{1-p}}(a,\infty)$. For $g \in L^q_{\psi_s}(a,\infty)$, define the formal inner of $T_a f$ and g as:

$$(T_a f, g) := \int_a^\infty \int_a^\infty k_\lambda(x,y) f(x) g(y) dxdy.$$

Hence the equivalent inequalities (2.10) and (2.11) may be rewritten as

$$(T_a f, g) < k_\lambda(r)||f||_{p,\phi_r}||g||_{q,\psi_s}; ||T_a f||_{p,\psi_s^{1-p}} < k_\lambda(r)||f||_{p,\phi_r},$$

where the constant factor is the best possible, T_a is obviously bounded and $||T_a|| = k_\lambda(r)$. We call T_a Hilbert-type integral operator with the homogeneous kernel of $-\lambda$-degree in the subinterval (a,∞).

Corollary 2.4. *Let the assumptions of Theorem 2.3 be fulfilled and additionally both* $k_\lambda(1,u), k_\lambda(u,1) \geq l_\lambda > 0$, $u \in (0,1]$. *Then we have the following equivalent inequalities:*

$$\begin{aligned}(T_a f, g) &< k_\lambda(r)\{\int_a^\infty [1 - \frac{sl_\lambda}{\lambda k_\lambda(r)}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x) f^p(x)dx\}^{\frac{1}{p}} \\ &\quad \times\{\int_a^\infty [1 - \frac{rl_\lambda}{\lambda k_\lambda(r)}(\frac{a}{y})^{\frac{\lambda}{r}}]\psi_s(y) g^q(y)dy\}^{\frac{1}{q}}; \qquad (2.18)\end{aligned}$$

$$\begin{aligned}&\int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{[1 - \frac{rl_\lambda}{\lambda k_\lambda(r)}(\frac{a}{y})^{\frac{\lambda}{r}}]^{p-1}}(\int_a^\infty k_\lambda(x,y) f(x)dx)^p dy \\ &< k_\lambda^p(r)\int_a^\infty [1 - \frac{sl_\lambda}{\lambda k_\lambda(r)}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x) f^p(x)dx, \qquad (2.19)\end{aligned}$$

where the constant factors $k_\lambda(r)$ *and* $k_\lambda^p(r)$ *are the best possible. We still have the following two pairs of equivalent inequalities:*

$$(T_a f, g) < k_\lambda(r)\{\int_a^\infty [1 - \frac{sl_\lambda}{\lambda k_\lambda(r)}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x) f^p(x)dx\}^{\frac{1}{p}}||g||_{q,\psi_s}, \qquad (2.20)$$

$$||T_a f||^p_{p,\psi_s^{1-p}} < k_\lambda^p(r)\int_a^\infty [1 - \frac{sl_\lambda}{\lambda k_\lambda(r)}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x) f^p(x)dx; \qquad (2.21)$$

$$(T_a f, g) < k_\lambda(r)||f||_{p,\phi_r}\{\int_a^\infty [1 - \frac{rl_\lambda}{\lambda k_\lambda(r)}(\frac{a}{y})^{\frac{\lambda}{r}}]\psi_s(y) g^q(y)dy\}^{\frac{1}{q}}, \qquad (2.22)$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{[1 - \frac{rl_\lambda}{\lambda k_\lambda(r)}(\frac{a}{y})^{\frac{\lambda}{r}}]^{p-1}}(\int_a^\infty k_\lambda(x,y) f(x)dx)^p dy < k_\lambda^p(r)||f||^p_{p,\phi_r}. \qquad (2.23)$$

Proof. By Lemma 2.1, setting $\kappa(y) = 1 - \frac{rl_\lambda}{\lambda k_\lambda(r)}(\frac{a}{y})^{\frac{\lambda}{r}}$ and $\widetilde{\mu}(x) = 1 - \frac{sl_\lambda}{\lambda k_\lambda(r)}(\frac{a}{x})^{\frac{\lambda}{s}}$ in (2.8) and (2.9), we have (2.18) and (2.19). Since $\kappa(y) \leq 1$, by (2.18) and (2.19), we have (2.20) and (2.21). Similarly, since $\widetilde{\mu}(x) \leq 1$, we have (2.22) and (2.23). The corollary is proved. □

Corollary 2.5. *Let the assumptions of Theorem 2.3 be fulfilled and additionally both* $k_\lambda(1,u)$ *and* $k_\lambda(u,1)$ *are derivable decreasing function in* $(0,1]$. *Then we have the following equivalent inequalities with the best constant factors:*

$$\begin{aligned}(T_a f, g) &< k_\lambda(r)\{\int_a^\infty [1 - \frac{\widetilde{\theta}_\lambda(s)}{k_\lambda(r)}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x) f^p(x)dx\}^{\frac{1}{p}} \\ &\quad \times\{\int_a^\infty [1 - \frac{\theta_\lambda(r)}{k_\lambda(r)}(\frac{a}{y})^{\frac{\lambda}{r}}]\psi_s(y) g^q(y)dy\}^{\frac{1}{q}}; \qquad (2.24)\end{aligned}$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{[1-\frac{\theta_\lambda(r)}{k_\lambda(r)}(\frac{a}{y})^{\frac{\lambda}{r}}]^{p-1}}(\int_a^\infty k_\lambda(x,y)f(x)dx)^p dy$$
$$< \ k_\lambda^p(r)\int_a^\infty [1-\frac{\widetilde{\theta}_\lambda(s)}{k_\lambda(r)}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x)f^p(x)dx. \tag{2.25}$$

If $k_\lambda(x,y)$ is symmetric and $\sigma(x) = 1-\frac{1}{2}(\frac{a}{x})^{\frac{\lambda}{2}}$ as (1.4), then we have the equivalent inequalities as:

$$I_\lambda(a) < k_\lambda\{\int_a^\infty \sigma(x)\phi_2(x)f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty \sigma(y)\psi_2(y)g^q(y)dy\}^{\frac{1}{q}}; \tag{2.26}$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{2}-1}}{[\sigma(y)]^{p-1}}(\int_a^\infty k_\lambda(x,y)f(x)dx)^p dy < k_\lambda^p\int_a^\infty \sigma(x)\phi_2(x)f^p(x)dx. \tag{2.27}$$

Proof. By Lemma 2.2, setting $\kappa(y) = 1-\frac{\theta_\lambda(r)}{k_\lambda(r)}(\frac{a}{y})^{\frac{\lambda}{r}}$ and $\widetilde{\mu}(x) = 1-\frac{\widetilde{\theta}_\lambda(s)}{k_\lambda(r)}(\frac{a}{x})^{\frac{\lambda}{s}}$ in (2.8) and (2.9), we have (2.24) and (2.25). For $r = s = 2$, by (2.6), we have (2.26) and (2.27). The corollary is proved. □

3. Applications to some particular kernels

In the following, we assume that $a,\lambda > 0, p, r > 1, \frac{1}{p}+\frac{1}{q} = 1, \frac{1}{r}+\frac{1}{s} = 1, \phi_r(x) := x^{p(1-\frac{\lambda}{r})-1}, \psi_s(x) := x^{q(1-\frac{\lambda}{s})-1}, \sigma(x) = 1-\frac{1}{2}(\frac{a}{x})^{\frac{\lambda}{2}}, f, g \geq 0, 0 < ||f||_{p,\phi_r}, ||g||_{q,\psi_s} < \infty$. Some words that the constants are the best possible are omitted.

Example 3.1. Let $k_\lambda(x,y) = \frac{1}{(x^\alpha+y^\alpha)^{\lambda/\alpha}}$ $(\alpha > 0)$, which is symmetric. Since both $k_\lambda(1,u)$ and $k_\lambda(u,1)$ are derivable decreasing in $(0,1]$, and $k_\lambda(u,1) = \frac{1}{(u^\alpha+1)^{\lambda/\alpha}} \geq l_\lambda = \frac{1}{2^{\lambda/\alpha}}$ $(u \in (0,1])$, setting $v = u^\alpha$, by (1.3), we have

$$\widetilde{k}_\lambda(r) := \int_0^\infty \frac{u^{\frac{\lambda}{r}-1}du}{(u^\alpha+1)^{\frac{\lambda}{\alpha}}} = \frac{1}{\alpha}\int_0^\infty \frac{v^{\frac{\lambda}{\alpha r}-1}dv}{(v+1)^{\frac{\lambda}{\alpha}}} = \frac{1}{\alpha}B(\frac{\lambda}{\alpha r},\frac{\lambda}{\alpha s}).$$

By (2.18), (2.19) and (2.26), (2.27), we have two pairs of equivalent inequalities as:

$$\begin{aligned} H(a) \ &:= \int_a^\infty\int_a^\infty \frac{f(x)g(y)dxdy}{(x^\alpha+y^\alpha)^{\frac{\lambda}{\alpha}}} \\ &< \widetilde{k}_\lambda(r)\{\int_a^\infty [1-\frac{s}{2^{\frac{\lambda}{\alpha}}\lambda\widetilde{k}_\lambda(r)}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x)f^p(x)dx\}^{\frac{1}{p}} \\ &\quad \times\{\int_a^\infty [1-\frac{r}{2^{\frac{\lambda}{\alpha}}\lambda\widetilde{k}_\lambda(r)}(\frac{a}{y})^{\frac{\lambda}{r}}]\psi_s(y)g^q(y)dy\}^{\frac{1}{q}}, \end{aligned}$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{[1-\frac{r}{2^{\lambda/\alpha}\lambda\widetilde{k}_\lambda(r)}(\frac{a}{y})^{\frac{\lambda}{r}}]^{p-1}}[\int_a^\infty \frac{f(x)}{(x^\alpha+y^\alpha)^{\frac{\lambda}{\alpha}}}dx]^p dy$$
$$< \ \widetilde{k}_\lambda^p(r)\int_a^\infty [1-\frac{s}{2^{\frac{\lambda}{\alpha}}\lambda\widetilde{k}_\lambda(r)}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x)f^p(x)dx;$$

$$H(a) < \widetilde{k}_\lambda(2)\{\int_a^\infty \sigma(x)\phi_2(x)f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty \sigma(y)\psi_2(y)g^q(y)dy\}^{\frac{1}{q}}, \qquad (3.1)$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{2}-1}}{\sigma^{p-1}(y)}[\int_a^\infty \frac{f(x)dx}{(x^\alpha+y^\alpha)^{\frac{\lambda}{\alpha}}}]^p dy < \widetilde{k}_\lambda^p(2)\int_a^\infty \sigma(x)\phi_2(x)f^p(x)dx.$$

Example 3.2. Let $k_\lambda(x,y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}$, which is symmetric. We find that both $k_\lambda(1,u)$ and $k_\lambda(u,1)$ are derivable decreasing in $(0,1]$, [16] and $k_\lambda(u,1) = \frac{\ln u}{u^\lambda - 1} \geq l_\lambda = \frac{1}{\lambda}$ $(u \in (0,1])$. Setting $v = u^\lambda$, we obtain [2]

$$k_\lambda(r) = \int_0^\infty \frac{(\ln u)u^{\frac{\lambda}{r}-1}}{u^\lambda - 1}du = \int_0^\infty \frac{(\ln v)v^{\frac{1}{r}-1}}{\lambda^2(v-1)}dv = [\frac{\pi}{\lambda\sin(\frac{\pi}{r})}]^2.$$

By (2.18), (2.19) and (2.26), (2.27), we have two pairs of equivalent inequalities as:

$$\begin{aligned} H'(a) &:= \int_a^\infty\int_a^\infty \frac{\ln(\frac{x}{y})f(x)g(y)}{x^\lambda - y^\lambda}dxdy \\ &< [\frac{\pi}{\lambda\sin(\frac{\pi}{r})}]^2\{\int_a^\infty [1 - s[\frac{\sin(\frac{\pi}{r})}{\pi}]^2(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x)f^p(x)dx\}^{\frac{1}{p}} \\ &\quad \times\{\int_a^\infty [1 - r[\frac{\sin(\frac{\pi}{r})}{\pi}]^2(\frac{a}{y})^{\frac{\lambda}{r}}]\psi_s(y)g^q(y)dy\}^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} &\int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{[1 - r[\frac{\sin(\frac{\pi}{r})}{\pi}]^2(\frac{a}{y})^{\frac{\lambda}{r}}]^{p-1}}(\int_a^\infty \frac{\ln(\frac{x}{y})f(x)}{x^\lambda - y^\lambda}dx)^p dy \\ &< [\frac{\pi}{\lambda\sin(\frac{\pi}{r})}]^{2p}\int_a^\infty [1 - s[\frac{\sin(\frac{\pi}{r})}{\pi}]^2(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x)f^p(x)dx; \end{aligned}$$

$$H'(a) < (\frac{\pi}{\lambda})^2\{\int_a^\infty \sigma(x)\phi_2(x)f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty \sigma(y)\psi_2(y)g^q(y)dy\}^{\frac{1}{q}},$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{2}-1}}{\sigma^{p-1}(y)}[\int_a^\infty \frac{\ln(\frac{x}{y})f(x)}{x^\lambda - y^\lambda}dx]^p dy < (\frac{\pi}{\lambda})^{2p}\int_a^\infty \sigma(x)\phi_2(x)f^p(x)dx.$$

Example 3.3. Let $k_\lambda(x,y) = \frac{1}{(\max\{x,y\})^\lambda}$, which is symmetric. Since both $k_\lambda(1,u)$ and $k_\lambda(u,1)$ are derivable decreasing in $(0,1]$, and $k_\lambda(u,1) = l_\lambda = 1$ $(u \in (0,1])$, we have

$$k_\lambda(r) = \int_0^\infty \frac{u^{\frac{\lambda}{r}-1}}{(\max\{u,1\})^\lambda}du = \int_0^1 u^{\frac{\lambda}{r}-1}du + \int_1^\infty \frac{u^{\frac{\lambda}{r}-1}}{u^\lambda}du = \frac{rs}{\lambda}.$$

Then by (2.18) and (2.19), we have two equivalent inequalities as follows:

$$\begin{aligned} \int_a^\infty\int_a^\infty \frac{f(x)g(y)dxdy}{(\max\{x,y\})^\lambda} &< \frac{rs}{\lambda}\{\int_a^\infty [1 - \frac{1}{r}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x)f^p(x)dx\}^{\frac{1}{p}} \\ &\quad \times\{\int_a^\infty [1 - \frac{1}{s}(\frac{a}{y})^{\frac{\lambda}{r}}]\psi_s(y)g^q(y)dy\}^{\frac{1}{q}}, \end{aligned}$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{[1-\frac{1}{s}(\frac{a}{y})^{\frac{\lambda}{r}}]^{p-1}}[\int_a^\infty \frac{f(x)dx}{(\max\{x,y\})^\lambda}]^p dy$$
$$< (\frac{rs}{\lambda})^p \int_a^\infty [1-\frac{1}{r}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x)f^p(x)dx;$$

Example 3.4. Let $k_\lambda(x,y) = \frac{|\ln(x/y)|}{(\max\{x,y\})^\lambda}$, which is symmetric. We find that

$$\begin{aligned} k_\lambda(r) &= \int_0^\infty \frac{|\ln u|u^{\frac{\lambda}{r}-1}du}{(\max\{u,1\})^\lambda} \\ &= \int_0^1 (-\ln u)u^{\frac{\lambda}{r}-1}du + \int_1^\infty \frac{(\ln u)u^{\frac{\lambda}{r}-1}}{u^\lambda}du = \frac{r^2+s^2}{\lambda^2}; \end{aligned}$$

$$\begin{aligned} \omega_\lambda(r,y,a) &= \int_{\frac{a}{y}}^\infty \frac{|\ln u|u^{\frac{\lambda}{r}-1}}{(\max\{u,1\})^\lambda}du = \frac{r^2+s^2}{\lambda^2} - \int_0^{\frac{a}{y}}(-\ln u)u^{\frac{\lambda}{r}-1}du \\ &= \frac{r^2+s^2}{\lambda^2} - \frac{r}{\lambda}\int_0^{\frac{a}{y}}(-\ln u)du^{\frac{\lambda}{r}} \le \frac{r^2+s^2}{\lambda^2}\kappa(y), \\ \kappa(y) &= 1 - \frac{r^2}{r^2+s^2}(\frac{a}{y})^{\frac{\lambda}{r}}; \end{aligned}$$

$$\varpi_\lambda(s,x,a) \le \frac{r^2+s^2}{\lambda^2}\widetilde{\mu}(x), \widetilde{\mu}(x) = 1 - \frac{s^2}{r^2+s^2}(\frac{a}{x})^{\frac{\lambda}{s}}.$$

Then by (2.8) and (2.9), we have two equivalent inequalities as follows:

$$\int_a^\infty\int_a^\infty \frac{|\ln(\frac{x}{y})|f(x)g(y)}{(\max\{x,y\})^\lambda}dxdy$$
$$< \frac{r^2+s^2}{\lambda^2}\{\int_a^\infty [1-\frac{s^2}{r^2+s^2}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x)f^p(x)dx\}^{\frac{1}{p}}$$
$$\times\{\int_a^\infty [1-\frac{r^2}{r^2+s^2}(\frac{a}{y})^{\frac{\lambda}{r}}]\psi_s(y)g^q(y)dy\}^{\frac{1}{q}};$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{[1-\frac{r^2}{r^2+s^2}(\frac{a}{y})^{\frac{\lambda}{r}}]^{p-1}}[\int_a^\infty \frac{|\ln(\frac{x}{y})|f(x)dx}{(\max\{x,y\})^\lambda}]^p dy$$
$$< (\frac{r^2+s^2}{\lambda^2})^p \int_a^\infty [1-\frac{s^2}{r^2+s^2}(\frac{a}{x})^{\frac{\lambda}{s}}]\phi_r(x)f^p(x)dx.$$

Remark 3.5. (i) For $\alpha = 1, p = q = 2$, inequality (3.1) deduces to (1.2). (ii) Inequality (2.8) is a refinement of (2.10), because of

$$I_\lambda(a) < k_\lambda(r)||f||_{p,\widetilde{\mu}\cdot\phi_r}||g||_{q,\kappa\cdot\psi_s} \le k_\lambda(r)||f||_{p,\phi_r},||g||_{q,\psi_s}.$$

(iii) When $a \to 0^+$, (2.10) deduces to a Hilbert-type integral inequality in $(0,\infty)$ with a best constant factor $k_\lambda(r)$ as:

$$I_\lambda(0) \le k_\lambda(r)\{\int_0^\infty \phi_r(x)f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty \psi_s(y)g^q(y)dy\}^{\frac{1}{q}}.$$

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