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FINITE-DIMENSIONAL HILBERT C^*-MODULES

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Abstract. In this paper we obtain a characterization of finite-dimensional Hilbert C^*-modules. It is known that those are the modules for which both underlying C^*-algebras are finite-dimensional. We show that such modules can be described by a certain property of bounded sequences of their elements. It turns out that similar property leads to another characterization of Hilbert C^*-modules over C^*-algebras of compact operators.

1. Introduction and preliminaries

Hilbert C^*-modules are straightforward generalization of Hilbert spaces where the field of complex numbers is replaced by a C^*-algebra. The concept was introduced by Kaplansky [10]. The origin of Hilbert C^*-modules is in operator theory, where they serve as a useful tool in areas like KK-theory, quantum groups and several other areas.

Although Hilbert C^*-modules behave like Hilbert spaces in some way, some fundamental and familiar Hilbert space properties do not hold. For example, given a closed submodule W of a Hilbert C^*-module V, we can define $W^\perp$ in a natural way. Then $W^\perp$ is a closed submodule, but usually $V \neq W \oplus W^\perp$ and $W \neq (W^\perp)^\perp$. However, this is always true in the class of Hilbert C^*-modules over a C^*-algebra of (not necessarily all) compact operators on some Hilbert space. Also, many other properties of Hilbert spaces that fail in general Hilbert C^*-modules are proved to be satisfied in Hilbert C^*-modules over C^*-algebras of

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compact operators. For results and, in particular, characterizations of this class of Hilbert C^*-modules we refer the reader to [2, 4, 8, 12, 18] and references therein. Also, some interesting properties of Hilbert C^*-modules over finite-dimensional C^*-algebras are obtained in [6, 9].

An interesting subclass consists of finite-dimensional Hilbert C^*-modules. A full Hilbert C^*-module is finite-dimensional if and only if both underlying C^*-algebras are finite-dimensional. We show in Theorem 2.5 that finite-dimensional Hilbert C^*-modules are also characterized by a certain property of bounded sequences of their elements. An analysis of that property combined with results of K. Ylinen ([20], [21]) enables us to obtain a new characterization of Hilbert C^*-modules over C^*-algebras of compact operators.

Before stating the results, we recall the definition of a Hilbert C^*-module and introduce our notation.

A *pre-Hilbert C^*-module V over a C^*-algebra $\mathcal{A}$*, or a *pre-Hilbert $\mathcal{A}$-module* is a right $\mathcal{A}$-module together with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathcal{A}$ satisfying the conditions:

- $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for $x, y, z \in V$, $\alpha, \beta \in \mathbb{C}$,
- $\langle x, ya\rangle = \langle x, y\rangle a$ for $x, y \in V$, $a \in \mathcal{A}$,
- $\langle x, y\rangle^* = \langle y, x\rangle$ for $x, y \in V$,
- $\langle x, x\rangle \geq 0$ for $x \in V$,
- $\langle x, x\rangle = 0$ if and only if $x = 0$.

We can define a norm on V by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. A pre-Hilbert $\mathcal{A}$-module V is called a *right Hilbert C^*-module over $\mathcal{A}$* (or a *right Hilbert $\mathcal{A}$-module*) if it is complete with respect to its norm. The notion of the *left Hilbert $\mathcal{A}$-module* is defined in a similar way.

Basic examples of Hilbert C^*-modules are as follows.

(I) Every Hilbert space is a left Hilbert $\mathbb{C}$-module.

(II) Every C^*-algebra $\mathcal{A}$ is a right Hilbert $\mathcal{A}$-module via $\langle a, b\rangle = a^*b$ for $a, b \in \mathcal{A}$.

(III) For every pair of Hilbert spaces H_1 and H_2, the space $\mathbb{B}(H_1, H_2)$ of all bounded linear operators from H_1 to H_2 is a right Hilbert $\mathbb{B}(H_1)$-module with the inner product $\langle T, S\rangle = T^*S$.

By $\langle V, V\rangle$ we denote the closure of the span of $\{\langle x, y\rangle : x, y \in V\}$. We say that V is *full* if $\langle V, V\rangle = \mathcal{A}$.

A mapping $T : V \to W$ between Hilbert $\mathcal{A}$-modules V and W is called *adjointable* if there exists a mapping $T^* : W \to V$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in V$, $y \in W$. It is easy to see that every adjointable operator T is a bounded linear $\mathcal{A}$-module mapping (that is, T is bounded, linear and satisfies $T(xa) = T(x)a$ for all $x \in V$, $a \in \mathcal{A}$). $\mathbb{B}(V, W)$ will stand for the space of all adjointable mappings from V into W.

By $\mathbb{K}(V, W)$ we denote the closed linear subspace of $\mathbb{B}(V, W)$ spanned by $\{\theta_{x,y} : x \in W, y \in V\}$, where $\theta_{x,y}$ is a mapping in $\mathbb{B}(V, W)$ defined by $\theta_{x,y}(z) = x\langle y, z\rangle$. Elements of $\mathbb{K}(V, W)$ are called 'compact' operators. When we say that a bounded linear operator T between Banach spaces is compact, we mean that it is compact in topological sense. Elements of $\mathbb{K}(V,W)$ considered as operators between the Banach spaces V and W need not be compact in topological sense.

We shall write $\mathbb{B}(V)$ for $\mathbb{B}(V,V)$, and $\mathbb{K}(V)$ for $\mathbb{K}(V,V)$. It is well known that $\mathbb{B}(V)$ is a C^*-algebra containing $\mathbb{K}(V)$ as a two-sided ideal.

By a finite-dimensional C^*-algebra (resp. Hilbert C^*-module) we understand a C^*-algebra (resp. Hilbert C^*-module) that is finite-dimensional as a vector space.

For a Banach space X, by X^* we denote the set of all bounded linear functionals on X. A sequence (x_n) in the Banach space X is said to be *weakly convergent* if there is $x_0 \in X$ such that $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all $f \in X^*$. A bounded (anti)linear mapping $T : X \to Y$ between Banach spaces X and Y is *weakly compact* if for every bounded sequence (x_n) in X, the sequence (Tx_n) has a weakly convergent subsequence in Y.

The basic theory of Hilbert C^*-modules can be found in [11, 13, 16, 19]. (For the general theory of C^*-algebras the reader is referred to [7, 14, 15, 17].)

2. Hilbert C^*-modules over finite-dimensional C^*-algebras

Let $(H, (\cdot,\cdot))$ be a Hilbert space, $\mathbb{B}(H)$ the algebra of all bounded linear operators, and $\mathbb{K}(H)$ the algebra of all compact linear operators acting on it. It is well known that for every bounded sequence (ξ_n) in H there exist a subsequence (ξ_{n_k}) of (ξ_n) and $\xi \in H$ such that

$$\lim_{k\to\infty} \|T\xi_{n_k} - T\xi\| = 0, \quad \forall T \in \mathbb{K}(H).$$

This follows from the fact that every bounded sequence in a Hilbert space has a weakly convergent subsequence, and that compact operators map weakly convergent sequences to the strongly convergent ones.

Suppose now that V is a Hilbert $\mathcal{A}$-module. One can ask whether for every bounded sequence (v_n) in V, there are a subsequence (v_{n_k}) of (v_n) and $v \in V$ for which

$$\lim_{k\to\infty} \|Tv_{n_k} - Tv\| = 0, \quad \forall T \in \mathbb{K}(V).$$

In general, the answer is negative. For example, let $\mathcal{A} = \mathbb{B}(H)$ for some infinite-dimensional Hilbert space H, and regard $\mathcal{A}$ as a Hilbert C^*-module over itself. Then the identity operator on H will also be 'compact'; however, since H is infinite-dimensional, the above cannot hold.

The following lemma will help us to characterize the class of Hilbert C^*-modules which possess the above property.

Lemma 2.1. *Let V be a right Hilbert $\mathcal{A}$-module. For a bounded sequence (v_n) in V and $v \in V$ the following statements are mutually equivalent.*

(i) $\lim_{n\to\infty} \|\langle y, v_n\rangle - \langle y, v\rangle\| = 0$ *for every* $y \in V$.

(ii) $\lim_{n\to\infty} \|Tv_n - Tv\| = 0$ *for every* $T \in \mathbb{K}(V)$.

Proof. (i)⇒(ii) From (i) it follows that

$$\lim_{n\to\infty} \|x\langle y, v_n\rangle - x\langle y, v\rangle\| = 0$$

for all $x, y \in V$, that is,

$$\lim_{n\to\infty} \|\theta_{x,y}(v_n) - \theta_{x,y}(v)\| = 0$$

for all $x, y \in V$. Since (v_n) is bounded and every $T \in \mathbb{K}(V)$ is a limit of finite linear combinations of mappings $\theta_{x,y}$, (ii) follows.

(ii)⇒(i) If (ii) holds, then for all $x \in V$ we have

$$\lim_{n\to\infty} \|\theta_{x,x}(v_n) - \theta_{x,x}(v)\| = 0,$$

that is,

$$\lim_{n\to\infty} \|x\langle x, v_n\rangle - x\langle x, v\rangle\| = 0.$$

This implies, for all $x \in V$,

$$\lim_{n\to\infty} \|\langle x, x\rangle\langle x, v_n\rangle - \langle x, x\rangle\langle x, v\rangle\| = 0,$$

which can be written in an equivalent form

$$\lim_{n\to\infty} \big\|\big\langle x\langle x, x\rangle, v_n\big\rangle - \big\langle x\langle x, x\rangle, v\big\rangle\big\| = 0.$$

To get (i) it remains to note that every $y \in V$ can be written as $y = x\langle x, x\rangle$ for some $x \in V$ (see e.g. [16, Proposition 2.31]). □

Remark 2.2. Observe that in the implication (ii)⇒(i) the sequence (v_n) does not have to be bounded.

In a recent paper [6] on perturbation of the Wigner equation in inner product C^*-modules, the main result is obtained for Hilbert $\mathcal{A}$-modules with the following property:

[H] *for every bounded sequence (v_n) in V there are a subsequence (v_{n_k}) of (v_n) and $v \in V$ such that for every $y \in V$*

$$\lim_{k\to\infty} \|\langle y, v_{n_k}\rangle - \langle y, v\rangle\| = 0.$$

It was proved in [6, Proposition 2.1] that condition [H] is satisfied in every Hilbert C^*-module over a finite-dimensional C^*-algebra. Later, in [3, Theorem 2.5], it was proved that if a full Hilbert $\mathcal{A}$-module satisfies condition [H], then $\mathcal{A}$ must be finite-dimensional. Therefore, condition [H] characterizes the class of Hilbert C^*-modules over finite-dimensional C^*-algebras, which, together with Lemma 2.1, gives us another characterization of this class of Hilbert C^*-modules.

Theorem 2.3. *Let V be a full right Hilbert $\mathcal{A}$-module. For every bounded sequence (v_n) in V there are a subsequence (v_{n_k}) of (v_n) and $v \in V$ such that*

$$\lim_{k\to\infty} \|Tv_{n_k} - Tv\| = 0, \quad \forall T \in \mathbb{K}(V)$$

if and only if $\mathcal{A}$ is a finite-dimensional C^*-algebra.

Since Theorem 2.3 also holds in the case of left Hilbert C^*-modules, one can reformulate its statement to get a characterization of full right Hilbert $\mathcal{A}$-modules V such that the C^*-algebra $\mathbb{K}(V)$ is finite-dimensional.

Theorem 2.4. *Let V be a full right Hilbert $\mathcal{A}$-module. For every bounded sequence (v_n) in V there are a subsequence (v_{n_k}) of (v_n) and $w \in V$ such that*

$$\lim_{k\to\infty} \|v_{n_k}a - wa\| = 0, \quad \forall a \in \mathcal{A}$$

if and only if $\mathbb{K}(V)$ is a finite-dimensional C^*-algebra.

Proof. Every right Hilbert C^*-module V over a C^*-algebra $\mathcal{A}$ can be regarded as a left Hilbert C^*-module over the C^*-algebra $\mathbb{K}(V)$, where the action of an operator $T \in \mathbb{K}(V)$ on a vector $x \in V$ is given by $T \cdot x = T(x)$, while the inner product is defined as $[x, y] = \theta_{x,y}$. By definition of $\mathbb{K}(V)$, V is full as a left Hilbert $\mathbb{K}(V)$-module. The ideal of all 'compact' operators acting on a left Hilbert $\mathbb{K}(V)$-module V is spanned by mappings $\varphi_{x,y}$, $x, y \in V$, where $\varphi_{x,y}(v) = [v, y]x, v \in V$. Since

$$\varphi_{x,y}(v) = [v, y]x = \theta_{v,y}(x) = v\langle y, x\rangle$$

for all $x, y, v \in V$, we deduce that every 'compact' operator on a left Hilbert $\mathbb{K}(V)$-module V is of the form $v \mapsto va$ for some $a \in \mathcal{A}$. It remains to apply Theorem 2.3. □

Finite-dimensional Hilbert C^*-modules can be now completely described in terms of the convergence of certain sequences.

Theorem 2.5. *Let V be a full right Hilbert $\mathcal{A}$-module. The following statements are mutually equivalent.*

(1) *V is finite-dimensional.*
(2) *$\mathcal{A}$ and $\mathbb{K}(V)$ are finite-dimensional.*
(3) *For every bounded sequence (v_n) in V there are a subsequence (v_{n_k}) of (v_n) and $v \in V$ such that*

$$\lim_{k\to\infty} \|v_{n_k}a - va\| = 0, \quad \forall a \in \mathcal{A},$$

$$\lim_{k\to\infty} \|\langle y, v_{n_k}\rangle - \langle y, v\rangle\| = 0, \quad \forall y \in V.$$

(4) *$\mathbb{K}(V)$ is a unital C^*-algebra, and for every bounded sequence (v_n) in V there are a subsequence (v_{n_k}) of (v_n) and $v \in V$ such that*

$$\lim_{k\to\infty} \|\langle y, v_{n_k}\rangle - \langle y, v\rangle\| = 0, \quad \forall y \in V.$$

(5) *$\mathcal{A}$ is a unital C^*-algebra, and for every bounded sequence (v_n) in V there are a subsequence (v_{n_k}) of (v_n) and $v \in V$ such that*

$$\lim_{k\to\infty} \|v_{n_k}a - va\| = 0, \quad \forall a \in \mathcal{A}.$$

Proof. Obviously, (1)⇒(2). To prove (2)⇒(1), first notice: when $\mathbb{K}(V)$ is finite-dimensional, it is necessarily unital and hence V is algebraically finitely generated. This, together with the assumption that $\mathcal{A}$ is finite-dimensional, immediately implies (1).

By [3, Theorem 2.5] and Theorem 2.4, (3)⇒(2). If (5) holds, then putting $a = e$ in the second condition of (5) we get that every bounded sequence in V has a convergent subsequence, so (1) holds. Similarly, (4)⇒(1). (2)⇒(4) and

(2)⇒(5) follow from [6, Proposition 2.1], resp. Theorem 2.4, and the fact that finite-dimensional C^*-algebras are unital.

Suppose that (2) holds. Then for every bounded sequence (v_n) there are a subsequence (v_{n_k}) of (v_n) and $v, w \in V$ such that

$$\lim_{k\to\infty} \|\langle y, v_{n_k}\rangle - \langle y, v\rangle\| = 0, \quad \forall y \in V,$$

$$\lim_{k\to\infty} \|v_{n_k}a - wa\| = 0, \quad \forall a \in \mathcal{A}.$$

Then for every $a \in \mathcal{A}$ and $y \in V$ we have

$$\lim_{k\to\infty} \|\langle y, v_{n_k}a\rangle - \langle y, va\rangle\| = 0,$$

$$\lim_{k\to\infty} \|\langle y, v_{n_k}a\rangle - \langle y, wa\rangle\| = 0.$$

Therefore $\langle y, va\rangle = \langle y, wa\rangle$ for all $a \in \mathcal{A}$ and $y \in V$, so $v = w$. This gives (3). □

If a C^*-algebra $\mathcal{A}$ is considered as a Hilbert C^*-module over itself, then conditions from the statement (3) of Theorem 2.5 coincide, and we have the following corollary.

Corollary 2.6. *A C^*-algebra $\mathcal{A}$ is finite-dimensional if and only if for every bounded sequence (a_n) in $\mathcal{A}$ there are a subsequence (a_{n_k}) of (a_n) and $a \in \mathcal{A}$ such that*

$$\lim_{k\to\infty} a_{n_k}b = ab, \quad \forall b \in \mathcal{A}.$$

Remark 2.7. Observe that if a full right Hilbert $\mathcal{A}$-module V satisfies the following two conditions:

(i) mappings $v \mapsto va$ from V into V are compact for all $a \in \mathcal{A}$;
(ii) for every bounded sequence (v_n) in V there are a subsequence (v_{n_k}) of (v_n) and $v \in V$ such that

$$\lim_{k\to\infty} \|\langle y, v_{n_k}\rangle - \langle y, v\rangle\| = 0, \quad \forall y \in V,$$

then V must be finite-dimensional. (Namely, (ii) means that the C^*-algebra $\mathcal{A}$ is finite-dimensional, so $\mathcal{A}$ is unital. From (i) it follows now then the identity operator on V is compact, that is, V is finite-dimensional.)

In a similar way we deduce that a full right Hilbert $\mathcal{A}$-module V satisfying the following two conditions:

(i) for every bounded sequence (v_n) in V there are a subsequence (v_{n_k}) of (v_n) and $v \in V$ such that

$$\lim_{k\to\infty} \|v_{n_k}a - va\| = 0, \quad \forall a \in \mathcal{A};$$

(ii) mappings $v \mapsto \langle y, v\rangle$ are compact from V into $\mathcal{A}$ for all $y \in V$,

must also be finite-dimensional.

However, if a full right Hilbert $\mathcal{A}$-module V satisfies conditions

(i) mappings $v \mapsto va$ from V into V are compact for all $a \in \mathcal{A}$, and
(ii) mappings $v \mapsto \langle y, v\rangle$ are compact from V into $\mathcal{A}$ for all $y \in V$,

then V does not have to be finite-dimensional. To see this, let H be a separable infinite-dimensional Hilbert space, and $\mathcal{A} \subset \mathbb{K}(H)$ the C^*-algebra of all diagonal (with respect to a fixed orthonormal basis) operators with diagonal entries converging to zero. Let us regard $\mathcal{A}$ as a Hilbert C^*-module over itself. Clearly, $\mathcal{A}$ is infinite-dimensional as a vector space. On the other hand, the mappings $v \mapsto va$, that is, $v \mapsto \langle y, v\rangle = y^*v = vy^*$, from $\mathcal{A}$ into $\mathcal{A}$ are compact for all $a, y \in \mathcal{A}$. (For details see Remark 2.6 of [3].)

3. Hilbert C^*-modules over C^*-algebras of compact operators

In this section we study Hilbert C^*-modules with the property that mappings $v \mapsto \langle y, v\rangle$ from V into $\mathcal{A}$ are weakly compact for all $y \in V$. We first consider some other mappings (related to every Hilbert C^*-module) whose weak (or norm) compactness is equivalent to the weak compactness of the mapping $v \mapsto \langle y, v\rangle$. We use results from [20] and [21] obtained in the setting of C^*-algebras. Combining this with results from [2], we get some new characterizations of Hilbert C^*-modules over compact operators.

Since we shall use linking algebras, we first recall relevant definitions.

Given a Hilbert C^*-module V over a C^*-algebra $\mathcal{A}$, the *linking algebra* $\mathcal{L}(V)$ is defined as the matrix algebra of the form

$$\mathcal{L}(V) = \begin{bmatrix} \mathbb{K}(\mathcal{A}) & \mathbb{K}(V, \mathcal{A}) \\ \mathbb{K}(\mathcal{A}, V) & \mathbb{K}(V) \end{bmatrix}.$$

Observe that $\mathcal{L}(V)$ is in fact the C^*-algebra of all 'compact' operators acting on the Hilbert C^*-module $\mathcal{A} \oplus V$ over $\mathcal{A}$. Each $v \in V$ induces the mappings $r_v \in \mathbb{B}(\mathcal{A}, V)$ and $l_v \in \mathbb{B}(V, \mathcal{A})$ given by $r_v(a) = va$ and $l_v(w) = \langle v, w\rangle$ such that $l_v^* = r_v$. The mapping $v \mapsto l_v$ is an isometric conjugate linear isomorphism from V to $\mathbb{K}(V, \mathcal{A})$, and $v \mapsto r_v$ is an isometric linear isomorphism from V to $\mathbb{K}(\mathcal{A}, V)$. Furthermore, every $a \in \mathcal{A}$ induces the mapping $T_a \in \mathbb{K}(\mathcal{A})$ given by $T_a(b) = ab$, and the mapping $a \mapsto T_a$ defines an isomorphism of C^*-algebras $\mathcal{A}$ and $\mathbb{K}(\mathcal{A})$. Therefore, we may write

$$\mathcal{L}(V) = \{ \begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix} : a \in \mathcal{A}, x, y \in V, T \in \mathbb{K}(V)\}$$

and identify:

$$\begin{aligned} \mathbb{K}(\mathcal{A}) &= \mathbb{K}(\mathcal{A} \oplus 0) \subseteq \mathbb{K}(\mathcal{A} \oplus V) = \mathcal{L}(V), \\ \mathbb{K}(V) &= \mathbb{K}(0 \oplus V) \subseteq \mathbb{K}(\mathcal{A} \oplus V) = \mathcal{L}(V). \end{aligned}$$

For details about linking algebras we refer to [16, 1, 5].

Theorem 3.1. *Let V be a full right Hilbert $\mathcal{A}$-module. For every $y \in V$ the following statements are mutually equivalent.*

(1) *$v \mapsto y\langle v, y\rangle$ is a compact mapping on V.*
(2) *$v \mapsto y\langle v, y\rangle$ is a weakly compact mapping on V.*
(3) *$v \mapsto \langle v, y\rangle$ is a weakly compact mapping from V into $\mathcal{A}$.*
(4) *$T \mapsto Ty$ is a weakly compact mapping from $\mathbb{K}(V)$ into V.*
(5) *$a \mapsto ya$ is a weakly compact mapping from $\mathcal{A}$ into V.*

(6) $v \mapsto \theta_{y,v}$ *is a weakly compact mapping from* V *into* $\mathbb{K}(V)$.

Proof. Let us take an arbitrary $y \in V$ and define $Y = \begin{bmatrix} 0 & 0 \\ r_y & 0 \end{bmatrix} \in \mathcal{L}(V)$. Then $v \mapsto y\langle v, y\rangle$ is a compact mapping on V if and only if the mapping $X \mapsto YXY$ is compact on $\mathcal{L}(V)$ (see the proof of Proposition 2 in [2]). Furthermore, by [21, Theorem 3.1], the mapping $X \mapsto YXY$ is compact on $\mathcal{L}(V)$ if and only if $X \mapsto XY$ is weakly compact on $\mathcal{L}(V)$ if and only if $X \mapsto YX$ is weakly compact on $\mathcal{L}(V)$. Writing $X \in \mathcal{L}(V)$ as $\begin{bmatrix} T_a & l_v \\ r_u & S \end{bmatrix}$ we have

$$XY = \begin{bmatrix} T_{\langle v,y\rangle} & 0 \\ r_{Sy} & 0 \end{bmatrix}.$$

We will now prove that the weak compactness of $X \mapsto XY$ on $\mathcal{L}(V)$ implies (3) and (4).

So, suppose that $X \mapsto XY$ is weakly compact on $\mathcal{L}(V)$. Let (v_n) and (S_n) be bounded sequences in V and $\mathbb{K}(V)$, respectively. Then $X_n = \begin{bmatrix} 0 & l_{v_n} \\ 0 & S_n \end{bmatrix}$ is a bounded sequence in $\mathcal{L}(V)$, so, by assumption, there are a subsequence (X_{n_k}) of (X_n) and $X_0 = \begin{bmatrix} T_{a_0} & l_{v_0} \\ r_{u_0} & S_0 \end{bmatrix} \in \mathcal{L}(V)$ such that

$$\lim_{k\to\infty} F(X_{n_k}Y) = \lim_{k\to\infty} F(\begin{bmatrix} T_{\langle v_{n_k},y\rangle} & 0 \\ r_{S_{n_k}y} & 0 \end{bmatrix}) = F(X_0), \quad \forall F \in \mathcal{L}(V)^*.$$

In particular, for $F \in \mathcal{L}(V)^*$ defined by $F(\begin{bmatrix} T_a & l_v \\ r_u & S \end{bmatrix}) = f(a)$, where $f \in \mathcal{A}^*$, we get that $f(\langle v_{n_k}, y\rangle)$ converges to $f(a_0)$ for every $f \in \mathcal{A}^*$, i.e., $v \mapsto \langle v, y\rangle$ is a weakly compact mapping from V into A. This proves that (1)⇒(3). Similarly, if we take $F \in \mathcal{L}(V)^*$ defined by $F(\begin{bmatrix} T_a & l_v \\ r_u & T \end{bmatrix}) = g(u)$, where $g \in V^*$, we get that $g(S_{n_k}y)$ converges to $g(u_0)$ for every $g \in V^*$, i.e., $T \mapsto Ty$ is a weakly compact mapping from $\mathbb{K}(V)$ into V, which gives (1)⇒(4).

Since

$$YX = \begin{bmatrix} 0 & 0 \\ r_{ya} & \theta_{y,v} \end{bmatrix},$$

one can prove in the same way that the weak compactness of $X \mapsto YX$ implies (5) and (6), i.e., (1)⇒(5) and (1)⇒(6).

Observe that the mapping $v \mapsto y\langle v, y\rangle$ from V into V can be written as a composition of the bounded mappings $v \mapsto \langle v, y\rangle$ from V into $\mathcal{A}$ and $a \mapsto ya$ from $\mathcal{A}$ into V. Since the composition of a bounded operator and a weakly compact operator is weakly compact, we conclude that (3)⇒(2) and (5)⇒(2). Another way to get the mapping $v \mapsto y\langle v, y\rangle$ is to compose bounded mappings from (4) and (6), so we analogously conclude that (4)⇒(2) and (6)⇒(2).

Since obviously (1)⇒(2), it only remains to show (2)⇒(1), that is, (2) implies compactness of the mapping $X \mapsto YXY$ on $\mathcal{L}(V)$. For this, it is enough to prove that (2) implies weak compactness of $X \mapsto YXY$ on $\mathcal{L}(V)$ since, by Theorem 3.1 of [20], such a mapping will be compact as well.

Observe that

$$YXY = \begin{bmatrix} 0 & 0 \\ r_{y\langle v,y\rangle} & 0 \end{bmatrix}.$$

Let (X_n) be a bounded sequence in $\mathcal{L}(V)$ and let $X_n = \begin{bmatrix} T_{a_n} & l_{v_n} \\ r_{u_n} & S_n \end{bmatrix}$ for $n \in \mathbb{N}$. Then (v_n) is a bounded sequences in V. If $v \mapsto y\langle v, y\rangle$ is weakly compact, then there are a subsequence (v_{n_k}) of (v_n) and $u_0 \in \mathcal{A}$ such that

$$\lim_{k\to\infty} g(y\langle v_{n_k}, y\rangle) = g(u_0), \quad \forall g \in V^*.$$

Then for $X_0 = \begin{bmatrix} 0 & 0 \\ r_{u_0} & 0 \end{bmatrix}$ we have

$$\lim_{k\to\infty} F(YX_{n_k}Y) = F(X_0), \quad \forall F \in \mathcal{L}(V)^*.$$

Indeed, every $F \in \mathcal{L}(V)^*$ can be written as

$$F(\begin{bmatrix} T_a & l_v \\ r_u & S \end{bmatrix}) = f_1(a) + \overline{f_2(v)} + f_3(u) + f_4(S),$$

where $f_1 \in \mathcal{A}^*$, $f_2, f_3 \in V^*$, $f_4 \in \mathbb{K}(V)^*$ are defined by

$$f_1(a) = F(\begin{bmatrix} T_a & 0 \\ 0 & 0 \end{bmatrix}), \quad \overline{f_2(v)} = F(\begin{bmatrix} 0 & l_v \\ 0 & 0 \end{bmatrix}),$$

$$f_3(u) = F(\begin{bmatrix} 0 & 0 \\ r_u & 0 \end{bmatrix}), \quad f_4(S) = F(\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}),$$

where $^-$ in the definition of f_2 stands for complex conjugation. We now have

$$\begin{aligned} \lim_{k\to\infty} F(YX_{n_k}Y) &= \lim_{k\to\infty} F(\begin{bmatrix} 0 & 0 \\ r_{y\langle v_{n_k},y\rangle} & 0 \end{bmatrix}) \\ &= \lim_{k\to\infty} f_3(y\langle v_{n_k}, y\rangle) \\ &= f_3(u_0) = F(X_0) \end{aligned}$$

which shows that $X \mapsto YXY$ is weakly compact on $\mathcal{L}(V)$. □

Observe that if we regard a C^*-algebra as a Hilbert C^*-module over itself, we get generalizations of [21, Theorem 3.1] and [20, Theorem 3.1].

As an immediate consequence of the preceding theorem and [2, Proposition 2], we obtain another characterization of Hilbert C^*-modules over compact operators.

Corollary 3.2. *Let V be a full right Hilbert $\mathcal{A}$-module. The following statements are mutually equivalent.*

(1) *There is a faithful representation $\pi : \mathcal{A} \to \mathbb{B}(H)$ such that $\pi(\mathcal{A}) \subseteq \mathbb{K}(H)$.*
(2) *For every $y \in V$ the mapping $v \mapsto y\langle v, y\rangle$ is compact on V.*
(3) *For every $y \in V$ the mapping $v \mapsto y\langle v, y\rangle$ is weakly compact on V.*
(4) *For every $y \in V$ the mapping $v \mapsto \langle v, y\rangle$ is weakly compact from V into $\mathcal{A}$.*

(5) *For every* $y \in V$ *the mapping* $T \mapsto Ty$ *is weakly compact from* $\mathbb{K}(V)$ *into* V.
(6) *For every* $y \in V$ *the mapping* $a \mapsto ya$ *is weakly compact from* $\mathcal{A}$ *into* V.
(7) *For every* $y \in V$ *the mapping* $v \mapsto \theta_{y,v}$ *is weakly compact from* V *into* $\mathbb{K}(V)$.

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