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WEIGHTED INEQUALITIES AND SPECTRAL PROBLEMS

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Dedicated to Professor Lars-Erik Persson on the occasion of his 65th birthday

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ABSTRACT. It is shown that the conditions of the validity of the Hardy inequality coincide with the conditions on the spectrum of some (nonlinear) differential operators to be bounded from below and discrete.

1. INTRODUCTION AND PRELIMINARIES

The aim of this paper is to show the mutual connection between the (N-dimensional) Hardy inequality

$$\left(\int_\Omega |f|^q u\,dx\right)^{1/q} \le C\left(\int_\Omega |\nabla f|^p v\,dx\right)^{1/p}, \quad f \in C_0^\infty(\Omega) \tag{1.1}$$

and the spectral problem

$$\begin{aligned} -\operatorname{div}(v|\nabla f|^{p-2}\nabla f) &= \lambda u|f|^{q-2}f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

Here Ω is a domain in $\mathbb{R}^N$ with boundary $\partial\Omega$, p, q are real parameters, $1 < p, q < \infty$, and u, v are weight functions on Ω, i.e. measurable and a.e. positive functions.

As an example, let us consider the special case $p = q = 2$, $u = v \equiv 1$. Then inequality (1.1) is the Friedrichs-Poincaré inequality

$$\int_\Omega |f|^2 dx \le C^2 \int_\Omega |\nabla f|^2 dx,$$

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and the equation in (1.2) takes the form

$$-\Delta f = \lambda f \text{ in } \Omega,$$

and it is well-known, that for the first eigenvalue λ_0 of the Laplace operator Δ we have

$$\lambda_0 = \frac{1}{C^2}.$$

A similar situation appears also in the general case: For the *weak solution* f of the boundary value problem (1.2) we have the characterization

$$\int_\Omega v|\nabla f|^{p-2}\nabla f \nabla g dx = \lambda \int_\Omega u|f|^{q-2} f g\, dx$$

for every $g \in C_0^\infty(\Omega)$. If we choose $g = f$, we have, using, moreover, inequality (1.1),

$$\int_\Omega v|\nabla f|^p dx = \lambda \int_\Omega u|f|^q dx \le \lambda C^q \Big(\int_\Omega v|\nabla f|^p dx \Big)^{q/p}$$

and, if $p = q$,

$$1 \le \lambda C^p, \text{ i.e. } \lambda \ge \frac{1}{C^p}.$$

Hence, we have shown, that *if the Hardy inequality (1.1) holds, then we have a lower bound for the spectrum of* (1.2).

If we denote (quite formally) by $W_0^{1,p}(\Omega; v)$ the space of functions f on Ω with the finite norm

$$\Big(\int_\Omega |\nabla f|^p v\, dx \Big)^{1/p}$$

and by $L^q(\Omega; u)$ the set of functions with the finite norm

$$\Big(\int_\Omega |f|^q u\, dx \Big)^{1/q},$$

then the Hardy inequality (1.1) describes the imbedding of $W_0^{1,p}(\Omega; v)$ into $L^q(\Omega; u)$, and the assertion above can be formulated as follows:

If the Hardy inequality (1.1) holds [i.e., if the corresponding imbedding is continuous:

$$W_0^{1,p}(\Omega; v) \hookrightarrow L^q(\Omega; u) \;]$$

then the spectrum of (1.2) is bounded from below.

Now, let us consider the case $N = 1$, and take for Ω the interval $(0, \infty)$. Then the Hardy inequality (1.1) can be rewritten as

$$\Big(\int_0^\infty |f(t)|^q u(t)\, dt \Big)^{1/q} \le C \Big(\int_0^\infty |f'(t)|^p v(t)\, dt \Big)^{1/p} \tag{1.3}$$

and we will consider functions $f = f(t)$ such that $f(0) = 0$. If we, moreover, consider the case

$$1 < p \le q < \infty,$$

then inequality (1.3) holds if and only if the so-called Muckenhoupt function

$$A_M(x) := \left(\int_x^\infty u(t)\,dt\right)^{1/q}\left(\int_0^x v^{1-p'}(t)\,dt\right)^{1/p'} \tag{1.4}$$

with $p' = \frac{p}{p-1}$ is bounded. Moreover, for the best constant C in (1.3) we have

$$C \approx \sup_{x\in(0,\infty)} A_M(x).$$

If we denote by $W_L^{k,p}(0,\infty;v)$ the set of all absolutely continuous functions f on $[0,\infty)$ such that $f(0)=0$ and that

$$\|f'\|_{p,v} := \left(\int_0^\infty |f'(t)|^p v(t)dt\right)^{1/p} < \infty,$$

then – similarly as in the N-dimensional case – the Hardy inequality (1.3) describes the fact, that the imbedding

$$W_L^{1,p}(0,\infty;v) \hookrightarrow L^q(0,\infty;u) \tag{1.5}$$

is *continuous*. Moreover, it is well-known (see, e.g., [4]) that this imbedding is *compact* if and only if

$$\lim A_M(x) = 0 \text{ for } x\to 0 \text{ and } x\to\infty.$$

Since the sixties, there are several schools dealing with the investigation of the Hardy inequality, e.g. in Sweden, Canada, Czech Republic, ..., but only the Kazakhstan school has had a direct connection with (and a motivation in) spectral problems of differential operators (compare, e.g., the titles of the books [6, 8]). Therefore, it was a surprise as we observed that in 1958, Kac and Krein [2] have obtained, during the investigation of the problem

$$\begin{gathered} -y'' = \lambda\rho(t)y \quad \text{on} \quad (0,\infty), \\ y(0)=0,\ y'(0)=1, \end{gathered} \tag{1.6}$$

the following result:

(i) The spectrum of (1.6) is *bounded from below* if and only if

$$x\int_x^\infty \rho(t)dt \le C < \infty$$

and $\lambda \ge \frac{1}{4C}$.

(ii) The spectrum is *discrete* if and only if

$$\lim_{x\to\infty} x\int_x^\infty \rho(t)dt = 0. \tag{1.7}$$

The data in (1.6) correspond to the data in the Hardy inequality (1.3) for the special choice

$$p=q=2,\ v(t)\equiv 1,\ u(t)=\rho(t).$$

Moreover, the expression $x\int_x^\infty \rho(t)dt$ is connected with the corresponding Muckenhoupt function (see (1.4)): we have

$$x\int_x^\infty \rho(t)dt = A_M^2(x).$$

Consequently, we can reformulate the result of Kac and Krein as follows:

(i) The imbedding (1.5) is *continuous* ($\equiv$ the Hardy inequality holds) if and only if the spectrum of (1.6) is *bounded from below.*

(ii) The imbedding (1.5) is *compact* if and only if the spectrum is *discrete.*

Remark 1.1. In our example, we have only the condition that $A_M(x) \to 0$ for $x \to \infty$ (see (1.7)) but the second condition of compactness, $A_M(x) \to 0$ for $x \to 0$, is satisfied – for reasonable functions ρ – automatically.

Remark 1.2. Let us emphasize that the Hardy inequality was *not mentioned* by Kac and Krein. On the other hand, conditions for the boundedness (from below) and discreteness, which appear in the literature, are often expressed in terms of the Muckenhoupt function, again *without mentioning* the Hardy inequality. Moreover, in all these cases, only *linear* spectral problems have been considered, which corresponds to the special choice

$$p = q = 2.$$

Therefore, it was our aim to show that also for a *nonlinear* problem we have a close connection between (i) the *continuity* of the corresponding imbedding (i.e., the validity of the Hardy inequality) and the *boundedness* of the spectrum, and (ii) the *compactness* of the imbedding and the *discreteness* of the spectrum.

Together with P. Drábek, we succeeded for the case

$$p = q\ (\neq 2).$$

Let us consider the following spectral problem on $(0,\infty)$ with $1 < p < \infty$:

$$\begin{gathered} (v(t)\varphi(x'(t)))' + \lambda u(t)\varphi(x(t)) = 0 \quad \text{on} \quad (0,\infty), \\ x'(0) = 0,\ x(\infty) = 0, \end{gathered} \tag{1.8}$$

where

$$\varphi(s) = |s|^{p-2}s = |s|^{p-1}\operatorname{sgn} s.$$

The corresponding Hardy inequality (1.3) has now the form (notice that $p = q$)

$$\left(\int_0^\infty |f(t)|^p u(t)dt\right)^{1/p} \le C\left(\int_0^\infty |f'(t)|^p v(t)dt\right)^{1/p},$$

and since we consider functions f such that $f(\infty) = 0$, the corresponding Muckenhoupt function has now a slightly modified form:

$$A_M(t) := \left(\int_0^t u(s)\,ds\right)^{1/p}\left(\int_t^\infty v^{1-p'}(s)\,ds\right)^{1/p'}.$$

We are looking for a *weak* solution of (1.8) in the space $W^{1,p}_R(0,\infty;v)$ of all functions $x = x(t)$ absolutely continuous on $[0,\infty)$ and such that $x(\infty) = 0$ and with finite norm

$$\|x\|_{1,p,v} := \left(\int_0^\infty v(t)|x'(t)|^p \, dt\right)^{1/p}.$$

The condition

$$\lim_{t\to\infty} A_M(t) = 0 \tag{1.9}$$

which is connected with the *compactness* of the imbedding

$$W^{1,p}_R(0,\infty;v) \hookrightarrow L^p(0,\infty;u) \tag{1.10}$$

allows to obtain the following nonlinear extension of the well known Sturm–Liouville theory (let us remark that the second condition of compactness, that is $\lim_{t\to 0} A_M(t) = 0$, is satisfied automatically for reasonable weight functions u, v):

Proposition 1.3. *The set of eigenvalues of the spectral problem (1.8) forms an increasing sequence* $\{\lambda_n\}_{n=1}^\infty$ *such that*

$$\lambda_1 > 0 \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \infty.$$

Every eigenvalue λ_n*,* $n = 1, 2, \ldots,$ *is simple (i.e., there exists a unique normalized eigenfunction* x_{λ_n} *associated with* λ_n*). Moreover, the eigenfunction* x_{λ_n} *has precisely* $n-1$ *zeros in* $(0,\infty)$*. In particular,* x_{λ_1} *does not change sign in* $(0,\infty)$*. For* $n \geq 3$*, between two consecutive zeros of* $x_{\lambda_{n-1}}$ *in* $(0,\infty)$*, there is exactly one zero of* x_{λ_n}*.*

The proof of this proposition is based on oscillatoricity properties of ordinary differential operators and uses some sophisticated tools from nonlinear functional analysis. All details can be found in [1].

Remark 1.4. If the important condition (1.9) is violated, but $A_M(t)$ is bounded (i.e. $\sup_{(0,\infty)} A_M(t) = A_M < \infty$), then only the *continuous* imbedding (1.10) holds, which guarantees the boundedness of possible eigenvalues from below.

If (1.9) is violated, then we have

- either *no* eigenvalue at all, or
- a *continuum* of eigenvalues (i.e. the spectrum is bounded from below, but *not discrete*).

Example 1.5. $p = 2$, $u = v \equiv 1$. No eigenvalue:

$$x''(t) + \lambda x(t) = 0, \;\; x'(0) = 0, \;\; x(\infty) = 0$$

$$A_M(t) = \left(\int_0^t ds\right)^{1/2}\left(\int_t^\infty ds\right)^{1/2} = \infty$$

Example 1.6. $p = 2$, $u \equiv 1$, $v(t) = (t+1)^2$. Every $\lambda \geq \frac{1}{4}$ is an eigenvalue:

$$((t+1)^2 x'(t))' + \lambda x(t) = 0,\ x'(0) = 0,\ x(\infty) = 0$$

$$A_M(t) = \left(\int_0^t ds\right)^{1/2} \left(\int_t^\infty (s+1)^{-2}\, ds\right)^{1/2} = \left(\frac{t}{t+1}\right)^{1/2} \to 1 \text{ for } t \to \infty.$$

Several authors have derived conditions (necessary and sufficient) for the spectrum of a (linear) ordinary differential operator to be bounded from below and discrete without mentioning any connection with the Hardy inequality (and maybe not being is some cases aware that such a connection exists). Besides the result of Kac and Krein mentioned above let us mention as a further example the following result of Lewis: in [5] he has shown that for the spectrum of the (higher order) equation

$$(-1)^n (v(x) y^{(n)})^{(n)} = \lambda u(x) y.$$

the corresponding condition reads

$$\lim_{x \to \infty} x^{2n-1} \int_x^\infty \frac{1}{v(t)} dt = 0$$

This condition is (for $n = 1$ and $u = 1$) a counterpart of the condition (1.9).

Remark 1.7. All details concerning the Hardy inequality can be found in the books [4, 3, 7].

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