



## ON PSEUDO HERMITE MATRIX POLYNOMIALS OF TWO VARIABLES

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Communicated by F. Kittaneh

**ABSTRACT.** The main aim of this paper is to define a new polynomial, say, pseudo hyperbolic matrix functions, pseudo Hermite matrix polynomials and to study their properties. Some formulas related to an explicit representation, matrix recurrence relations are deduced, differential equations satisfied by them is presented, and the important role played in such a context by pseudo Hermite matrix polynomials are underlined.

### 1. INTRODUCTION

The pseudo hyperbolic and pseudo trigonometric functions have been introduced on the eve of seventies by Ricci [14] in applications has been recognized only recently, within the context of problems involving arbitrary order coherent states [7, 13] and the emission of electromagnetic radiation by accelerated charges [7, 13]. This class of functions providing a fairly natural generalization of the ordinary exponential, hyperbolic and trigonometric functions [3, 4], offers the possibility of exploring, from a more general and unifying point of view, the theory of special functions including generalized cases. The pseudo-Laguerre, pseudo-Hermite polynomials and pseudo Bessel functions have been introduced in [5, 6]. Moreover, the Hermite matrix polynomials have been introduced and studied in

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*Date:* Received: 29 December 2009; Accepted: 4 April 2010.

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2000 *Mathematics Subject Classification.* Primary 33C47; Secondary, 33-00, 33C45, 33E20, 35A22, 45P05, 47G10.

*Key words and phrases.* Operational calculus, Pseudo hyperbolic functions, Hermite matrix polynomials, Matrix differential equation, Pseudo Hermite Polynomials.

[2, 10, 11, 12] for matrices in  $\mathbb{C}^{N \times N}$  whose eigenvalues are all situated in the right open half-plane.

If  $D_0$  is the complex plane cut along the negative real axis and  $\log(z)$  denotes the principal logarithm of  $z$ , then  $z^{\frac{1}{2}}$  represents  $\exp(\frac{1}{2} \log(z))$ . If  $A$  is a matrix in  $\mathbb{C}^{N \times N}$ , its two-norm denoted  $\|A\|$  is defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

where for a vector  $y$  in  $\mathbb{C}^N$ ,  $\|y\|_2 = (y^T y)^{\frac{1}{2}}$  denotes the usual Euclidean norm of  $y$ . The set of all the eigenvalues of  $A$  is denoted by  $\sigma(A)$ . If  $f(z)$  and  $g(z)$  are holomorphic functions of the complex variable  $z$ , which are defined in an open set  $\Omega$  of the complex plane, and  $A$  is a matrix in  $\mathbb{C}^{N \times N}$  such that  $\sigma(A) \subset \Omega$ , then from the properties of the matrix functional calculus [10], it follows that

$$f(A)g(A) = g(A)f(A).$$

If  $A$  is a matrix with  $\sigma(A) \subset D_0$ , then  $A^{\frac{1}{2}} = \sqrt{A} = \exp(\frac{1}{2} \log(A))$  denotes the image by  $z^{\frac{1}{2}} = \sqrt{z} = \exp(\frac{1}{2} \log(z))$  of the matrix functional calculus acting on the matrix  $A$ . If  $A$  is a positive stable matrix in  $\mathbb{C}^{N \times N}$  [2, 10, 11, 12]

$$\operatorname{Re}(z) > 0, \quad \text{for all } z \in \sigma(A). \quad (1.1)$$

If  $A(k, n)$  and  $B(k, n)$  are matrices on  $\mathbb{C}^{N \times N}$  for  $n \geq 0$ ,  $k \geq 0$ , in an analogous way to the proof of Lemma 11 of [10], it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) &= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{1}{2}n]} A(k, n - 2k), \\ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n) &= \sum_{n=0}^{\infty} \sum_{k=0}^n B(k, n - k). \end{aligned} \quad (1.2)$$

Similarly to (1.2), we can write

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{1}{2}n]} A(k, n) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n + 2k), \\ \sum_{n=0}^{\infty} \sum_{k=0}^n B(k, n) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n + k). \end{aligned} \quad (1.3)$$

In the following we will make great use of the operator  $D_x^{-1}$ , which is the inverse of the derivative operator. By exploiting the notion of Cauchy repeated integral [8, 9], we can write

$$D_x^{-n} f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt$$

and it is also easy on unity to realize that

$$D_x^{-n}(1) = \frac{x^n}{n!}.$$

The following two identities are a fairly direct consequence of the previous considerations, it is indeed easily checked that

$$D_x^{-i}(D_x^{-n})(1) = \frac{x^{n+i}}{(n+i)!}$$

and that

$$e^{x\sqrt{rA}} = \sum_{n=0}^{\infty} \frac{(x\sqrt{rA})^n}{n!} = \sum_{n=0}^{\infty} D_x^{-n}(\sqrt{rA})^n = (I - D_x^{-1}(\sqrt{rA}))^{-1}.$$

We extend the definition of exponential by introducing in the following a new family of functions, characterized by an integer  $r$

$$E_0(x, A; r) = \sum_{n=0}^{\infty} \frac{(x\sqrt{rA})^{nr}}{(nr)!} = \sum_{n=0}^{\infty} D_x^{-nr}(\sqrt{rA})^{nr}. \quad (1.4)$$

We can infer directly from their definition that the functions  $E_0(x, A; r)$ , called from now on pseudo hyperbolic functions, can be complemented by

$$E_i(x, A; r) = (\sqrt{rA})^i D_x^{-i} E_0(x, A; r) = \sum_{n=0}^{\infty} \frac{(x\sqrt{rA})^{nr+i}}{(nr+i)!} = \sum_{m=0}^{\infty} D_x^{-nr}(\sqrt{rA})^{nr}$$

all linearly independent if  $i < r$  and satisfying the identities

$$\frac{d}{dx} E_i(x, A; r) = \sqrt{rA} E_{i-1}(x, A; r)$$

this can be combined to get

$$\left(\frac{d}{dx}\right)^r E_i(x, A; r) = (\sqrt{rA})^r E_{i-r}(x, A; r).$$

In the forthcoming sections of the paper we will show how a proper combination of the points of view of [14] offers the possibility of developing the theory of pseudo Hermite matrix polynomials. We will show a further natural complement of the functions defined by the families of pseudo Hermite matrix polynomials. We will discuss the properties of these new families of polynomials and we will analyze possible developments and applications of the theory.

## 2. PSEUDO HERMITE MATRIX POLYNOMIALS

The pseudo Hermite matrix polynomials of two variables defined by the series

$$\mathbb{k}_n(x, y, A; r, 0) = n! \sum_{k=0}^n \frac{(-y)^{n-k} (x\sqrt{rA})^{rk}}{(rk)!(n-k)!} = n! \sum_{k=0}^n \frac{(-y)^{n-k} D_x^{-rk}(\sqrt{rA})^{rk}}{(n-k)!} \quad (2.1)$$

where  $A$  satisfying the condition (1.1). By using (1.3), (1.4) and (2.1), we consider the series

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{k}_n(x, y, A; r, 0) &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-y)^{n-k} t^n (x\sqrt{rA})^{rk}}{(rk)!(n-k)!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-y)^n t^{n+k} (x\sqrt{rA})^{rk}}{n!(rk)!} \\ &= \sum_{n=0}^{\infty} \frac{(-y)^n t^n}{n!} \sum_{k=0}^{\infty} \frac{t^k (x\sqrt{rA})^{rk}}{(rk)!} \\ &= \exp(-yt) E_0(xt^{\frac{1}{r}}, A, r). \end{aligned}$$

Thus, we obtain the new generating function which represents an explicit representation for the pseudo Hermite matrix polynomials in the form

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{k}_n(x, y, A; r, 0) = \exp(-yt) E_0(xt^{\frac{1}{r}}, A, r) \quad (2.2)$$

It is clear that

$$\mathbb{k}_{-1}(x, y, A; r, 0) = 0 \quad \text{and} \quad \mathbb{k}_0(x, y, A; r, 0) = I.$$

Also, we can write

$$\mathbb{k}_n(x, y, A; r, 0) = y^n \mathbb{k}_n\left(\frac{x}{\sqrt{y}}, 1, A; r, 0\right) \quad \text{and} \quad \mathbb{k}_n(x, 1, A; r, 0) = \mathbb{k}_n(x, A; r, 0).$$

Special case: It should be observed that, in view of the explicit representation (2.1), the pseudo Hermite matrix polynomials  $\mathbb{k}_n(x, 1, A; r, 0)$  reduces to the pseudo Hermite matrix polynomials  $\mathbb{k}_n(x, A; r, 0)$ . Before getting into the main body of the paper, let us recall some important properties of pseudo Hermite matrix polynomials of the addition, multiplication theorems, which will be used in the forthcoming papers.

**Theorem 2.1.** *Multiplication Theorem*

$$\mathbb{k}_n(x, \alpha y, A; r, 0) = \alpha^n \mathbb{k}_n\left(\frac{x}{\sqrt{\alpha}}, y, A; r, 0\right) \quad (2.3)$$

and

$$\mathbb{k}_n(\alpha x, \alpha^r y, A; r, 0) = \alpha^{nr} \mathbb{k}_n(x, y, A; r, 0) \quad (2.4)$$

where  $\alpha$  is constant.

*Proof.* Using (2.1), we have

$$\begin{aligned} \alpha^n \mathbb{k}_n\left(\frac{x}{\sqrt{\alpha}}, y, A; r, 0\right) &= \alpha^n n! \sum_{k=0}^n \frac{(-y)^{n-k}}{(rk)!(n-k)!} \left(\frac{x\sqrt{rA}}{\sqrt{\alpha}}\right)^{rk} \\ &= n! \sum_{k=0}^n \frac{(-\alpha y)^{n-k}}{(rk)!(n-k)!} (x\sqrt{rA})^{rk} \\ &= \mathbb{k}_n(x, \alpha y, A; r, 0). \end{aligned}$$

From (2.1) yields the pseudo Hermite matrix polynomials as given in the following form

$$\mathbb{k}_n(\alpha x, \alpha^r y, A; r, 0) = n! \sum_{k=0}^n \frac{(-\alpha^r y)^{n-k}}{(rk)!(n-k)!} (x\alpha\sqrt{rA})^{rk} = \alpha^{nr} \mathbb{k}_n(x, y, A; r, 0).$$

Therefore, the expressions (2.3) and (2.4) are established. □

In the following theorem, we obtain the addition properties of pseudo Hermite matrix polynomials as follows

**Theorem 2.2.** *Addition Theorem*

$$\mathbb{k}_n(x, y + z, A; r, 0) = n! \sum_{k=0}^n \frac{(-1)^k z^k}{k!(n-k)!} \mathbb{k}_{n-k}(x, y, A; r, 0). \tag{2.5}$$

*Proof.* From (1.3), (2.1), (2.2) and the properties of exponential matrix, the generating pseudo Hermite polynomials reduces to

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{k}_n(x, y + z, A; r, 0) &= \exp(- (y + z)t) E_0(xt^{\frac{1}{r}}, A, r) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-z)^k t^{n+k}}{n!k!} \mathbb{k}_n(x, y, A; r, 0) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-z)^k t^n}{k!(n-k)!} \mathbb{k}_{n-k}(x, y, A; r, 0) \end{aligned}$$

by comparing the coefficients of  $t^n$ , we get (2.5) and the proof is completed. □

Some recurrence relation is carried out on the pseudo Hermite matrix polynomials. We obtain the following theorem.

**Theorem 2.3.** *The pseudo Hermite matrix polynomials of two variables satisfy the following relations*

$$\frac{\partial^r}{\partial x^r} \mathbb{k}_n(x, y, A; r, 0) = n(\sqrt{rA})^r \mathbb{k}_{n-1}(x, y, A; r, 0) \tag{2.6}$$

and

$$\frac{\partial^m}{\partial y^m} \mathbb{k}_n(x, y, A; r, 0) = \frac{(-1)^m n!}{(n-m)!} \mathbb{k}_{n-m}(x, y, A; r, 0); \quad 0 \leq m \leq n. \tag{2.7}$$

*Proof.* Differentiating the identity (2.1) with respect to  $x$  yields

$$\frac{\partial}{\partial x} \mathbb{k}_n(x, y, A; r, 0) = n! \sqrt{rA} \sum_{k=0}^n \frac{(-y)^{n-k} (x\sqrt{rA})^{rk-1}}{(rk-1)!(n-k)!}. \tag{2.8}$$

Iteration (2.8), for  $0 \leq r \leq n$ , implies (2.6). Differentiating the identity (2.1) with respect to  $y$ , we get

$$\frac{\partial}{\partial y} \mathbb{k}_n(x, y, A; r, 0) = -n! \sum_{k=0}^n \frac{(-y)^{n-k-1} (x\sqrt{rA})^{rk}}{(rk)!(n-k-1)!} = -n \mathbb{k}_{n-1}(x, y, A; r, 0) \tag{2.9}$$

and, in general,

$$\frac{\partial^m}{\partial y^m} \mathbb{k}_n(x, y, A; r, 0) = \frac{(-1)^m n!}{(n-m)!} \mathbb{k}_{n-m}(x, y, A; r, 0).$$

Therefore, the expression (2.7) is established and the proof of Theorem 2.3 is completed.  $\square$

The following corollary is a consequence of Theorem 2.3 to satisfy the differential equations.

**Corollary 2.4.** *The pseudo Hermite matrix polynomials satisfy the following relations*

$$\frac{\partial^r}{\partial x^r} \mathbb{k}_n(x, y, A; r, 0) + (\sqrt{rA})^r \frac{\partial}{\partial y} \mathbb{k}_n(x, y, A; r, 0) = 0. \quad (2.10)$$

*Proof.* From (2.6) and (2.9) the equation (2.10) follows directly. According to (2.10), it is clear that the  $\mathbb{k}_n(x, y, A; r, 0)$  are the natural solutions of the heat partial differential equation.  $\square$

The above terms the differential recurrence relation will be used in the following corollary is a consequence of Theorem 2.3.

**Corollary 2.5.** *Let  $A$  be a matrix in  $\mathbb{C}^{N \times N}$  satisfying (1.1), then, we have*

$$nr \mathbb{k}_n(x, y, A; r, 0) = x \frac{\partial}{\partial x} \mathbb{k}_n(x, y, A; r, 0) - nry \mathbb{k}_{n-1}(x, y, A; r, 0), \quad n \geq 1. \quad (2.11)$$

*Proof.* Replacing  $n$  by  $n-1$  in (2.1), multiply by  $y$  and multiply of (2.8) by  $x$ , we obtain differential recurrence relation (2.11) follows directly.  $\square$

In the following result, the pseudo Hermite matrix polynomials appear as finite series solutions of the  $r$ -th order matrix differential equations.

**Corollary 2.6.** *The pseudo Hermite matrix polynomials are solutions of the matrix differential equations of the  $r$ -th order in the form*

$$\left[ y \frac{\partial^r}{\partial x^r} - \frac{x(\sqrt{rA})^r}{r} \frac{\partial}{\partial x} + n(\sqrt{rA})^r \right] \mathbb{k}_n(x, y, A; r, 0) = 0. \quad (2.12)$$

*Proof.* From (2.6) and (2.11) to obtain (2.12). Thus the proof of Corollary 2.3 is completed.  $\square$

In the following theorem, we obtain another representation for the pseudo Hermite matrix polynomials as follows theorem.

**Theorem 2.7.** *Suppose that  $A$  is a matrix in  $\mathbb{C}^{N \times N}$  satisfying (1.1). Then the pseudo Hermite matrix polynomials has the following representation*

$$\mathbb{k}_n(x, y, A; r, 0) = \exp\left(-\frac{y}{(\sqrt{rA})^r} \frac{\partial^r}{\partial x^r}\right) \left(\frac{n!}{(nr)!} (x\sqrt{rA})^{nr}\right).$$

*Proof.* It is clear by (1.2) and (2.1) that

$$\begin{aligned}
& \exp\left(-\frac{y}{(\sqrt{rA})^r} \frac{\partial^r}{\partial x^r}\right) \left(\frac{n!}{(nr)!} (x\sqrt{rA})^{nr}\right) \\
&= \sum_{k=0}^{\infty} \frac{(-y)^k}{k!(\sqrt{rA})^{rk}} \frac{\partial^{rk}}{\partial x^{rk}} \left(\frac{n!}{(nr)!} (x\sqrt{rA})^{nr}\right) \\
&= \sum_{k=0}^{\infty} \frac{(-y)^k (\sqrt{rA})^{rk} (nr)!}{k!(nr-rk)! (\sqrt{rA})^{rk}} \left(\frac{n!}{(nr)!} (x\sqrt{rA})^{nr-rk}\right) \\
&= n! \sum_{k=0}^n \frac{(-y)^{n-k} (x\sqrt{rA})^{nr}}{(nr)!(n-k)!} = \mathbb{k}_n(x, y, A; r, 0).
\end{aligned}$$

Therefore, the result is established.  $\square$

In the following corollary, we obtain another recurrence formula for the pseudo Hermite matrix polynomials as follows.

**Corollary 2.8.** *The pseudo Hermite matrix polynomials of two variables satisfy the following*

$$\mathbb{k}_n(x, y+z, A; r, 0) = \exp\left(-\frac{z}{(\sqrt{rA})^r} \frac{\partial^r}{\partial x^r}\right) \mathbb{k}_n(x, y, A; r, 0). \quad (2.13)$$

*Proof.* By using the Theorem 2.4, we get directly the equation (2.13). Hence the Corollary 2.4 is established.  $\square$

By recalling that the Hermite matrix polynomials  $H_{n,m}(x, y, A)$  are defined through [15]

$$\begin{aligned}
H_{n,m}(x, y, A) &= \exp\left(-\frac{y}{(\sqrt{mA})^m} \frac{\partial^m}{\partial x^m}\right) (x\sqrt{mA})^n \\
&= n! \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-1)^k y^k}{k!(n-mk)!} (x\sqrt{mA})^{n-mk}
\end{aligned}$$

we can identify the  $\mathbb{k}_n(x, y, A; r, 0)$  with

$$\mathbb{k}_n(x, y, A; r, 0) = \frac{n!}{(nr)!} H_{nr,r}(x, y, A).$$

The generating function of the polynomials (2.1) defined from now on pseudo Hermite matrix polynomials, can be derived, by recalling that, according to [15], the following identity holds

$$\sum_{n=0}^{\infty} \frac{t^{nr}}{(nr)!} H_{nr,m}(x, y, A) = \exp\left(xt^r \sqrt{mA} - yIt^m\right).$$

Now, we can see state that the expansion of pseudo Hermite matrix polynomials with on their properties and prove the following theorem.

**Theorem 2.9.** *Let  $A$  be a positive stable matrix in  $\mathbb{C}^{N \times N}$  satisfy (1.1), and then we have*

$$(x\sqrt{rA})^{nr} = \sum_{k=0}^n \frac{(nr)!}{k!(n-k)!} y^k \mathbb{k}_{n-k}(x, y, A; r, 0), \quad -\infty < x < \infty. \tag{2.14}$$

*Proof.* By (1.2) and (2.3), we can write the following

$$\begin{aligned} E_0(xt^{\frac{1}{r}}, A; r) &= \sum_{n=0}^{\infty} \frac{(x\sqrt{rA})^{nr}}{(nr)!} t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{y^k \mathbb{k}_n(x, y, A; r, 0)}{n!k!} t^{n+k} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{y^k \mathbb{k}_{n-k}(x, y, A; r, 0)}{k!(n-k)!} t^n. \end{aligned} \tag{2.15}$$

By expanding the left-hand side of (2.15) in powers of  $t$  and identification of the coefficients of  $t^n$  in both sides gives (2.14). Therefore, the expression (2.14) is established and the proof of Theorem 2.5 is completed.  $\square$

The above relations will be used, along with the generalized pseudo Hermite matrix polynomials can be shown to satisfy the property, to derive new properties of the family generated by (2.1) yields as given in the following section.

### 3. GENERALIZED PSEUDO HERMITE MATRIX POLYNOMIALS

It goes by itself that we can introduce the matrix polynomials

$$\mathbb{k}_n(x, y, A; r, i) = (\sqrt{rA})^i D_x^{-i} \mathbb{k}_n(x, y, A; r, 0) = n! \sum_{k=0}^n \frac{(-y)^{n-k} (x\sqrt{rA})^{rk+i}}{(n-k)!(rk+i)!}$$

which in terms of generalized two index Hermite matrix polynomials of two variables

$$\mathbb{k}_n(x, y, A; r, i) = \frac{n!}{(nr+i)!} H_{nr+i,r}(x, y, A),$$

and the relevant generating function can be written in terms of pseudo hyperbolic matrix function as

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{k}_n(x, y, A; r, i) = t^{-\frac{i}{r}} e^{-yt} E_i(xt^{\frac{1}{r}}, A; r).$$

This last identity completes the first part of the paper. We have indeed proved the existence of new families of polynomials linked to pseudo hyperbolic matrix functions in the same way in which ordinary polynomials Hermite matrix are linked to the hyperbolic matrix functions.

We believe interesting to consider a further example relevant to the family of matrix polynomials

$$\mathbb{k}_{n,m}(x, y, A; r, i) = n! \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-y)^{n-mk} (x\sqrt{rA})^{rk+i}}{(rk+i)!(n-mk)!}; \quad m < r. \tag{3.1}$$



satisfying the differential equations

$$\frac{\partial^r}{\partial x^r} \mathbb{k}_{n,m}(x, y, A; r, i) + (\sqrt{rA})^r \frac{\partial^m}{\partial y^m} \mathbb{k}_{n,m}(x, y, A; r, i) = 0.$$

According to the so far developed discussion it is easily realized that the polynomials (3.1) too are linked to the pseudo hyperbolic matrix functions by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{k}_{n,m}(x, y, A; r, i) = t^{-\frac{mi}{r}} e^{-yt} E_i(xt^{\frac{m}{r}}, A; r)$$

this can be used as a useful starting point to study the properties of this family of matrix polynomials.

**Acknowledgement.** The author wishes to express their gratitude to the unknown referee for several helpful comments and suggestions.

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