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CONDITIONAL MULTIPLIERS AND ESSENTIAL NORM OF uC_φ BETWEEN L^p SPACES

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ABSTRACT. In this paper the conditional multipliers acting between L^p spaces are characterized by using some properties of conditional expectation operator. Also, we determine the essential norm of uC_φ on L^p for $1 < p < \infty$.

1. INTRODUCTION AND PRELIMINARIES

Let (X, Σ, μ) be a sigma finite measure space. By $L^0(\Sigma)$, we denote the linear space of all Σ-measurable functions on X. For any complete sigma finite sub-algebra $\mathcal{A} \subseteq \Sigma$ with $1 \leq p \leq \infty$ the L^p-space $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\|.\|_p$. We understand $L^p(\mathcal{A})$ as a Banach subspace of $L^p(\Sigma)$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The support of a measurable function f is defined as $\sigma(f) = \{x \in X; f(x) \neq 0\}$. A Σ-measurable function u on X for which $uf \in L^q(\Sigma)$ for each $f \in L^p(\mathcal{A})$, is called a conditional multiplier.

Associated with each complete sigma finite sub-algebra $\mathcal{A} \subseteq \Sigma$, there exists an operator $E = E^{\mathcal{A}}$, which is called conditional expectation operator, on the set of all non-negative measurable functions f or for each $f \in L^p(\Sigma)$ ($1 \leq p \leq \infty$), and is uniquely determined by the following conditions:

(1) (i) $E(f)$ is $\mathcal{A}$- measurable, and
(2) (ii) if A is any $\mathcal{A}$- measurable set for which $\int_A f d\mu$ exists, we have $\int_A f d\mu = \int_A E(f) d\mu$.

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We shall make repeated use of the following properties of conditional expectations:
E1. $E(f) = f$ if and only if f is $\mathcal{A}$-measurable.
E2. If f is $\mathcal{A}$-measurable then $E(fg) = fE(g)$.
E3. $|E(f)|^p \leq E(|f|^p)$.
E4. If $f \geq 0$ then $E(f) \geq 0$; if $f > 0$ then $E(f) > 0$.

Properties E1. and E2. imply that E is idempotent and $E(L^p(\Sigma)) = L^p(\mathcal{A})$. For more details on the properties of E see [10, 12, 14].

Recall that an $\mathcal{A}$-atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure with no atoms is called non-atomic. It is well-known fact that every σ-finite measure space $(X, \mathcal{A}, \mu_{|\mathcal{A}})$ can be partitioned uniquely as $X = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathcal{A}$-atoms and B, being disjoint from each A_n, is non-atomic (see [19]).

Combination of conditional expectation operator E and multiplication operator M_u appears more often in the service of the study of other operators such as multiplication and weighted composition operators (see [11, 12]). These operators are closely related to averaging operators on order ideals in Banach lattices and to operators called conditional expectation-type operators introduced in [1]. For a beautiful exposition of the study of weighted conditional expectation operators on L^p-spaces, see [3] and the references therein.

Some results of this article is a generalization of the work done in [4, 11, 17]. In section 2, we will determine the symbol functions $u \in L^0(\Sigma)$ that induce bounded multiplication (weighted composition) operators from $L^p(\mathcal{A})$ into $L^q(\Sigma)$, for $1 \leq p, q \leq \infty$. In section 3, for a bounded weighted composition operator uC_φ on $L^p(\Sigma)$ with $1 < p < \infty$, we determine its essential norm. In addition, in section 4 there are a number of examples which they can describe reasonably some of the results of this paper.

2. Characterization of conditional multipliers between L^p-spaces

Let $1 \leq p, q \leq \infty$. We denote the set of all conditional multipliers by $\mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$ where is defined as follows:

$$\mathcal{K}_{p,q} = \mathcal{K}_{p,q}(\mathcal{A}, \Sigma) = \{u \in L^0(\Sigma) : uL^p(\mathcal{A}) \subseteq L^q(\Sigma)\}.$$

$\mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$ is a vector subspace of $L^0(\Sigma)$. In fact, $u \in \mathcal{K}_{p,q}$ if and only if the corresponding multiplication operator $M_u : L^p(\mathcal{A}) \to L^q(\Sigma)$ is bounded. Put $\mathcal{K}_{p,p} = \mathcal{K}_p$. It is easy to check that $L^\infty(\Sigma) \subseteq \mathcal{K}_p(\mathcal{A}, \Sigma)$ and $\mathcal{K}_p(\Sigma, \Sigma) = L^\infty(\Sigma)$. In this section we characterize the members of $u \in \mathcal{K}_{p,q}$ in terms of the conditional expectation induced by $\mathcal{A}$.

Lambert in [11], proved that $u \in \mathcal{K}_p(\mathcal{A}, \Sigma)$ if and only if $E(|u|^p) \in L^\infty(\mathcal{A})$, where $1 \leq p < \infty$. Also Takagi and Yokouchi in [17] characterized the members of $\mathcal{K}_{p,q}(\Sigma,\Sigma)$ in the case $1 \le p, q \le \infty$. In the rest of this section we consider remain cases.

Cases: $1 \le q < p < \infty$ and $1 \le p < q < \infty$

Although, the parts (a) and (b) of the following theorem was given in [5, 6], however, we rewrite them with modifications in their proofs which will be needed later (Theorem 2.3).

Theorem 2.1. *(a) Suppose* $1 \le q < p < \infty$ *and* $u \in L^0(\Sigma)$. *Then* $u \in \mathcal{K}_{p,q}$ *if and only if* $(E(|u|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$, *where* $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$.

(b) Suppose $1 \le p < q < \infty$ *and* $u \in L^0(\Sigma)$. *Then* $u \in \mathcal{K}_{p,q}$ *if and only if*
(i) $E(|u|^q) = 0$ *on* B, *and*
(ii) $M := \sup_{n\in\mathbb{N}} \frac{E(|u|^q)(A_n)}{\mu(A_n)^{\frac{q}{r}}} < \infty$, *where* $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$.

Proof. (a) Let $u \in \mathcal{K}_{p,q}$. So the multiplication operator $M_u : L^p(\mathcal{A}) \to L^q(\Sigma)$ is bounded. Define $\Lambda : L^{\frac{p}{q}}(\mathcal{A}) \to \mathbb{C}$ given by $\Lambda(f) = \int_X E(|u|^q) f d\mu$. We shall show that the linear functional Λ is bounded. For each $f \in L^{\frac{p}{q}}(\mathcal{A})$ we have

$$|\Lambda(f)| \le \|E(|u|^q)|f|\ \|_1 = \|E(|u|^q|f|)\|_1 = \|u|f|^{\frac{1}{q}}\|_q^q$$
$$= \|M_u|f|^{\frac{1}{q}}\|_q^q \le \|M_u\|^q\ \|\ |f|^{\frac{1}{q}}\|_p^q = \|M_u\|^q\|f\|_{\frac{p}{q}} .$$

It follows that $\|\Lambda\| \le \|M_u\|^q$. By the Riesz representation theorem, there exists the unique function $g \in L^{\frac{r}{q}}(\mathcal{A})$ such that $\Lambda(f) = \int_X g f d\mu$, for each $f \in L^{\frac{p}{q}}(\mathcal{A})$. Therefore $g = E(|u|^q)$ a.e. on X, and hence $(E(|u|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$.

Now, if $(E(|u|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$, it follows easily from the Hölder's inequality that $\|uf\|_q \le \|(E(|u|^q))^{\frac{1}{q}}\|_r\|f\|_p$, for each $f \in L^p(\mathcal{A})$. Thus, $u \in \mathcal{K}_{p,q}$.

(b) Suppose that both (i) and (ii) hold. Then, for each $f \in L^p(\mathcal{A})$ with $\|f\|_p \le 1$ we have

$$\|uf\|_q^q = \|E(|u|^q)|f|^q\|_1 = (\int_B + \int_{\cup A_n})(E(|u|^q)|f|^q)d\mu$$
$$= \sum_{n\in\mathbb{N}} \int_{A_n} E(|u|^q)|f|^q d\mu = \sum_{n\in\mathbb{N}} (E(|u|^q)(A_n)|f(A_n)|^q\mu(A_n)$$
$$= \sum_{n\in\mathbb{N}} \frac{(E(|u|^q)(A_n)}{\mu(A_n)^{\frac{q}{r}}}(|f(A_n)|^p\ \mu(A_n))^{\frac{q}{p}} \le M\|f\|_p^q$$

where we have used the fact that $E(|u|^q)$ is constant $\mathcal{A}$-measurable function on each A_n (see [9, Theorem I.7.3]). Hence $u \in \mathcal{K}_{p,q}$.

Conversely, suppose $u \in \mathcal{K}_{p,q}$. So the multiplication operator $M_u : L^p(\mathcal{A}) \to L^p(\Sigma)$ is bounded. First we show that $E(|u|^q) = 0$ a.e. on B. Assuming the contrary, we can find some $\delta > 0$ such that $\mu(\{x \in B : E(|u|^q)(x) \ge \delta\}) > 0$. Put $F = \{x \in B : E(|u|^q)(x) \ge \delta\}$. Since $(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is a σ-finite measure space, we can suppose that $\mu(F) < \infty$. Also since F is non-atomic, then for all $n \in \mathbb{N}$, there exists $F_n \subseteq F$ such that $\mu(F_n) = \frac{\mu(F)}{2^n}$. For any $n \in \mathbb{N}$, put $f_n = \mu(F_n)^{\frac{-1}{p}} \chi_{F_n}$. It is clear that $f_n \in L^p(\mathcal{A})$ and $\|f_n\|_p = 1$. Since $\frac{q}{p} > 1$, we have

$$\infty > \|M_u\|^q \geq \|uf_n\|_q^q = \frac{1}{\mu(F_n)^{\frac{q}{p}}} \|u\chi_{F_n}\|_q^q = \frac{1}{\mu(F_n)^{\frac{q}{p}}} \int_{F_n} E(|u|^q) d\mu$$

$$\geq \frac{\delta\mu(F_n)}{\mu(F_n)^{\frac{q}{p}}} = \delta\mu(F_n)^{1-\frac{q}{p}} = \delta \left(\frac{2^n}{\mu(F)}\right)^{\frac{q}{p}-1} \longrightarrow \infty, \quad \text{as } n \to \infty,$$

which is a contradiction. Hence we conclude that $\mu(\{x \in B : E(|u|^q)(x) \neq 0\}) = 0$. Now, we examine the superimum in (ii). For any $n \in \mathbb{N}$, put $f_n = \mu(A_n)^{\frac{-1}{p}} \chi_{A_n}$. Then it is clear that $f_n \in L^p(\mathcal{A})$ and $\|f_n\|_p = 1$. Similar argument shows that $M \leq \|M_u\|^q < \infty$. □

Cases: $1 \leq q < \infty = p$, $1 \leq p < \infty = q$ and $p = q = \infty$

Theorem 2.2. *(a) If $1 \leq q < \infty = p$, then $u \in \mathcal{K}_{\infty,q}$ if and only if $E(|u|^q) \in L^1(\mathcal{A})$.*

(b) If $1 \leq p < \infty = q$, then $u \in \mathcal{K}_{p,\infty}$ if and only if $E(|u|^p) = 0$ a.e. on B and $\sup_{n\in\mathbb{N}} \frac{E(|u|^p)(A_n)}{\mu(A_n)} < \infty$.

(c) $u \in \mathcal{K}_\infty$ if and only if $E(|u|) \in L^\infty(\mathcal{A})$.

Proof. (a) Let $E(|u|^q) \in L^1(\mathcal{A})$ and take $f \in L^\infty(\mathcal{A})$. Then we have

$$\|uf\|_q^q \leq \int_X E(|u|^q)|f|^q d\mu \leq \|f\|_\infty^q \ \|E(|u|^q)\|_1 \ .$$

Hence $u \in \mathcal{K}_{\infty,q}$. Now suppose only that the multiplication operator $M_u : L^\infty(\mathcal{A}) \to L^q(\Sigma)$ is bounded. Then $\|E(|u|^q)\|_1 = \int_X |u|^q d\mu \leq \|M_u\|^q < \infty$.

(b) Suppose that $u \in \mathcal{K}_{p,\infty}$. By the same argument in the proof of the Theorem 2.1(b), if we put $f_n = \chi_{F_n}$, we have then

$$\infty > \|M_u\|^p \geq \frac{\|M_u\chi_{F_n}\|_{L^\infty(\mathcal{A})}^p}{\|\chi_{F_n}\|_p^p} = \frac{\sup_{A\in\mathcal{A},\ 0<\mu(A)<\infty} \frac{1}{\mu(A)} \int_A E(|u|^p)\chi_{F_n} d\mu}{\mu(F_n)}$$

$$\geq \frac{\delta}{\mu(F_n)} = \frac{\delta 2^n}{\mu(F)} \longrightarrow \infty, \quad \text{as } n \to \infty,$$

which is a contradiction. Hence we conclude that $E(|u|^p) = 0$ a.e. on B. Also, for any $n \in \mathbb{N}$ we have

$$E(|u|^p)(A_n) \leq \|E(|u|^p)\|_{L^\infty(\mathcal{A})} = \sup_{n\in\mathbb{N}} \frac{1}{\mu(A_n)} \int_{A_n} E(|u|^p) d\mu$$

$$= \sup_{n\in\mathbb{N}} \frac{1}{\mu(A_n)} \int_{A_n} |u\chi_{A_n}|^p d\mu = \|M_u\chi_{A_n}\|_{L^\infty(\mathcal{A})}^p \leq \|M_u\|^p \mu(A_n).$$

It follows that $M := \sup_{n\in\mathbb{N}} \frac{E(|u|^p)(A_n)}{\mu(A_n)} < \infty$. Conversely, suppose that $E(|u|^p) = 0$ a.e. on B and $M < \infty$. For each $f \in L^p(\mathcal{A})$, we have

$$\|uf\|^p_{L^\infty(\mathcal{A})} = \sup_{n\in\mathbb{N}} \frac{1}{\mu(A_n)} \int_{A_n} |uf|^p d\mu \leq \sup_{n\in\mathbb{N}} \frac{1}{\mu(A_n)} \int_{A_n} E(|u|^p)|f|^p d\mu$$

$$= \sup_{n\in\mathbb{N}} \left(\frac{E(|u|^p)(A_n)}{\mu(A_n)} \right) |f(A_n)|^p \mu(A_n) \leq M \sum_{n\in\mathbb{N}} |f(A_n)|^p \mu(A_n) \leq M\|f\|^p_p .$$

It follows that $M_u(L^p(\mathcal{A})) \subseteq L^\infty(\mathcal{A}) \subseteq L^\infty(\Sigma)$, and hence $u \in \mathcal{K}_{p,\infty}$.

(c) Suppose that $u \in L^\infty(\mathcal{A})$ and take $f \in L^\infty(\mathcal{A})$. Then we have

$$\|uf\|_{L^\infty(\mathcal{A})} = \sup_{A\in\mathcal{A},\ 0<\mu(A)<\infty} \frac{1}{\mu(A)} \int_A |uf| d\mu$$

$$\leq \|u\|_\infty \sup_{A\in\mathcal{A},\ 0<\mu(A)<\infty} \frac{1}{\mu(A)} \int_A |f| d\mu = \|u\|_\infty \|f\|_{L^\infty(\mathcal{A})}.$$

It follows that $M_u(L^\infty(\mathcal{A})) \subseteq L^\infty(\mathcal{A}) \subseteq L^\infty(\Sigma)$, and hence $u \in \mathcal{K}_\infty$. For the other direction, suppose that for each $f \in L^\infty(\mathcal{A})$, $uf \in L^\infty(\Sigma)$. Then we obtain $\|u\|_\infty = \|u\chi_X\|_\infty \leq \|M_u\| < \infty$. Thus the proposition is proved. □

Let $u \in \mathcal{K}_{p,q}$. We define

$$\|u\|_{\mathcal{K}_{p,q}} = \begin{cases} \|(E(|u|^q))^{\frac{1}{q}}\|_{\frac{pq}{p-q}} & 1 \leq q < p < \infty \\ \|E(|u|^p)\|^{\frac{1}{p}}_\infty & 1 \leq q = p < \infty \\ \|E(|u|^q)\|^{\frac{1}{q}}_1 & 1 \leq q < \infty = p \\ \left\{ \sup_n \frac{E(|u|^q)(A_n)}{\mu(A_n)^{\frac{q-p}{p}}} \right\}^{\frac{1}{q}} & 1 \leq p < q < \infty \\ \left\{ \sup_n \frac{E(|u|^p)(A_n)}{\mu(A_n)} \right\}^{\frac{1}{p}} & 1 \leq p < \infty = q \\ \|u\|_\infty & p = q = \infty. \end{cases}$$

Theorem 2.3. *$(\mathcal{K}_{p,q}, \|.\|_{\mathcal{K}_{p,q}})$ is a Banach space.*

Proof. According to the procedure used in the proof of the previous results, it is easy to see that $\|u\|_{\mathcal{K}_{p,q}} = \|M_u\|$ and hence $\mathcal{K}_{p,q}$ is a Banach space with respect to this norm. Nevertheless, we shall illustrate the case $1 \leq q < p < \infty$. The same method used in the proof of Theorem 2.1 yields $\|M_u\| \leq \|(E(|u|^q))^{\frac{1}{q}}\|_r$ and $\sup\{\int_X E(|u|^q)|f|d\mu : f$ a unit vector in $L^{\frac{p}{q}}(\Sigma)\} \leq \|M_u\|^q$. It follows that

$$\|(E(|u|^q))^{\frac{1}{q}}\|_r = \|E(|u|^q)\|^{\frac{1}{q}}_{\frac{r}{q}} \leq \|M_u\|,$$

where $r = \frac{pq}{p-q}$. □

Put $\mathcal{M}_{p,q} = \{M_u : L^p(\mathcal{A}) \to L^q(\Sigma);\ u \in \mathcal{K}_{p,q}\}$. It follows from the above theorem that the mapping $\Gamma : u \mapsto M_u$ is then an isometry from $\mathcal{K}_{p,q}$ onto $\mathcal{M}_{p,q}$. Let $M_u \in \mathcal{M}_{p,q}$. It is easy to see that for $1 \leq p,\ q < \infty$, the adjoint operator

$M_u^*: L^{q'}(\Sigma) \to L^{p'}(\mathcal{A})$ is given by $M_u^* f = E(uf)$, where p' and q' are the conjugate exponent to p and q respectively. Such operators played a central role in the classification project undertaken in [1]. It turns out that this class of operators includes a number of interesting special cases such as kernel operators, partial integral operators and Rieze homomorphisms (see [2, 3]).

In what follows, $\varphi : X \to X$ will be a non-singular measurable transformation of X, namely, a mapping from X into itself with the properties that the measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ, and $\varphi^{-1}(\Sigma)$ is sigma finite. We set $h = d\mu \circ \varphi^{-1}/d\mu$. Let $1 \leq p, q \leq \infty$ and $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$. Define $\mathcal{K}^{\varphi}_{p,q}$ the set of all multipliers of the range of composition operators from $L^p(\mathcal{A})$ into $L^q(\Sigma)$, as follows

$$\mathcal{K}^{\varphi}_{p,q} = \{w \in L^0(\Sigma) : \quad u.\mathcal{R}(C_\varphi) \subset L^q(\Sigma)\},$$

where C_φ is a composition operator on $L^p(\mathcal{A})$ and $\mathcal{R}(C_\varphi)$ is the range of C_φ. In other words, $u \in \mathcal{K}^{\varphi}_{p,q}$ if and only if the corresponding weighted composition operator $W : L^p(\mathcal{A}) \to L^q(\Sigma)$ defined as $Wf = u.f \circ \varphi$ is bounded. Put $\mathcal{K}^{\varphi}_{p,p} = \mathcal{K}^{\varphi}_{p}$ and $\|u\|_{\mathcal{K}^{\varphi}_{p,q}} = \|W\|$, for $u \in \mathcal{K}^{\varphi}_{p,q}$. Since $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$, $f \circ \varphi$ is an $\mathcal{A}$-measurable whenever f is an $\mathcal{A}$-measurable function. Suppose $1 \leq p, q < \infty$. As an application of the properties of the conditional expectation operator and using the change of variable formula, for each $f \in L^p(\mathcal{A})$, we have

$$\|uC_\varphi f\|_q^q = \int_X E^{\mathcal{A}}(|u|^q)|f \circ \varphi|^q d\mu = \int_X E^{\varphi^{-1}(\Sigma)}(E^{\mathcal{A}}(|u|^q))|f|^q \circ \varphi d\mu$$

$$= \int_X hE^{\varphi^{-1}(\Sigma)}(E^{\mathcal{A}}(|u|^q)) \circ \varphi^{-1}|f|^q d\mu = \int_X |\sqrt[q]{S}f|^q d\mu = \|M_{\sqrt[q]{S}}f\|_q^q,$$

where $S := hE^{\varphi^{-1}(\Sigma)}(E^{\mathcal{A}}(|u|^q)) \circ \varphi^{-1}$. Thus $u \in \mathcal{K}^{\varphi}_{p,q}$ if and only if $\sqrt[q]{S} \in \mathcal{K}_{p,q}$. Note that if $\mathcal{A} = \Sigma$, then $E = I$ and $\|uC_\varphi f\|_q = \|M_{\sqrt[q]{J}}f\|_q$, where $J := hE^{\varphi^{-1}(\Sigma)}(|u|^q) \circ \varphi^{-1}$. Using this fact and previous results we have the following theorem.

Theorem 2.4. *Assume that $\varphi : X \to X$ is a non-singular measurable transformation and $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$.*

(a) Let $1 \leq p = q < \infty$. Then $u \in \mathcal{K}^{\varphi}_{p}$ if and only if $S \in L^\infty(\Sigma)$. In this case $\|u\|_{\mathcal{K}^{\varphi}_{p}} = \|S\|_\infty^{\frac{1}{p}}$.

(b) Let $1 \leq q < p < \infty$. Then $u \in \mathcal{K}^{\varphi}_{p,q}$ if and only if $S \in L^{\frac{p}{p-q}}(\Sigma)$. In this case $\|u\|_{\mathcal{K}^{\varphi}_{p,q}} = \|S\|_{\frac{p}{p-q}}^{\frac{1}{q}}$.

(c) Let $1 \leq q < \infty = p$. Then $u \in \mathcal{K}^{\varphi}_{\infty,q}$ if and only if $u \in L^q(\Sigma)$. In this case $\|u\|_{\mathcal{K}^{\varphi}_{\infty,q}} = \|u\|_q$.

(d) Let $1 \leq p < q < \infty$. Then $u \in \mathcal{K}^{\varphi}_{p,q}$ if and only if $S = 0$ a.e. on B and $\sup_n \frac{S(A_n)}{\mu(A_n)^{\frac{q-p}{p}}} < \infty$. In this case $\|u\|_{\mathcal{K}^{\varphi}_{p,q}} = \{\sup_n \frac{S(A_n)}{\mu(A_n)^{\frac{q-p}{p}}}\}^{\frac{1}{q}}$.

(e) Let $1 \leq p < \infty = q$. Then $u \in \mathcal{K}^{\varphi}_{p,\infty}$ if and only if $J = 0$ a.e. on B and $\sup_n \frac{S(A_n)}{\mu(A_n)} < \infty$. In this case $\|u\|_{\mathcal{K}^{\varphi}_{p,\infty}} = \left\{\sup_n \frac{S(A_n)}{\mu(A_n)}\right\}^{\frac{1}{p}}$.

(f) Let $p = q = \infty$. Then $u \in \mathcal{K}^{\varphi}_{\infty}$ if and only $u \in L^{\infty}(\Sigma)$. In this case $\|u\|_{\mathcal{K}^{\varphi}_{\infty}} = \|u\|_{\infty}$.

Note that when $\mathcal{A} = \Sigma$, all parts of the Theorem 2.4 can be easily follows from [17].

3. Essential norm of weighted composition operators

In this section by using conditional expectation operator we determine the essential norm of uC_{φ} on the spaces $L^p(\Sigma)$ for $1 < p < \infty$ in terms of the set $\{x \in X : J(x) \geq r > 0\}$. First, we collect some materials and facts that will be needed in the sequel.

Let $\mathbf{B}$ be a Banach space and $\mathcal{K}$ be the set of all compact operators in $\mathbf{B}$. For any bounded linear operator T on $\mathbf{B}$, the essential norm of T means the distance from T to $\mathcal{K}$ in the operator norm, namely

$$\|T\|_e = \inf\{\|T - S\| : \ S \in \mathcal{K}\}.$$

Clearly, T is compact if and only if $\|T\|_e = 0$. As is seen in [16], the essential norm plays an interesting role in the compact problem of concrete operators. Many people have computed the essential norm of various concrete operators. For these studies about (weighted) composition operators, refer to [13, 18, 7].

We are concerned with the case that T is a weighted composition operator $W := uC_{\varphi}$ on $L^p(\Sigma)$. In [8], Chan has showed that uC_{φ} is compact on $L^p(\Sigma)$ if and only if

$$\text{for any } \varepsilon > 0, \{x \in X : J(x) \geq \varepsilon\} \text{ consists of finitely many atoms.} \tag{3.1}$$

From this point of view, we compute the essential norm of uC_{φ}.

Theorem 3.1. *Let $1 < p < \infty$ and $uC_{\varphi} : L^p(\Sigma) \to L^p(\Sigma)$ be a bounded weighted composition operator. The essential norm of uC_{φ} is given by*

$$\|uC_{\varphi}\|_e = \inf\{r > 0 : G_r \text{ consists of finitely many atoms}\}, \tag{3.2}$$

where $G_r = \{x \in X : \sqrt[p]{J(x)} \geq r\}$. Considering the case $\|uC_{\varphi}\|_e = 0$ in (3.2), we know that (3.1) is necessary and sufficient for uC_{φ} to be compact.

Proof. Denote the right side of (3.2) by α. We first show that $\|uC_{\varphi}\|_e \geq \alpha$. If $\alpha = 0$, there is nothing to prove, and so we assume that $\alpha > 0$. Take $\varepsilon > 0$ arbitrarily. The definition of α implies that $F = G_{\alpha - \varepsilon/2}$ either contains a non-atomic subset or has infinitely many atoms. If F contains a non-atomic subset, then there are measurable sets F_n, $n \in \mathbb{N}$, such that $F_{n+1} \subseteq F_n \subseteq F$, $0 < \mu(F_n) < \frac{1}{n}$. Define $f_n = \mu(F_n)^{-\frac{1}{p}} \chi_{F_n}$. Then $\|f_n\|_p = 1$ for all $n \in \mathbb{N}$. We claim that $f_n \to 0$ weakly. For this we show that $\int_X f_n g \to 0$ for all $g \in L^q(\Sigma)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Let $A \subseteq F$ with $0 < \mu(A) < \infty$ and $g = \chi_A$. Then

$$\left| \int_X f_n \chi_A d\mu \right| = \mu(F_n)^{-\frac{1}{p}} \ \mu(A \cap F_n) \leq \left(\frac{1}{n}\right)^{1-\frac{1}{p}} \longrightarrow 0, \quad \text{as } n \to \infty.$$

Since simple functions are dense in $L^q(\Sigma)$, thus f_n is proved to converge to 0 weakly. Now assume that F consists of infinitely many atoms. Let $\{F_n\}_{n=1}^{\infty}$ be disjoint atoms in F. Again put f_n as above. It is easy to see that for $A \subseteq F$ with $0 < \mu(A) < \infty$ we have $\mu(A \cap F_n) = 0$, for sufficiently large n. So in both case $\int_X f_n g \to 0$. Now, take a compact operator T on $L^p(\Sigma)$ such that $\|uC_\varphi - T\| < \|uC_\varphi\|_e + \frac{\varepsilon}{2}$. Then we have

$$\|uC_\varphi\|_e > \|uC_\varphi - T\| - \frac{\varepsilon}{2} \geq \|uC_\varphi f_n - Tf_n\|_p - \frac{\varepsilon}{2}$$

$$\geq \|uC_\varphi f_n\|_p - \|Tf_n\|_p - \frac{\varepsilon}{2} = \|M_{\sqrt[p]{J}} f_n\|_p - \|Tf_n\|_p - \frac{\varepsilon}{2}$$

$$\geq \left(\int_{F_n} |\sqrt[p]{J} f_n|^p d\mu\right)^{\frac{1}{p}} - \|Tf_n\|_p - \frac{\varepsilon}{2} \geq (\alpha - \frac{\varepsilon}{2}) - \|Tf_n\|_p - \frac{\varepsilon}{2}$$

for all $n \in \mathbb{N}$. Since a compact operator maps weakly convergent sequences into norm convergent ones, it follows $\|Tf_n\|_p \to 0$. Hence $\|uC_\varphi\|_e \geq \alpha - \varepsilon$. Since ε was arbitrary, we obtain $\|uC_\varphi\|_e \geq \alpha$.

For the opposite inequality, take ε arbitrarily. Put $K = G_{\alpha+\varepsilon}$ and $v = \chi_K u$. The definition of α implies that K consists of finitely many atoms. So we can write $K = \{K_1, K_2, \ldots, K_m\}$, where $K_1, K_2, \ldots, K_m$ are distinct. Since $vC_\varphi f(X) = \sum_{i=1}^m v(K_i) f(\varphi(K_i))$, for all $f \in L^p(\Sigma)$, hence vC_φ has finite rank. Noting that vC_φ is a compact operator, so we have

$$\|uC_\varphi - vC_\varphi\| = \|(1 - \chi_K)uC_\varphi\| = \sup_{\|f\|_p \leq 1} \|(\chi_{X\setminus K} u)C_\varphi f\|_p$$

$$= \sup_{\|f\|_p \leq 1} \left(\int_X hE^{\varphi^{-1}(\Sigma)}(\chi_{X\setminus K}|u|^p) \circ \varphi^{-1} |f|^p \, d\mu\right)^{\frac{1}{p}}$$

$$= \sup_{\|f\|_p \leq 1} \left(\int_{X\setminus K} hE^{\varphi^{-1}(\Sigma)}(|u|^p) \circ \varphi^{-1} |f|^p \, d\mu\right)^{\frac{1}{p}}$$

$$= \sup_{\|f\|_p \leq 1} \left(\int_{X\setminus K} |\sqrt[p]{J} f|^p \, d\mu\right)^{\frac{1}{p}} \leq (\alpha + \varepsilon) \sup_{\|f\|_p \leq 1} \left(\int_{X\setminus K} |f|^p \, d\mu\right)^{\frac{1}{p}} \leq (\alpha + \varepsilon).$$

Since ε was arbitrary, we get $\|uC_\varphi\|_e \leq \alpha$. □

Note that, in case of non-atomic measures, the equality $\|uC_\varphi\| = \|uC_\varphi\|_e$ was proved in much greater generality by Schep in [15].

4. Examples

In this section examples are then given to illustrating some of the previous results and to show that how the conditional expectation operators work in practice.

Example 4.1. Let $w = \{m_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Consider the space $l^p(w) = L^p(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and μ is a measure on $2^{\mathbb{N}}$ defined by $\mu(\{n\}) = m_n$. Let $u = \{u_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Put $\mathcal{A} = \varphi^{-1}(2^{\mathbb{N}})$, where $\varphi : \mathbb{N} \to \mathbb{N}$ is a non-singular measurable transformation. Direct computation shows that

$$E(f)(k) = \frac{\sum_{n\in\varphi^{-1}(\varphi(k))} f_n m_n}{\sum_{n\in\varphi^{-1}(\varphi(k))} m_n},$$

for all non-negative sequence $f = \{f_n\}_{n=1}^{\infty}$ and $k \in \mathbb{N}$. By using the previous results, $u \in K_{p,q}$ if and only if there exists $M_i > 0$ $(1 \le i \le 4)$ such that for all $k \in \mathbb{N}$, we have

$$(a) \quad \sum_{n\in\varphi^{-1}(\varphi(k))} |u_n|^p m_n \le M_1 \sum_{n\in\varphi^{-1}(\varphi(k))} m_n \qquad (1 \le p = q < \infty).$$

$$(b) \quad \sum_{k=1}^{\infty} \left\{ \frac{\sum_{n\in\varphi^{-1}(\varphi(k))} |u_n|^q m_n}{\sum_{n\in\varphi^{-1}(\varphi(k))} m_n} \right\}^{\frac{p}{p-q}} m_k < \infty \qquad (1 \le q < p < \infty).$$

$$(c) \quad \sum_{n\in\varphi^{-1}(\varphi(k))} |u_n|^q m_n \le M_3 m_n^{\frac{q-p}{p}} \sum_{n\in\varphi^{-1}(\varphi(k))} m_n \quad (1 \le p < q < \infty).$$

$$(d) \quad \sum_{k=1}^{\infty} \left\{ \frac{\sum_{n\in\varphi^{-1}(\varphi(k))} |u_n|^q m_n}{\sum_{n\in\varphi^{-1}(\varphi(k))} m_n} \right\} m_k < \infty \qquad (1 \le q < \infty = p).$$

$$(e) \quad \sum_{n\in\varphi^{-1}(\varphi(k))} |u_n|^p m_n \le M_4 m_k \sum_{n\in\varphi^{-1}(\varphi(k))} m_n \qquad (1 \le q < \infty = p).$$

Example 4.2. Let $X = (-\pi/2, \pi/2)$, $d\mu = dx$, Σ the Lebesgue sets, and $\mathcal{A}$ the σ-subalgebra generated by the symmetric sets about the origin. Put $0 < a < \pi/2$. Then for each $u \in L^0(\Sigma)$ we have

$$\int_{-a}^{a} E(|u|^p)(x)dx = \int_{-a}^{a} |u(x)|^p dx$$

$$= \int_{-a}^{a} \left\{ \frac{|u(x)|^p + |u(-x)|^p}{2} + \frac{|u(x)|^p - |u(-x)|^p}{2} \right\} dx = \int_{-a}^{a} \frac{|u(x)|^p + |u(-x)|^p}{2} dx.$$

Consequently, $E(|u|^p)(x) = (|u(x)|^p + |u(-x)|^p)/2$. This example is due to Lambert [11]. Now, since $|u(x)|^p \le 2(|u(x)|^p + |u(-x)|^p)/2 = 2E(|u|^p)(x)$, then $\mathcal{K}_p \subseteq L^\infty(\Sigma)$. On the other hand we always have $L^\infty(\Sigma) \subseteq \mathcal{K}_p$. Thus $\mathcal{K}_p = L^\infty(\Sigma)$. Note that if $u(x) = x^2 + \tan x$, then $E(u)(x) = x^2 \in L^\infty(\mathcal{A})$ but $u \notin L^\infty(\Sigma)$.

Example 4.3. Let $X = [-1, 1]$, $d\mu = \frac{1}{2}dx$ and Σ the Lebesgue sets. Define the non-singular transformations $\varphi_i : X \to X$ by $\varphi_1(x) = \sqrt[3]{3x}$ and $\varphi_2(x) = (\sqrt{1+x} - 1)\chi_{[-1,0]} + (1 - \sqrt{1-x})\chi_{(0,1]}$. Put $h_{\varphi_i} = d\mu \circ \varphi_i^{-1}/d\mu$ and $\mathcal{A} = \varphi_2^{-1}(\Sigma)$. It is easy to see that $E^{\varphi_1^{-1}(\Sigma)} = I$ and $E^{\mathcal{A}}(f) = (f(x) + f(-x))/2$, for all positive measurable function f on X. Put $u(x) = \sqrt{x^2 + x + 1}$. Direct computations show that $h_{\varphi_1}(x) = x^2$, $h_{\varphi_2}(x) = (2+2x)\chi_{[-1,0]} + (2-2x)\chi_{(0,1]}$ and $E^{\mathcal{A}}(u^2)(x) = x^2+1$. Therefore we get that

$$S_1(x) := h_{\varphi_1}(x)E^{\varphi_1^{-1}(\Sigma)}(E^{\mathcal{A}}(u^2)) \circ \varphi_1^{-1}(x) = x^2 + \frac{1}{9}x^8,$$

$$J_1(x) = x^2 + \frac{1}{3}x^5 + \frac{1}{9}x^8,$$

$$J_2(x) = (2+2x)\left((2x+x^2)^2+1\right)\chi_{[-1,0]} + (2-2x)\left((2x-x^2)^2+1\right)\chi_{(0,1]},$$

where $J_i := h_{\varphi_i}E^{\varphi_i^{-1}(\Sigma)}(u^2) \circ \varphi_i^{-1}$. If $W_i = u.f \circ \varphi_i$, then we get that

$$\|W_1\|_{L^2(\mathcal{A}) \to L^2(\Sigma)} = \frac{\sqrt{10}}{3}, \quad \|W_1\|_{L^2(\Sigma) \to L^2(\Sigma)} = \frac{\sqrt{13}}{3}, \quad \|W_2\|_{L^2(\Sigma) \to L^2(\Sigma)} = 2\sqrt{10}.$$

Example 4.4. Let $l^p(w)$, u and φ be the same as stated informally in the hypotheses of Example 4.1. Put $\mathcal{A} = \varphi^{-2}(2^{\mathbb{N}})$. Direct computations show that

$$h(k) = \frac{1}{m_k}\sum_{j \in T^{-1}(k)} m_j\ , \qquad h_2(k) = \frac{d\mu \circ \varphi^{-2}}{d\mu}(k) = \frac{1}{m_k}\sum_{j \in \varphi^{-1}(k)} h(j)m_j\ ,$$

$$J(k) = \frac{1}{m_k}\sum_{j \in \varphi^{-1}(k)} (u(j))^2 m_j\ , \qquad (h_2E^{\mathcal{A}}(u^2) \circ \varphi^{-2})(k) = \frac{1}{m_k}\sum_{j \in \varphi^{-2}(k)} (u(j))^2 m_j\ ,$$

$$E^{\varphi^{-1}(\Sigma)}(u^2)(k) = \frac{\sum_{j \in \varphi^{-1}(\varphi(k))}(u(j))^2 m_j}{\sum_{j \in \varphi^{-1}(\varphi(k))} m_j}\ , \quad E^{\mathcal{A}}(u^2)(k) = \frac{\sum_{j \in \varphi^{-2}(\varphi^2(k))}(u(j))^2 m_j}{\sum_{j \in \varphi^{-2}(\varphi^2(k))} h(j)m_j},$$

$$S(k) = h(k)E^{\varphi^{-1}(\Sigma)}(E^{\mathcal{A}}(u^2)) \circ \varphi^{-1}(k) = \frac{1}{m_k}\sum_{j \in \varphi^{-1}(k)} (E^{\mathcal{A}}(u^2))(j)m_j.$$

Example 4.5. Let $X = (-\infty, 0] \cup \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Let μ be the Lebesque measure on $(-\infty, 0]$ and $\mu(\{n\}) = \frac{1}{2^n}$, if $n \in \mathbb{N}$. Define $u(x) = 1$ and $\varphi : \mathbb{N} \to \mathbb{N}$ as: $\varphi(1) = \varphi(2) = \varphi(3) = 1$, $\varphi(4) = 2$, $\varphi(5) = \varphi(6) = 3$, $\varphi(2n+1) = 5$, for $n \geq 3$, $\varphi(2n) = 2n-2$, for $n \geq 4$, and $\varphi(x) = 5x$ for all $x \in (-\infty, 0]$. Then a simple computation gives $J = h = \frac{7}{4}\chi_{\{1\}} + \frac{1}{4}\chi_{\{2\}} + \frac{3}{8}\chi_{\{3\}} + \frac{1}{3}\chi_{\{2n+1:\ n \geq 3\}} + \frac{1}{4}\chi_{\{2n:\ n \geq 4\}} + \frac{1}{5}\chi_{(-\infty,0]}$. Thus $\|C_\varphi\|_e = 3^{-\frac{1}{p}}$ on $L^p(X, \Sigma, \mu)$ for $1 < p < \infty$.

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References

1. P. Dodds, C. Huijsmans and B. de Pagter, *Characterizations of conditional expectation-type operators*, Pacific J. Math. **141** (1990), 55–77.
2. J.J. Grobler and B. de Pagter, *Operators representable as multiplication-conditional expectation operators*, J. Operator Theory **48** (2002), 15–40.
3. J. Herron, *Weighted conditional expectation operators on L^p spaces*, UNC Charlotte Doctoral Dissertation, 2004.
4. H. Hudzik, R. Kumar and R. Kumar, *Matrix multiplication operators on Banach function spaces*, Proc. Indian Acad. Sci. Math. Sci. **116** (2006), 71–81.
5. M.R. Jabbarzadeh and E. Pourreza, *A note on weighted composition operators on L^p-spaces*, Bull. Iranian Math. Soc. **29** (2003), 47–54.

6. M.R. Jabbarzadeh, *Weighted composition operators between L^p-spaces*, Bull. Korean Math. Soc. **42** (2005), 369–378.
7. M.R. Jabbarzadeh, *The essential norm of a composition operator on Orlicz spaces*, Turkish J. Math., to appear.
8. J.-T. Chan, *A note on compact weighted composition operators on $L^p(\mu)$*, Acta Sci. Math. **56** (1992), 165–168.
9. S. Kantorovitz, *Introduction to modern analysis*, Oxford University Press, 2003.
10. A. Lambert, *L^∞ norm estimates for conditional expectations*, Proc. Roy. Irish Acad. Sect. A **97** (1997), 169–173.
11. A. Lambert, *L^p multipliers and nested sigma-algebras*, Oper. Theory Adv. Appl. **104** (1998), 147–153.
12. A. Lambert and B.M. Weinstock, *A class of operator algebras induced by probabilistic conditional expectations*, Michigan Math. J. **40** (1993), 359–376.
13. A. Montes-Rodríguez, *The essential norm of a composition operator on Bloch spaces*, Pacific J. Math. **188** (1999), 339–351.
14. M.M. Rao, *Conditional measure and applications*, Marcel Dekker, New York, 1993.
15. A.R. Schep, *The measure of non-compactness of a disjointness preserving operator*, J. Operator Theory **21** (1989), 397–402.
16. J.H. Shapiro, *The essential norm of a composition operator*, Analls of Math. **125** (1987), 375–404.
17. H. Takagi and K. Yokouchi, *Multiplication and composition operators between two L^p-spaces*, Contemporary Math. **232** (1999), 321–338.
18. H. Takagi, T. Miura and S.-E. Takahasi, *Essential norm and stability constants of weighted composition operators on $C(X)$*, Bull. Korean Math. Soc. **40** (2003), 583–591.
19. A.C. Zaanen, *Integration*, 2nd ed., North-Holland, Amsterdam, 1967.

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