FUNCTIONAL HOMOMORPHISMS AND DYNAMICAL SYSTEMS

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ABSTRACT. This article presents an interaction between functional homomorphisms, dynamical systems and differential equations. The exponential functions play pivotal role in this interaction.

1. INTRODUCTION

Let X and Y be topological rings. Then consider the following four types of mappings from X to Y.

(1) $h_1 : X \to Y$ such that $h_1(x + y) = h_1(x) + h_1(y)$
(2) $h_2 : X \to Y$ such that $h_2(x + y) = h_1(x).h_2(y)$
(3) $h_3 : X \to Y$ such that $h_3(x.y) = h_1(x) + h_2(y)$
(4) $h_4 : X \to Y$ such that $h_4(x.y) = h_4(x).h_4(y)$, for every $x, y \in X$.

These mapping are homomorphisms under different (or same) algebraic operations on X and Y. For example, a mapping satisfying (1) and (4) is a ring homomorphism. If it satisfies only (1), then it is a group homomorphism from $(X, +)$ to $(Y, +)$. The mapping satisfying (2) is a homomorphism from the additive group $(X, +)$ to multiplicative semigroup $(Y, \cdot)$. These types of mappings and their combinations give rise to what is known as functional equations in

analysis and they have connections with several areas of mathematics and mathematical physics. They have been the subject matter of study for the last two centuries or so and a lot of mathematics has been developed around them. The homomorphisms of type (2) are of special significance as they make contact with time dependent systems describing changes in nature. In this article we shall restrict our attention to this type of homomorphisms. By a functional homomorphism we shall mean a function $f : X \to Y$ satisfying to the equation (FH) $f(x + y) = f(x)f(y)$ for every $x, y \in X$. A solution of this equation is a homomorphism from additive group $(X, +)$ to multiplicative semigroup $(Y, \cdot)$. In order to have more fruitful interactions with application oriented mathematics we shall consider only continuous homomorphisms, i.e. continuous functions from X to Y satisfying (FH). In this article, which is expository in nature, we shall describe a relation between functional homomorphisms, semigroups of operators on Banach spaces and dynamical systems.

2. Preliminaries

Let G be a topological group with e as the identity, let X be a topological space and $\pi : G \times X \to X$ be a continuous map such that

(1) $\pi(e, x) = x$ for every $x \in X$
(2) $\pi(st, x) = \pi(s, \pi(t, x))$ for every $s, t \in G$ and $x \in X$

Then the triple (G, X, π) is called a transformation group, X is called the state space (or phase space) and π is called an action or a motion on X. For $x \in X$, let the map $\pi_x : G \to X$ be defined as $\pi_x(t) = \pi(t, x)$ for every $t \in G$. Then π_x is called the motion through x and the range of π_x is called the orbit of x. Thus orbit $x = \{\pi(t, x) : t \in G\}$. For $t \in G$ define the map $\pi^t : X \to X$ as $\pi^t(x) = \pi(t, x)$ for every $x \in X$. The map π^t is a homeomorphism and $(\pi^t)^{-1} = \pi^{t^{-1}}$. The set $\{\pi^t : t \in G\}$ is a subgroup of the group of all homeomorphisms of X. The study of motions, orbits and orbit spaces comes under topological dynamics [2]. If $G = (\mathbb{R}, +)$ or $G = (\mathbb{Z}, +)$, the corresponding transformation group is called a dynamical system. The transformation group $(\mathbb{R}, X, \pi)$ is known as continuous dynamical system, whereas $(\mathbb{Z}, X, \pi)$ is called a discrete dynamical system. It is well known that every discrete dynamical system on X comes from a homeomorphism of X. Thus there is one-to-one correspondence between the set of all homeomorphisms of X and the set of all discrete dynamical systems on X. For details see [2]. If $\mathbb{R}$ is replaced by $\mathbb{R}^+$ or $\mathbb{Z}$ is replaced by $\mathbb{Z}^+$, then we get a future dependent semidynamical system. Every discrete semidynamical system comes from a continuous self map of X. In this note our concern is with semidynamical system on Banach spaces which we shall also call dynamical system. If X is a Banach space and $\pi(t, \alpha\beta + \beta y) = \alpha\pi(t, x) + \beta\pi(t, y)$, for $t \in \mathbb{R}, \alpha, \beta \in \mathbb{C}$ and $x, y, \in X$, then $(\mathbb{R}^+, X, \pi)$ is called a linear dynamical system. If X is a Banach space, then by $B(X)$ we shall denote the Banach algebra of all continuous linear operators on X. If $A \in B(X)$, then e^{tA} is defined as $e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$. This series is convergent in $B(X)$ and thus $e^{tA} \in B(X)$ for every $t \in \mathbb{R}$. If we define, $\pi_A : \mathbb{R} \times X \to X$ as $\pi_A(t, x) = e^{tA}(x)$, then $(\mathbb{R}^+, X, \pi_A)$ is a linear dynamical system on X. If $\pi_A^d : \mathbb{Z} \times X \to X$ is defined

as $\pi_A^d(m, x) = A^m(x)$, then $(\mathbb{Z}^+, X, \pi_A^d)$ is a discrete linear dynamical system. Thus every bounded linear operator on a Banach space X induces at least two dynamical systems. It turns out that most of the linear dynamical systems are induced by operators. The functional homomorphisms play important role in this process. By a semigroup of operators on a Banach space X, we mean a functional homomorphism $T : \mathbb{R}^+ \to B(X)$ such that $T(0) = I$, the identity operator. The range $\{T(t) : t \geq 0\}$ of T is a multiplicative abelian semigroup of $B(X)$. The semigroup T is called c_0−semigroup if it is continuous with respect to the strong operator topology on $B(X)$. If T is continuous with respect to norm topology on $B(X)$, then it is called uniformly continuous semigroup. Usually, a semigroup T is denoted as $\{T(t) : t \geq 0\}$, which is nothing but the image of T in $B(X)$. It is a family of bounded operators $\{T_t : t \geq 0\}$ on X such that $T_0 = I$ and $T_{s+t} = T_s T_t$ for $s, t \in \mathbb{R}^+$. If $A \in B(X)$, and $T(t) = e^{tA}$, then T is a c_0−semigroup on X. Every c_0−semigroup T on X gives rise to a dynamical system, where action is given by $\pi_T(t, x) = T(t)(x) = T_t(x)$ for $t \in \mathbb{R}^+$ and $x \in X$ If $A \in B(X)$, then the following initial value problem is known as Abstract Cauchy Problem. $\frac{d}{dt}x = Ax, x(0) = x_0$ Here x is a differentiable function from $\mathbb{R}^+$ to X. If A is the generator of a c_0− semigroup $T : \mathbb{R}^+ \to B(X)$, then $x(t) = T(t)x_0$ is a solution of the Abstract Cauchy Problem.

3. Some classical Solutions of (FH)

In 1821, A. Cauchy made study of functional homomorphisms from $\mathbb{R}$ to $\mathbb{C}$. He suggested that all such homomorphisms are exponential functions. In the following theorem we record the result of A. Cauchy.

Theorem 3.1. *(A. Cauchy)* *If $T : \mathbb{R} \to \mathbb{C}$ is a continuous functional homomorphism, such that $T(0) = 1$, then there exists $a \in \mathbb{C}$, such that $T(t) = e^{at}$ for every $t \in \mathbb{R}$. It is clear that the $T(t) = e^{at}$ is a functional homomorphism and satisfies the initial-value problem, $\frac{d}{dt}T = aT, T(0) = 1$. These are the only functions satisfying the initial value problem and $a = T'(0)$, where T' denotes the derivative of T.*

Note. Every continuous functional homomorphism from $\mathbb{R}^+$ to $\mathbb{C}$ is also differentiable. The above result of Cauchy is true for all continuous functional homomorphisms from the additive group $\mathbb{C}$ to multiplicative group $\mathbb{C}\backslash\{0\}$. Every functional homomorphism T gives rise to dynamical system, where orbit of a state $x_0 \in \mathbb{C}$ is given by $\{T(t)x_0 : t \in \mathbb{R}\}$.

4. Functional homomorphisms in higher dimensions

The result of the last section has been extended to higher dimensions. Let $X = \mathbb{C}^n$, the n−dimensional Banach space of n−tupples of complex numbers with the pointwise vector operations and the usual norm. Let A be an $n \times n$ matrix. then A is a bounded linear operator from the $\mathbb{C}^n$ to $\mathbb{C}^n$. Let $B_n(\mathbb{C})$ denote the Banach algebra of all $n \times n$ matrices with operator norm, i.e. $\|A\| = \sup\limits_{x \neq 0} \frac{\|Ax\|}{\|x\|}$.

If $A \in B_n(\mathbb{C})$, then e^{tA} is also a matrix given by $e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$. It is very difficult to compute the matrix e^{tA} If A is a diagonal matrix with diagonal $(\alpha_1, \alpha_2, \ldots \alpha_n)$, then e^{tA} is also a diagonal matrix with diagonal $(e^{\alpha_1 t}, e^{\alpha_2 t}, \ldots \ldots, e^{\alpha_n t})$. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then e^{tA} is rotational matrix $\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. The matrix e^{tA} for every $A \in B_2(\mathbb{C})$ has been computed [3]. Let $T : \mathbb{R} \to B_n(\mathbb{C})$ be a functional homomorphism taking 0 to the identity matrix I. Then the same question regarding characterization of such homomorphisms can be asked, i.e. find all continuous functional homomorphisms $T : \mathbb{R} \to B_n(\mathbb{C})$. This question was answered in 1887 by Peano. This is recorded in the following theorem.

Theorem 4.1. (*Peano*). *Let $T : \mathbb{R} \to B_n(\mathbb{C})$ be a continuous functional homomorphism such that $T(0) = I$. Then there exists a unique $A \in B_n(\mathbb{C})$ such that $T(t) = e^{tA}$ for every $t \in \mathbb{R}$. Futhermore, T is differentiable function and is a solution of $\frac{d}{dt}T = AT, T(0) = I$ and any solution of the above initial value problem is of this type* [3]

Let $(\mathbb{R}, \mathbb{C}^n, \pi)$ be a linear dynamical system. For every $t \in \mathbb{R}$, define the map $\pi^t : \mathbb{C}^n \to \mathbb{C}^n$ by $\pi^t(x) = \pi(t, x)$ for $x \in \mathbb{C}^n$. Then π^t is a linear operator on $\mathbb{C}^n$, i.e. π^t is an invertible $n \times n$ matrix. Let $T : \mathbb{R} \to B_n(\mathbb{C})$ be defined as $T(t) = \pi^t$, for $t \in \mathbb{R}$. Then T is a functional homomorphism $T(t) = e^{tA}$ for some unique A. It is interesting to note that the set of functional homomorphisms, the set of linear dynamical systems on $\mathbb{C}^n$, and the set of solutions of the initial value problem $\frac{d}{dt}T = AT, T(0) = I, A \in B_n(\mathbb{C})$, are equipotent and they are closely related. Let $T : \mathbb{R} \to B_n(\mathbb{C})$ be a continuous functional homomorphism, let $\pi_T : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$ be defined as $\pi_T(t, x) = T(t)x$. Then π_T is a linear dynamical system. Conversely, every linear dynamical system on $\mathbb{C}^n$ is of this type. This we shall show in the following theorem,

Theorem 4.2. *(a) Let $\pi : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$ be a continuous map. Then $(\mathbb{R}, \mathbb{C}^n, \pi)$ is a linear dynamical system if and only if there exists a continuous functional homomorphism $T : \mathbb{R} \to \mathbb{C}^n$ such that $\pi = \pi_T$.*
(b) Every continuous functional homomorphism $T : \mathbb{R} \to B_n(\mathbb{C})$ is the only solution of $\frac{d}{dt}T = AT, T(0) = I$ where $A = T(0)$.
(c) A solution of the initial value problem gives rise to a linear dynamical system on $\mathbb{C}^n$.

Proof. (a) If π is a motion on $\mathbb{C}^n$, then let $\pi^t : \mathbb{C}^n \to \mathbb{C}^n$, defined as $\pi^t(x) = \pi(t, x)$ is a linear operator on $\mathbb{C}^n$ and hence $\pi^t \in B_n(\mathbb{C})$. Let $T : \mathbb{R} \to B_n(\mathbb{C})$ be defined as $T(t) = \pi^t$. Then T is a continuous functional homomorphism and $\pi_T(t, x) = T(t)x = \pi^t(x) = \pi(t, x)$ for all $t \in \mathbb{R}$ and $x \in X$. Hence $\pi = \pi_T$. The converse is obvious.
(b) If $T : \mathbb{R} \to B_n(\mathbb{C})$ is a continuous homomorphism, then result follows from Peano's theorem. Consider the initial value problem, $\frac{d}{dt}T = AT, T(0) = I$, $A \in B_n(\mathbb{C})$. By Peano's theorem we know that $T(t) = e^{tA}$ is a solution of the initial value problem. Let $\pi = \pi_T$ Then π is a motion on $\mathbb{C}^n$ and $u(t) = e^{tA}x_0$

is the orbit function of state x_0 and satisfies the Abstract Cauchy Problem on $\mathbb{C}^n$, $\frac{d}{dt}u = Au, u(0) = x_0$. ∎

Note. The above results are valid for $n-$dimensional Banach spaces also.

5. Functional homomorphisms in infinite dimensional Banach spaces

In this section we assume that X is an infinite dimensional Banach space. There are three nice operator topologies on $B(X)$, namely uniform operator topology, strong operator topology and weak operator topology. To include a larger class of functional homomorphisms we shall take their domain as $\mathbb{R}^+$, the semigroup of positive reals with usual topology. If $T : \mathbb{R}^+ \to B(X)$ is a functional homomorphism, then uniform, strong and weak operator topologies on $B(X)$ give rise to three continuities of T, namely uniform continuity, strong continuity and weak continuity. The characterization of uniformly continuous functional homomorphisms is similar to the finite dimensional case . This is given in the following theorem.

Theorem 5.1. *Let X be any Banach space and let $T : \mathbb{R}^+ \to B(X)$ be a uniformly continuous functional homomorphism with $T(0) = I$. Then there exists a unique $A \in B(X)$ such that*
(a) $T : \mathbb{R}^+ \to B(X)$ is differentiable
(b) $T(t) = e^{tA}$ for every $t \in \mathbb{R}^+$
(c) The orbit function $u : \mathbb{R}^+ \to X$, given by $u(t) = T(t)x$ is differentiable for every $x \in X$ and satisfies the initial value problem, $\frac{du}{dt} = Au, u(0) = x$

An outline of the proof.

If T is uniformly continuous, then the operator $S(t)$, given by $S(t) = \int\limits_0^t T(s)ds$

is invertible for small t. Now $T(t) = S(t_0)^{-1}S(t_0)T(t) = S(t_0)^{-1} \int\limits_t^{t+t_0} T(s+t)ds$

for $t \geq 0$. Now $\frac{d}{dt}T(t) = \lim\limits_{h \to 0^+} \frac{T(t+h)-T(t)}{h} = T'(0)T(t)$. This shows that T is differentiable and satisfies the differential equation $\frac{d}{dt}T(t) = AT, T(0) = I$, where $A = T'(0)$. Hence by uniqueness of solution we have $T(t) = e^{tA}$. This proves (a) and(c). The proof of (c) is easy.

Note. The operator $A(= T'(0))$ is called the generator of the semigroup T. There is one-to-one correspondence between the set of all uniformly continuous functional homomorphisms and linear (semi)dynamical systems on X. Thus the concept of uniformly continuous semigroups is equivalent to linear dynamical systems on a Banach space X. If $T : \mathbb{R}^+ \to B(X)$ is strongly continuous functional homomorphism with $T(0) = I$, then the generator of T may not be a bounded operator, and hence representation of T as $T(t) = e^{tA}$ may not be possible as was done in case T is uniformly continuous. We shall present an analogous characterization for strongly continuous functional homomorphisms. The (infinitesimal)

generator A of $T : \mathbb{R}^+ \to B(X)$ is defined as $Ax = \lim\limits_{t \to 0^+} \frac{T(t)x - x}{t}$. Then the domain $D(A)$ of A is the set $D(A) = \{x \in X : \lim\limits_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists}\}$. A is a linear and closed operator, not necessarily bounded and $D(A)$ is a dense subspace of X. For details see [1, 3]. A is called the generator of T. The resolvent set $\rho(A)$ is the set of all those complex numbers λ such that the operator $\lambda I - A : D(A) \to X$ is a bijection. The spectrum of A, denoted by $\sigma(A)$ is $\mathbb{C} \backslash \rho(A)$. The resolvent operator $R(\lambda, A)$ for $\lambda \in \rho(A)$ is defined by $R(\lambda, A) = (\lambda I - A)^{-1}$. By the closed graph theorem $R(\lambda, A)$ is a bounded operator. If λ and μ are in $\rho(A)$, then it is easy to show that $(\mu - \lambda)R(\lambda, A)R(\mu, A) = R(\lambda, A) - R(\mu, A)$. Let T be a strongly continuous homomorphism from $\mathbb{R}^+$ to $B(X)$ with $T(0) = I$ and let A be the infinitesimal generator of T. Let $x \in D(A)$. Then the orbit map $T_x : \mathbb{R}^+ \to X$ defined as $T_x(t) = T(t)x$ is differentiable and $\frac{d}{dt}T_x(t) = T(t)Ax = AT(t)x \ (T(t)x \in D(A))$. If S is another strongly continuous homomorphism with $S(0) = I$ having the same infinitesimal generator A, then it turns out that $S(t)x = T(t)x$ for all $x \in D(A)$. Since $D(A)$ is dense, we conclude that $S = T$. Thus the infinitesimal generator determine the strongly continuous functional homomorphism uniquely. In case the infinitesimal generator A is bounded, the strongly continuous functional homomorphism is uniformly continuous and hence $T(t) = e^{tA}$ for $t \in \mathbb{R}^+$. For every strongly continuous functional homomorphism there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for every $t \in \mathbb{R}^+$ [6, 9]. Every strongly continuous functional homomorphism from $\mathbb{R}^+$ to $B(X)$ gives rise to a densely defined closed operator on X, namely its generator. The following theorem characterizes such operators generating strongly continuous functional homomorphisms.

Theorem 5.2. (Miyadera and Phillips). *Let A be a linear operator on a Banach space and let $M \geq 1$ and $\omega \in \mathbb{R}$. Then the following are equivalent.*

(a) A generates a strongly continuous homomorphism $T : \mathbb{R}^+ \to B(X)$ such that $\|T(t)\| \leq Me^{\omega t}$.

(b) A is closed, densely defined and $\lambda > \omega$, it is true that $\lambda \in \rho(A)$ and $\|[(\lambda - \omega)R(\lambda, A)]^n\| \leq M$ for every $n \in N$.

(c) A is closed, densely defined operator and for every complex number λ such that $\operatorname{Re} \lambda > \omega$, it is true that $\lambda \in \rho(A)$ and $\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re} \lambda - \omega)^n}$ for every $n \in N$.

Note. The above theorem is a generalization of famous Hille–Yosida theorem proved in 1948 [4], which characterizes strongly continuous functional homomorphisms. If $\{A_n\}$ is a sequence of bounded operators on X such that $\lim\limits_{n} A_n x = Ax$, for every $x \in D(A)$, then the generalized exponential function for A is defined as $e^{tA} = \lim\limits_{n} e^{tA_n}$ and it may give rise to a strongly continuous functional homomorphism. This generalization of exponential function for unbounded operator on X help in giving the following characterization of strongly continuous functional homomorphisms [3].

Theorem 5.3. *Let $T : \mathbb{R}^+ \to B(X)$ be a strongly continuous function. Then T is a functional   homomorphism with $T(0) = I$ and $\|T(t)\| \leq 1$, if and only if*

$T(t) = e^{tA}$ for some closed, densely defined operator A on X with property that $\lambda > 0$ implies $\lambda \in \rho(A)$ and $\|\lambda R(\lambda, A)\| \leq 1$.

An outline of the proof.

If $T : \mathbb{R}^+ \to B(X)$ is a strongly continuous homomorphism with $T(0) = I$, then infinitesimal generator A of T is closed, densely defined operator on X with the property given above. If $A_n = nAR(n, A)$, then A_n is a bounded operator on X and $e^{tA_n} \in B(X)$ for every $t \in \mathbb{R}^+$. Let $T_n(t) = e^{tA_n}$. Since $A_n \to A$ pointwise on $D(A)$, we have $T(t)x = \lim_n T_n(t)x = \lim e^{tA_n}x = e^{tA}x$. Thus $T(t) = e^{tA}$. Conversely, If $T(t) = e^{tA}$ for some closed, densely defined operator A, then by definition of e^{tA}, $T(t) = \lim_n e^{tA_n}$, where $A_n = nAR(n, A)$. Since e^{tA_n} is uniformly continuous functional homomorphism and $A_n \to A$ pointwise, we conclude that T is strongly continuous, $T(t + s) = T(t)T(s)$ and $T(0) = I$. This takes care of the outline of the proof. An interaction of strongly continuous homomorphisms, linear dynamical systems and abstract cauchy problem is given in the following theorem.

Theorem 5.4. *(a) Every strongly continuous homomorphism $T : \mathbb{R}^+ \to B(X)$ gives rise to the linear dynamical system π_T on X defined as $\pi_T(t, x) = T(t)x$ for every $t \in \mathbb{R}^+$ and $x \in X$. Conversely, for every dynamical system π on X, there exists a strongly continuous homomorphism $T : \mathbb{R}^+ \to B(X)$ such that $\pi = \pi_T$.*

(b) If $T : \mathbb{R}^+ \to B(X)$ is a strongly continuous homomorphism and A is infinitesimal generator of T, then the Abstract Cauchy Problem $\frac{d}{dt}u(t) = Au(t), u(0) = x$ has a solution given by $u(t) = T(t)x$.

(c) The Abstract Cauchy Problem has a unique solution if A is densely defined closed operator with property $\lambda > 0$ implies $\lambda \in \rho(A)$ and $\|\lambda R(\lambda, A)\| \leq 1$ (i.e. A is a generator of a strongly continuous homomorphism).

This survey completes the characterization of continuous homomorphisms from $\mathbb{R}^+ \to B(X)$. A lot of work has been done in the last 50 years or so and many applications of concrete homomorphisms have been obtained. For details we refer to [5, 7, 8].

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