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# $G$-CONVERGENCE AND HOMOGENIZATION OF MONOTONE DAMPED HYPERBOLIC EQUATIONS

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*Dedicated to Professor Lars-Erik Persson on the occasion of his 65th anniversary*

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ABSTRACT. Multiscale stochastic homogenization is studied for quasilinear hyperbolic problems. We consider the asymptotic behaviour of a sequence of realizations of the form $\frac{\partial^2 u_\varepsilon^\omega}{\partial t^2} - \operatorname{div}\left(a\left(T_1(\frac{x}{\varepsilon_1})\omega_1, T_2(\frac{x}{\varepsilon_2})\omega_2, t, Du_\varepsilon^\omega\right)\right) - \Delta(\frac{\partial u_\varepsilon^\omega}{\partial t}) + G\left(T_3(\frac{x}{\varepsilon_3})\omega_3, t, \frac{\partial u_\varepsilon^\omega}{\partial t}\right) = f$. It is shown, under certain structure assumptions on the random maps $a(\omega_1, \omega_2, t, \xi)$ and $G(\omega_3, t, \eta)$, that the sequence $\{u_\epsilon^\omega\}$ of solutions converges weakly in $L^p(0,T; W_0^{1,p}(\Omega))$ to the solution $u$ of the homogenized problem $\frac{\partial^2 u}{\partial t^2} - \operatorname{div}(b(t, Du)) - \Delta(\frac{\partial u}{\partial t}) + \overline{G}(t, \frac{\partial u}{\partial t}) = f$.

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## 1. Introduction and preliminaries

In this paper we consider the homogenization problem for the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial^2 u_\varepsilon^\omega}{\partial t^2} - \operatorname{div}\left(a\left(T_1(\dfrac{x}{\varepsilon_1})\omega_1, T_2(\dfrac{x}{\varepsilon_2})\omega_2, t, Du_\varepsilon^\omega\right)\right) - \Delta(\dfrac{\partial u_\varepsilon^\omega}{\partial t}) \\ \qquad\qquad\qquad + G\left(T_3(\frac{x}{\varepsilon_3})\omega_3, t, \frac{\partial u_\varepsilon^\omega}{\partial t}\right) = f \text{ in } Q, \\ u_\varepsilon^\omega(x,0) = u_0(x) \text{ in } \Omega, \\ \dfrac{\partial u_\varepsilon^\omega}{\partial t}(x,0) = u_1(x) \text{ in } \Omega, \\ u_\varepsilon^\omega(x,t) = 0 \text{ in } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where $\Omega$ is an open bounded set in $\mathbb{R}^n$, $T$ is a positive real number and $Q = \Omega \times (0,T)$. For each fixed $\omega_i \in X_i$, $i = 1, 2, 3$, the realization (1.1) is an initial-boundary value problem. Following the framework in [15] and [16], see also [7] we associate three probability spaces $(X_k, \mathcal{F}_k, \mu_k)$, $k = 1,2,3$. Each $\mathcal{F}_k$ is a complete $\sigma$-algebra and each $\mu_k$ is the associated countably additive non-negative probability measure on $\mathcal{F}_k$ normalized by $\mu_k(X_k) = 1$. For every $x \in \mathbb{R}^n$ we associate the dynamical system

$$T_k(x) : X_k \to X_k.$$

For the random fields

$$a\,(x, \omega_1, \omega_2, t, \xi) \quad \text{and} \quad G\,(\omega_3, t, \eta)$$

we can then, for fixed $\omega_1$, $\omega_2$ and $\omega_3$, consider the realization

$$a\,(T_1(x)\omega_1, T_2(x)\omega_2, t, \xi) \quad \text{and} \quad G\,(T_3(x)\omega_3, t, \eta)$$

and the "speeded up" realizations

$$a\left(T_1(\frac{x}{\varepsilon_1})\omega_1, T_2(\frac{x}{\varepsilon_2})\omega_2, t, \xi\right) \quad \text{and} \quad G\left(T_3(\frac{x}{\varepsilon_3})\omega_3, t, \eta\right)$$

respectively. With this construction which will be precisely defined below the random field becomes stationary due to the invariance properties of the associated probability measure and therefore the Birkhoff ergodic theorem applies and we can define the limit of the speeded up realization in terms of the expectation (the mean value) over the probability space. We also assume that $\varepsilon_1$ and $\varepsilon_2$ are two well separated functions (scales) of $\varepsilon > 0$ which converge to zero as $\varepsilon$ tends to zero. Well separatedness means

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_2}{\varepsilon_1} = 0.$$

Further we assume that the map $a = a(\omega_1, \omega_2, t, \xi)$ satisfies certain coercivity and growth conditions in $\xi$ and is measurable in $(\omega_1, \omega_2, t)$. We assume that the map $G = G(\omega_3, t, \eta) = \rho(\omega_3, t)|\eta|^\gamma sgn(\eta)$, $0 < \gamma \leq 1$. Under the structure conditions (to be precisely defined below) it follows that for given data $f \in L^2(Q)$, $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$ there exists a unique solution $u_\epsilon^\omega \in L^p(0,T; W_0^{1,p}(\Omega))$ to (1.1) with time derivative $\frac{\partial u_\varepsilon^\omega}{\partial t} \in L^2(Q)$ and $\dfrac{\partial^2 u_\varepsilon^\omega}{\partial t^2} \in L^q(0,T; W^{-1,q}(\Omega))$ for every fixed $\varepsilon > 0$ and almost all $(\omega_1, \omega_2) \in X_1 \times X_2$, where $p$ and $q$ are the dual exponents.

The multiscale stochastic homogenization problem for (1.1) consists in studying the asymptotic behavior of the solutions $u_\varepsilon^\omega$ as $\varepsilon$ tends to zero.

Periodic homogenization problems with more than one oscillating scale was first introduced in [2] for linear elliptic problems. Recently stochastic homogenization for the monotone hyperbolic case with a linear damping has been studied in [16]. For related recent results on deterministic homogenization beyond the periodic setting of hyperbolic problems we refer to [9] and [10]. In those papers the concept of algebra with mean value, denoted homogenizaton algebra, and non-periodic multiscale convergence, denoted Sigma convergence, are the main tools.

In this work we will introduce the reader to the general framework of $G$-convergence, which can be thought of as a non-periodic "homogenization" or stabilization of sequences of operator equations. Here we show that the general theory also applies to the situation of multiple scales and e.g. multiscale stochastic homogenization. The result of Theorem 10.3 is that the sequence $\{u_\epsilon\}$ of solutions to (1.1) converges weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ to the solution $u$ in $L^p(0,T;W_0^{1,p}(\Omega))$ to a homogenized problem of the form

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \text{div}\,(b\,(t,Du)) - \Delta(u') + \overline{G}(t,u') = f \ \text{ in } Q, \\ u(x,0) = u_0(x) \ \text{ in } \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \ \text{ in } \Omega, \\ u(x,t) = 0 \ \text{ in } \partial\Omega \times (0,T), \end{cases}$$

where $b$ and $\overline{G}$ depend on $t$ but is no longer oscillating in space with $\varepsilon$. It is also seen from the framework that the result easily extends to any number of well separated scales. A typical situation where periodic and random scales occur is the modeling of porous media. A meso-scale can be modeled as a periodic distribution of solid parts whereas a sub-scale on a finer level can be modeled by a certain random distribution. The homogenization problem for random fields in the linear elliptic case is studied in [8]. The extension to monotone operators in the random setting is studied in [12] and has been further studied in a series of papers by Efendiev and Pankov, see [7] and the references therein. They consider single spatial and temporal scales. For homogenization problem for random fields of monotone parabolic problems we refer to [15]. A prototype application of results in this paper is damped elastic wave propagation in heterogeneous media of e.g. Voigth type. We will use the general framework of $G$-convergence for monotone parabolic operators developed in [14] to prove a stochastic homogenization result for (1.1) (Theorem 10.3). The result will rely on a number of well-known results for elliptic and parabolic $G$-convergence. For the benefit of the reader we review these results with references to the original proofs. This makes the present work more self-contained. The paper is organized as follows: In Section 2 we recall the basic terminology of $G$-convergence of parabolic operators, in Section 3 we introduce some basic facts about monotone operators in reflexive Banach Spaces. In Section 4 we state a theorem for existence and uniqueness of weak solution to a quasilinear hyperbolic problem. Sections 5, 6 and 7 review some basic results for elliptic and parabolic $G$-convergence that will be needed in the proof of convergence of hyperbolic problems which is presented in Section 8. Section 9 is a preparation on multiscale stochastic operators where the framework is based on a dynamical systems setup and in Section 10 we prove a homogenization result for the nonlinearly damped quasilinear hyperbolic problem (1.1).

## 2. General setting - $G$-convergence

Let us say that we are interested in the asymptotic behaviour (as $h \to \infty$) for a sequence of parabolic initial-boundary value problems of the form

$$\begin{cases} u_h' - \operatorname{div}(a_h(x,t,Du_h)) = f \ \text{ in } \ Q, \\ u_h(0) = u_0 \ \text{ in } \Omega, \\ u_h \in L^p(0,T;W_0^{1,p}(\Omega)), \end{cases}$$

where $\Omega$ is an open bounded set in $\mathbb{R}^n$, $T$ is a positive real number, $Q = \Omega \times (0,T)$ and $2 \leq p < \infty$. The maps $a_h$ are assumed to be monotone and to satisfy certain boundedness and coercivness assumptions uniformly in $h$. We will see that the general theory yields a subsequence still denoted by $\{a_h\}$ and a map $a$ with the same qualitative properties as the maps $\{a_h\}$ such that, as $h \to \infty$,

$$\begin{aligned} &u_h \to u \text{ weakly in } \ L^p(0,T;W_0^{1,p}(\Omega)) \\ &\text{and} \\ &a_h(x,t,Du_h) \to a(x,t,Du) \text{ weakly in } \ L^q(0,T;L^q(\Omega;\mathbb{R}^n)), \end{aligned}$$

respectively, where $1/p + 1/q = 1$ and where $u$ is the solution of the following initial-boundary value problem:

$$\begin{cases} u' - \operatorname{div}(a(x,t,Du)) = f \ \text{ in } \ Q \\ u(0) = u_0 \ \text{ in } \Omega, \\ u \in L^p(0,T;W_0^{1,p}(\Omega)), \end{cases}$$

where the map $a$ only depends on the subsequence $\{a_h\}$. This yields $G$-convergence of quasilinear parabolic operators. For a complete treatment of $G$-convergence of monotone parabolic operators we refer to [14].

## 3. Some notations

Let us introduce some function spaces related to the differential equations studied in this paper. For a nice introduction to partial differential operators in Banach spaces we refer to the monograph [1] by Barbu. Let $V$ be a reflexive real Banach space with dual $V'$ and let H be a real Hilbert space. We introduce the evolution triple

$$V \subseteq H \subseteq V',$$

with dense embeddings. Further, for positive real-valued T and for $2 \leq p < \infty$, let us introduce the spaces $\mathcal{V} = L^p(0,T;V)$, $\mathcal{H} = L^2(0,T;H)$ and $\mathcal{V}' = L^q(0,T;V')$, where $1/p + 1/q = 1$. Then we can consider the corresponding evolution triple

$$\mathcal{V} \subseteq \mathcal{H} \subseteq \mathcal{V}'$$

also with dense embeddings where the duality pairing $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ between $\mathcal{V}$ and $\mathcal{V}'$ is given by

$$\langle f, u \rangle_{\mathcal{V}} = \int_0^T \langle f(t), u(t) \rangle_V \, dt, \text{ for } u \in \mathcal{V}, \ f \in \mathcal{V}'.$$

We define the spaces $\mathcal{W}$ and $\mathcal{W}_0$ as

$$\mathcal{W} = \{v \in \mathcal{V} : v' \in \mathcal{V}'\} \text{ and } \mathcal{W}_0 = \{v \in \mathcal{W} : v(0) \in H\}.$$

Here $v'$ denotes the time derivative of $v$, where this derivative is taken in distributional sense. Equipped with the graph norm

$$\|v\|_{\mathcal{W}_0} = \|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}'}$$

$\mathcal{W}_0$ becomes a real reflexive Banach space. Moreover, since the embedding $\mathcal{W}_0 \to C(0, T; H)$ is continuous, every function in $\mathcal{W}_0$, with possible modification on a set of measure zero, can be considered as a continuous function with values in $H$. We also define the space $\mathcal{Z}_0$ as

$$\mathcal{Z}_0 = \{v \in \mathcal{V} : v' \in \mathcal{H}, \ v'' \in \mathcal{V}', \ v(0) \in V, \ v'(0) \in H\}.$$

Here $v''$ denotes the second time derivative of $v$, where this derivative is taken in distributional sense. Equipped with the graph norm

$$\|v\|_{\mathcal{Z}_0} = \|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{H}} + \|v''\|_{\mathcal{V}'}$$

$\mathcal{Z}_0$ becomes a real reflexive Banach space. Let us define the operator $\frac{d}{dt} : \mathcal{V} \to \mathcal{V}'$ given by

$$\frac{d}{dt} u = u' \text{ for } u \in D(\frac{d}{dt}) = \mathcal{W}_0$$

and the operator $\frac{d^2}{dt^2} : \mathcal{V} \to \mathcal{V}'$ given by

$$\frac{d^2}{dt^2} u = u'' \text{ for } u \in D(\frac{d^2}{dt^2}) = \mathcal{Z}_0.$$

We will denote by $\Omega$ a bounded open set in $\mathbb{R}^n$ and, if nothing else is said, $V = W_0^{1,p}(\Omega)$ with norm $\|u\|_V^p = \int_\Omega |Du|^p \, dx$, $H = L^2(\Omega)$ and $V' = W^{-1,q}(\Omega)$. Then the evolution triples considered above are well-defined with dense embeddings. We also define the spaces

$$U = L^p(\Omega; \mathbb{R}^n) \ \text{ and } \ U' = L^q(\Omega; \mathbb{R}^n)$$

and the spaces

$$\mathcal{U} = L^p(0, \mathrm{T}; U) \ \text{ and } \ \mathcal{U}' = L^q(0, T; U').$$

Further, we define the pairing $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ between $\mathcal{U}'$ and $\mathcal{U}$ as

$$\langle u, v \rangle_{\mathcal{U}} = \int_0^T \int_\Omega (u, v) \, dx dt, \quad \text{for } u \in \mathcal{U}' \text{ and } v \in \mathcal{U},$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^n$. By $|\cdot|$ we understand the usual Euclidean norm in $\mathbb{R}^n$. Moreover, by $\{h\}$ we understand a sequence in $\mathbb{N}$ tending to $+\infty$. We denote by $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ the modulus of continuity, i.e. an increasing function which is continuous and vanishes at the origin.

**Definition 3.1.** Given $0 < \beta \leq 1$, $2 \leq p < \infty$ and three positive real constants $c_0$, $c_1$ and $c_2$, we define the class $S = S(c_0, c_1, c_2, \beta)$ of maps

$$a : Q \times \mathbb{R}^n \to \mathbb{R}^n,$$

satisfying

(i) $|a(x,t,0)| \leq c_0$ a.e in $Q$.
(ii) $a(\cdot,\cdot,\xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^n$.
(iii) $|(a(x,t,\xi_1) - a(x,t,\xi_2)| \leq c_1(1 + |\xi_1| + |\xi_2|)^{p-1-\beta}|\xi_1 - \xi_2|^\beta$, a.e. in $Q$ for all $\xi_1, \xi_2 \in \mathbb{R}^n$.
(iv) $(a(x,t,\xi_1) - a(x,t,\xi_2), \xi_1 - \xi_2) \geq c_2|\xi_1 - \xi_2|^p$, a.e. in $Q$ for all $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$.

## 4. Existence of weak solution

Let us consider the hyperbolic initial-boundary value problem

$$\begin{cases} u'' - \operatorname{div}(a(x,t,Du)) - \Delta(u') + G(x,t,u') = f \text{ in } Q, \\ u(x,0) = u_0(x) \text{ in } \Omega, \\ u'(x,0) = u_1(x) \text{ in } \Omega, \\ u(x,t) = 0 \text{ in } \partial\Omega \times (0,T), \end{cases} \tag{4.1}$$

where $\Omega$ is an open bounded set in $\mathbb{R}^n$, T is a positive real number and $2 \leq p < \infty$. We assume that $a \in S$ and that $G : Q \times \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$G(y,s,\eta) = \theta(y,s)|\eta|^\gamma sgn(\eta),$$

where $\theta \in L^\infty(Q)$ is strictly positive a.e. in $Q$. In order to obtain the existence and uniqueness for (4.1) we will have to add some hypotheses on $a$. Following [3] we assume that there are three positive real functions $\sigma$, $\psi_1$ and $\psi_2$ such that $\psi_1$ and $\psi_2$ are continuous and $\psi_1(s) \to +\infty$ as $s \to +\infty$ and

$$\psi_1(|v|_x) \leq \sigma(v) \leq \psi_2(|v|_x)$$

for $v \in V$. We now let

$$\mathcal{A}v = \operatorname{div}(a(x,t,Du))$$

and assume that

$$\langle \mathcal{A}u(t), u'(t)\rangle \geq \frac{d}{dt}\sigma(u(t)).$$

We have the following existence and uniqueness result.

**Theorem 4.1.** *Consider the hyperbolic initial-boundary value problem (4.1) above. There exists a unique weak solution $u \in \mathcal{V}$ to (4.1) for every $f \in \mathcal{H}$, $u_0 \in V$ and $u_1 \in H$.*

*Proof.* We refer to [3] or [11] for a full proof. □

*Remark* 4.2. By the results of Theorem 4.1 the definition of the function space $\mathcal{Z}_0$ makes sense and we can now write the hyperbolic initial-boundary value problem (4.1) as

$$\begin{cases} u'' - \operatorname{div}(a(x,t,Du)) - \Delta(u') + G(x,t,u') = f \text{ in } Q, \\ u \in \mathcal{Z}_0. \end{cases}$$

## 5. Parabolic $G$-convergence

Let $\{a_h\} \subset S$ and define $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ via

$$\mathcal{A}u = -\mathrm{div}(a(x,t,Du(x,t)))$$

and consider the sequence of parabolic problems

$$\begin{cases} u_h' + \mathcal{A}_h u_h = f \ \text{ in } \ Q, \\ u_h \in \mathcal{W}_0. \end{cases} \tag{5.1}$$

We define "parabolic" $G$-convergence in the following way:

**Definition 5.1.** The sequence $\{a_h\}$ is said to G-converge to $a$ if, for every $f \in \mathcal{V}'$, the sequence $\{u_h\}$ of solutions to (5.1) satisfies

$$\begin{aligned} &u_h \rightharpoonup u \text{ weakly in } \ \mathcal{W}_0 \\ &\text{and} \\ &a_h(\cdot,\cdot,Du_h) \rightharpoonup a(\cdot,\cdot,Du) \text{ weakly in } \ \mathcal{U}', \end{aligned}$$

respectively, where $u$ is the unique solution of the problem

$$\begin{cases} u' + \mathcal{A}u = f \ \text{ in } \ Q, \\ u \in \mathcal{W}_0. \end{cases}$$

**Proposition 5.2.** *Assume that $a_h \in S$. Then there exists a unique solution $u_h \in \mathcal{W}_0$ to (5.1) for every $f \in \mathcal{V}'$ and for every $h \in \mathbb{N}$.*

The following G-compactness result is proved in [14].

**Theorem 5.3.** *Assume that $\{a_h\} \subset S$. Then, there exists a subsequence still denoted by $\{a_h\}$ and a map $a$ such that $\{a_h\}$ G-converges to $a \in S$.*

## 6. Elliptic $G$-convergence

For a complete treatment of a large class of (possibly multivalued) elliptic operators we refer to [4] and [5]. We consider the following sequence of Dirichlet boundary value problems

$$\begin{cases} -\mathrm{div}(a_h(x,Du_h)) = f_h \text{ in } \Omega, \\ u_h \in V. \end{cases} \tag{6.1}$$

**Definition 6.1.** The sequence $\{a_h\}$ is said to $G$-converge, in the elliptic sense, to $a$ if, for every $f \in \mathcal{V}'$, the sequence $\{u_h\}$ of solutions to (6.1) satisfy

$$\begin{aligned} &u_h \rightharpoonup u \text{ weakly in } \ V \\ &\text{and} \\ &a_h(\cdot,Du_h) \rightharpoonup a(\cdot,Du) \text{ weakly in } \ U', \end{aligned}$$

respectively, where $u$ is the unique solution to the problem

$$\begin{cases} -\mathrm{div}(a(x,Du)) = f \text{ in } \Omega \\ u \in V. \end{cases}$$

The following elliptic G-compactness result holds true:

**Theorem 6.2.** *Suppose that the sequence $\{a_h\}$ belongs to $S^E$. Then there exists a subsequence, still denoted by $\{h\}$, such that, for every $f \in \mathrm{V}'$, the sequence $\{a_h\}$ G-converges to a map $a \in S^E$.*

*Proof.* See [5]. □

Here $S^E$ denotes the subclass of the class $S$ of maps which does not depend on $t$.

## 7. Parameter-dependent elliptic $G$-convergence

We begin by stating a compactness result with respect to elliptic $G$-convergence for parameter dependent elliptic problems:

**Theorem 7.1.** *Suppose that the sequence $\{a_h\}$ belongs to $S$. Suppose, in addition, that*

$$|a_h(x,t,\xi) - a_h(x,s,\xi)| \leq \rho(t-s)(1+|\xi|^{p-1})$$

*for every $\xi \in \mathbb{R}^n$ and for every $t \in (0,T)$ a.e. in $\Omega$, then, there exists a subsequence still denoted by $\{h\}$, such that $\{a_h(\cdot,t,\cdot)\}$ G-converges in the elliptic sense to a map $a(\cdot,t,\cdot)$ for every $t \in (0,T)$.*

*Proof.* See [14]. □

We also recall an important comparison result which plays a crucial role in the homogenization of hyperbolic problems in Section 10.

**Theorem 7.2.** *Suppose that*

$$|a_h(x,t,\xi) - a_h(x,s,\xi)| \leq \rho(t-s)(1+|\xi|^{p-1})$$

*for all $\xi \in \mathbb{R}^n$ a.e. in $\Omega$. Suppose that $\{a_h\}$ G-converges to $a$ in the parabolic sense and that $\{a_h(\cdot,t,\cdot)\}$ G-converges to $b(\cdot,t,\cdot)$ in the elliptic sense for every $t \in (0,T)$, then $a = b$.*

*Proof.* See [14]. □

## 8. Convergence of hyperbolic problems

We begin by stating a general compensated compactness theorem for quasilinear monotone hyperbolic problems.

**Theorem 8.1.** *Let $\{u_h\}$ and $\{v_h\}$ be two sequences in $\mathcal{V}$, such, that, $\{u_h''\}$ and $\{v_h''\}$ belong to $\mathcal{V}'$, satisfying*

$$u_h \rightharpoonup u \text{ and } v_h \rightharpoonup v \text{ in } \mathcal{V},$$

$$u_h'' \rightharpoonup u'' \text{ and } v_h'' \rightharpoonup v'' \text{ in } \mathcal{V}',$$

$$\Delta u_h' \rightharpoonup \Delta u' \text{ and } \Delta v_h' \rightharpoonup \Delta v' \text{ in } \mathcal{V}',$$

$$a_h(\cdot,\cdot,Du_h) \rightharpoonup A^* \text{ and } b_h(\cdot,\cdot,Dv_h) \rightharpoonup B^*, \text{ in } \mathcal{U}'.$$

*If, in addition,*

$$u_h'' - \operatorname{div}(a_h(\cdot,\cdot,Du_h) - \Delta u_h' \text{ and } v_h'' - \operatorname{div}(b_h(\cdot,\cdot,Dv_h)) - \Delta v_h'$$

*are compact in $\mathcal{V}'$, then for arbitrary $\varphi \in \mathrm{C}_0^\infty(Q)$, as $h \to \infty$,*

$$\int_0^T \int_\Omega (a_h(x,t,Du_h) - b_h(x,t,Dv_h), Du_h - Dv_h)\varphi \, dxdt \\ \to \int_0^T \int_\Omega (A^* - B^*, Du - Dv)\varphi \, dxdt.$$

*Proof.* For the proof we refer to [16].

We also recall the following compactness result recently proved in [16].

**Theorem 8.2.** *Let us consider a sequence of parameter dependent hyperbolic problems*

$$\begin{cases} u_h'' - \operatorname{div}(a_h(x,t,Du_h)) - \Delta u_h' = f \ \text{ in } \ Q, \\ u_h \in \mathcal{Z}_0, \end{cases}$$

*where $a_h(x,t,\xi) = \alpha_h(x,t)|\xi|^{p-2}\xi$ has the same properties as the map $a$ in (4.1). For every $f \in \mathcal{H}$, $u_0 \in V$ and $u_1 \in H$ there exist subsequences $\{u_h\}$ and $\{a_h(x,t,Du_h)\}$ such that*

$$u_h \rightharpoonup u \ \text{ in } \mathcal{Z}_0,$$

$$a_h(\cdot,\cdot,Du_h) \rightharpoonup a(\cdot,\cdot,Du) \ \text{ in } \mathcal{U}',$$

*where $u$ is the unique weak solution to the limit problem*

$$\begin{cases} u'' - \operatorname{div}(a(x,t,Du)) - \Delta u' = f \ \text{ in } \ Q, \\ u \in \mathcal{Z}_0. \end{cases}$$

We now extend our result to the damped nonlinear wave equation with nonlinear damping.

**Theorem 8.3.** *Let us consider a sequence of parameter dependent damped hyperbolic problems*

$$\begin{cases} u_h'' - \operatorname{div}(a_h(x,t,Du_h)) - \Delta(u_h') + G_h(x,t,u_h') = f \ \text{ in } \ Q, \\ u_h \in \mathcal{Z}_0, \end{cases}$$

*where $a_h$ and $G_h$ are defined as above. For every $f \in \mathcal{H}$, $u_0 \in V$ and $u_1 \in H$ there exist subsequences $\{u_h\}$, $\{a_h(x,t,Du_h)\}$ and $\{G_h(x,t,u_h')\}$ such that*

$$u_h \rightharpoonup u \ \text{ in } \mathcal{Z}_0$$

$$a_h(x,t,Du_h) \rightharpoonup a(x,t,Du) \ \text{ in } \mathcal{U}',$$

*and*

$$G_h(x,t,u_h') \rightharpoonup G(x,t,u') \ \text{ in } \mathcal{U}',$$

*where $u$ is the unique weak solution to the limit problem*

$$\begin{cases} u'' - \operatorname{div}(a(x,t,Du)) - \Delta(u') + G(x,t,u') = f \ \text{ in } \ Q, \\ u \in \mathcal{Z}_0. \end{cases}$$

*Proof.* We first observe that by the existence Theorem 4.1 there exists a unique solution $u_h \in \mathcal{Z}_0$ for every $h \in \mathbb{N}$.

We write the equation as

$$\begin{cases} u_h'' - \operatorname{div}(a_h(x,t,Du_h)) - \Delta(u_h') = f - G_h(x,t,u_h') \text{ in } Q, \\ u_h \in \mathcal{Z}_0. \end{cases} \tag{8.1}$$

By Theorem 8.2 and the definition of $G$-convergence a passage to the limit in (8.1) yields

$$\begin{cases} u'' - \operatorname{div}(a(x,t,Du)) - \Delta(u') = f - N \text{ in } Q, \\ u \in \mathcal{Z}_0. \end{cases}$$

We are done if we can prove that $N = G(x,t,u')$. By the a priori estimates on the sequence $\{u_h'\}$ an application of the Aubin-Lions lemma yields

$$u_h' \to u' \text{ in } \mathcal{H}.$$

Up to a subsequence $h_j$ we then have $u_{h_j}'(x,t) \to u'(x,t)$ a.e. in $Q$. Therefore, by the Egoroff theorem, there exists a Lebesgue measurable set $Q_\mu \subset Q$ with $|Q \setminus Q_\mu| < \mu$ such that $u_{h_j}' \to u'$ uniformly in $Q_\mu$. Now let $\chi_{Q_\mu}$ be the characteristic function of $Q_\mu$. By the uniform convergence of the sequence $\{u_{h_j}'\}$ a limit passage in

$$\int_Q \chi_{Q_\mu} G_h(x,t,u_h')\psi \, dx$$

letting $h_j \to \infty$, yields

$$\int_Q \chi_{Q_\mu} G(x,t,u')\psi \, dx,$$

for any bounded and continuous function $\psi$ on $\Omega$. A passage to the limit ($\mu \to 0$), using that $|Q \setminus Q_\mu| \to 0$, yields the limit

$$\int_Q G(x,t,u')\psi \, dx.$$

□

## 9. A dynamical systems approach to stochastic multiscale analysis

For a nice exposition of the framework below we refer to the monograph [8]. Let $(X, \mathcal{F}, \mu)$ denote a probability space, where $\mathcal{F}$ is a complete $\sigma$-algebra and $\mu$ is a probability measure. For each $x \in \mathbb{R}^n$ the dynamical system

$$T(x) : X \to X$$

is such that both $T(x)$ and $T(x)^{-1}$ are measurable. Moreover it is assumed that the following (measure preserving) properties are satisfied:

- $T(0)\omega = \omega$ for each $\omega \in X$.
- $T(x+y) = T(x)T(y)$ for $x, y \in \mathbb{R}^n$.

- $\mu(T(x)^{-1}F) = \mu(F)$, for each $x \in \mathbb{R}^n$ and $F \in \mathcal{F}$.
- The set $\{(x,\omega) \in \mathbb{R}^n \times X : \ T(x)\omega \in F\}$ is a $dx \times d\mu(\omega)$ measurable subset of $\mathbb{R}^n \times X$ for each $F \in \mathcal{F}$ where $dx$ denotes the Lebesgue measure.
- For any measurable function $f(\omega)$ defined on $X$, the function $f(T(x)\omega)$ defined on $\mathbb{R}^n \times X$ is also measurable where $\mathbb{R}^n$ is endowed with the Lebesgue measure.

The dynamical system is said to be *ergodic* if every invariant function $f$, (i.e functions $f$ which satisfies $f(T(x)\omega) = f(\omega)$) is constant almost everywhere in $X$.

**Example 9.1.** (periodic case) As a special case we recover the periodic functions by letting

$$\Omega = \{\omega \in \mathbb{R}^n : 0 \le \omega_k \le 1,\, k = 1, \dots, n\} \ \text{ and } \ T(x) : \Omega \to \Omega$$

given by

$$T(x)\omega = x + \omega (mod 1).$$

For a random field $f(x,\omega)$ the "periodic" realization is given by $f(x+\omega)$.

**Definition 9.2.** We say that a vector field $f \in [L^p_{loc}(\mathbb{R}^n)]^n$ is a potential field if there exists a function $g \in W^{1,p}_0(\mathbb{R}^n)$ such that $f = Dg$.

**Definition 9.3.** We say that a random vector field $f \in [L^p(X)]^n$ is a potential field if almost all its realizations are potential fields. We denote this field by $L^p_{pot}$.

**Definition 9.4.** We define the space of vector fields with mean value zero.

$$V_{pot}(X) = \{f \in [L^p(X)]^n : \ \int_X f(\omega)\, d\mu(\omega) = 0\}.$$

We observe that by the Fubini theorem it follows that if $f \in L^p(X)$ then almost all realizations $f(T(x)\omega) \in L^p_{loc}(\mathbb{R}^n)$.

**Definition 9.5.** Let $f \in L^1_{loc}(\mathbb{R}^n)$. The number $M(f)$ is called the mean value of $f$ if

$$\lim_{\epsilon \to 0} \int_K f(x/\epsilon)\, dx = |K| M(f)$$

for any Lebesgue measurable bounded set $K \in \mathbb{R}^n$. Alternatively the mean can be expressed in terms of weak convergence. If the family $\{f(\cdot/\epsilon)\}$ is in $L^p(\Omega)$, $p \ge 1$ then $M(f)$ is called the mean value of $f$ if

$$\{f(\cdot/\epsilon)\} \rightharpoonup M(f) \ \ in \ \ L^p(\Omega).$$

We can now formulate

**Theorem 9.6.** *(Birkhoff Ergodic Theorem) Let $f \in L^p(X)$, $p \ge 1$. Then for almost all $\omega \in X$ the realization $f(T(x)\omega)$ possesses a mean value $M(f(T(x)\omega))$.*

*Moreover, as a function of $\omega \in X$, this mean value $M(f(T(x)\omega))$ is invariant and*

$$\int_X f(\omega)\,d\mu(\omega) = \int_X M(f(T(x)\omega))\,d\mu(\omega).$$

*If the system $T(x)$ is ergodic then*

$$\int_X f(\omega)\,d\mu(\omega) = M(f(T(x)\omega)).$$

*Proof.* We refer to [6]. □

Now let $\{(\Omega_k, \mathcal{F}_k, \mu_k)\}_{k=1}^M$ denote a family of probability spaces, where each $\mathcal{F}_k$ is a complete $\sigma$-algebra and each $\mu_k$ is the associated probability measure. For every $x \in \mathbb{R}^n$ we also associate the dynamical system

$$T_k(x) : \Omega_k \to \Omega_k.$$

We also associate

$$\mathbf{T} = (T_1, \ldots, T_M)$$

as a dynamical system on the product space $\Omega_1 \times \ldots \times \Omega_M$. We can now state a multidimensional extension of the Birkhoff ergodic theorem (see [6]).

**Theorem 9.7.** *Let $f \in L^p(\Omega_1 \times \ldots \times \Omega_M)$ and let $p \geq 1$. Then for almost all $\omega_k \in \Omega_k$ the realization $f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M)$ possesses a mean value $M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M))$. Moreover, as a function of $\omega_k \in \Omega_k$, this mean value $M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M))$ is invariant and*

$$\langle f \rangle \equiv \int_{\Omega_1} \cdots \int_{\Omega_M} f(\omega_1, \ldots, \omega_M)\,d\mu_1(\omega_1)\ldots d\mu_M(\omega_M) =$$

$$\int_{\Omega_1} \cdots \int_{\Omega_M} M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M))\,d\mu_1(\omega_1)\ldots d\mu_M(\omega_M).$$

*If in addition the system $\mathbf{T}$ is ergodic on $\Omega_1 \times \ldots \times \Omega_M$ then*

$$\langle f \rangle = M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M)).$$

*Proof.* We refer to [6]. □

## 10. An application to homogenization

The idea now is to first study the homogenization problem for the corresponding elliptic problem and then use the comparison results from above. We begin by setting the appropriate structure conditions:

**Definition 10.1.** Let $(X_k, \mathcal{F}_k, \mu_k)$, $k = 1, 2$, be two probability spaces. Given $0 < \beta \leq 1$, $2 \leq p < \infty$ and three positive real constants $c_0$, $c_1$ and $c_2$, we define the class $S^\omega = S^\omega(c_0, c_1, c_2, \beta)$ of maps

$$a : X_1 \times X_2 \times (0, T) \times \mathbb{R}^n \to \mathbb{R}^n,$$

satisfying

(i) $|a(\omega_1, \omega_2, t, 0)| \leq c_0$ a.e in $X_1 \times X_2 \times (0, T)$.

(ii) $a(\cdot, \cdot, \xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^n$.

(iii) $|(a(\omega_1,\omega_2,t,\xi_1)-a(\omega_1,\omega_2,t,\xi_2)|\le c_1(1+|\xi_1|+|\xi_2|)^{p-1-\beta}|\xi_1-\xi_2|^{\beta}$, a.e. in $X_1\times X_2\times(0,T)$ for all $\xi_1,\xi_2\in\mathbb{R}^n$.

(iv) $(a(\omega_1,\omega_2,t,\xi_1)-a(\omega_1,\omega_2,t,\xi_2),\xi_1-\xi_2)\ge c_2|\xi_1-\xi_2|^p$, a.e. in $X_1\times X_2\times(0,T)$ for all $\xi_1,\xi_2\in\mathbb{R}^n,\ \xi_1\ne\xi_2$.

Let us define the operator $A^{\omega}_{\epsilon}:\mathcal{V}\to\mathcal{U}'$ as

$$A^{\omega}_{\epsilon}(x,t,\xi)=a(T_1(\frac{x}{\epsilon_1})\omega_1,T_2(\frac{x}{\epsilon_2})\omega_2,t,\xi).$$

With some abuse of notation we will say that $A^{\omega}_{\epsilon}$ belongs to $S^{\omega}$ if the corresponding map $a$ does.

**Theorem 10.2.** *Consider the sequence of parameter dependent elliptic boundary value problems*

$$\begin{cases}-\mathrm{div}(A^{\omega}_{\epsilon}(x,t,Du^{\omega}_{\epsilon}))=f_{\epsilon}\ \ in\ \ \Omega\\ u^{\omega}_{\epsilon}(\cdot,t)\in W^{1,p}_0(\Omega),\quad t\in[0,T].\end{cases}$$

*where it is assumed that $f_{\epsilon}\to f$ strongly in $V'$. Assume that $A^{\omega}_{\epsilon}\in S^{\omega}$ and that*

$$|A^{\omega}_{\epsilon}(x,t,\xi)-A^{\omega}_{\epsilon}(x,s,\xi)|\le\rho(t-s)(1+|\xi|^{p-1})$$

*Also assume that the underlying dynamical systems $T_1(x)$ and $T_2(x)$ are ergodic, that the system $\mathbf{T}(x)=(T_1(x),T_2(x))$ is ergodic on $X_1\times X_2$ and that the realizations $T_1(x)\omega_1$ and $T_2(x)\omega_2$ are measurable. Then*

$$u^{\omega}_{\epsilon}(\cdot,t)\rightharpoonup\ u\ \ in\ \ W^{1,p}_0(\Omega)$$

*and*

$$A^{\omega}_{\epsilon}(\cdot,t,Du^{\omega}_{\epsilon})\rightharpoonup\ b(t,Du)\ \ in\ \ [L^q(\Omega)]^n$$

*where $u$ is the solution to the homogenized problem*

$$\begin{cases}-\mathrm{div}(b(t,Du))=f\ \ \text{in}\ \ \Omega,\\ u(\cdot,t)\in W^{1,p}_0(\Omega),\quad t\in[0,T].\end{cases}$$

*The operator $b$ is defined as*

$$b(t,\xi)=\int_{X_1}b_1(\omega_1,t,\xi+z^{\xi}_1(\omega_1,t))\,d\mu_1(\omega_1)= \tag{10.1}$$

$$\int_{X_1}\int_{X_2}a(\omega_1,\omega_2,t),\xi+z^{\xi}_1+z^{\xi}_2)\,d\mu_2(\omega_2)d\mu_1(\omega_1),$$

*where $z^{\xi}_1(\omega_1,t)\in V_{pot}(X_1)$ is the solution to the $\epsilon_1$-scale local problem*

$$\langle b_1(\omega_1,t,\xi+z^{\xi}_1(\omega_1,t),\Phi_1(\omega_1)\rangle=0$$

*for all $\Phi_1(\omega_1)\in V_{pot}(X_1)$, $t\in[0,T]$. The operator $b_1$ is defined as*

$$b_1(\omega_1,t,\xi)=\int_{X_2}a(\omega_1,\omega_2,t,\xi+z^{\xi}_2(\omega_1,\omega_2,t))\,d\mu_2(\omega_2)=$$

$$\int_{X_2}a(\omega_1,\omega_2,t),\xi+z^{\xi}_2)\,d\mu_2(\omega_2),$$

*where $z^{\xi}_2(\omega_1,\omega_2,t)\in V_{pot}(X_2)$ is the solution to the $\epsilon_2$-scale local problem*

$$\langle a(\omega_1,\omega_2,t,\xi+z^{\xi}_2(\omega_1,\omega_2,t),\Phi_2(\omega_2)\rangle=0$$

*for all* $\Phi_2(\omega_2) \in V_{pot}(X_2)$ *a.e.* $\omega_1 \in X_1$, $t \in [0,T]$.

*Proof.* For a proof we refer to [15]. □

Let us now also define the operator $G_\epsilon^\omega : \mathcal{V} \to \mathcal{V}'$ given by

$$G_\epsilon^\omega(x,t,\eta) = \rho(T_3(\frac{x}{\epsilon_3})\omega_3, t)|\eta|^\gamma sgn(\eta).$$

where $T_3(x)$ is assumed to be ergodic on the probability space $X_3$ with respect to the measure $\mu_3$. We close the present study by presenting a reiterated stochastic homogenization result for quasilinear hyperbolic problems:

**Theorem 10.3.** *Consider the hyperbolic problem*

$$\begin{cases} (u_\epsilon^\omega)'' - \text{div}\,(A_\epsilon^\omega(x,t,Du_\epsilon^\omega)) - \Delta((u')_\epsilon^\omega) + G_\epsilon^\omega(x,t,(u')_\epsilon^\omega) = f \ \text{ in } \ Q, \\ u_\epsilon^\omega \in \mathcal{Z}_0, \end{cases} \tag{10.2}$$

*Assume the same hypotheses as in Theorem 10.2 with the addition that the product system* $\hat{\mathbf{T}}(x) = (T_1(x), T_2(x), T_3(x))$ *is ergodic on* $X_1 \times X_2 \times X_3$. *Then, as* $\epsilon \to 0$, *the sequence of solutions*

$$u_\epsilon^\omega \rightharpoonup u \ \ in \ \ \mathcal{Z}_0,$$

$$A_\epsilon^\omega(x,t,Du_\epsilon^\omega) \rightharpoonup \ b(t,Du) \ \ in \ \ \mathcal{U}',$$

$$G_\epsilon^\omega(x,t,(u')_\epsilon^\omega) \rightharpoonup \ \overline{G}(t,u') \ \ in \ \ \mathcal{U}',$$

*where* $u$ *is the unique solution to the homogenized problem*

$$\begin{cases} u'' - \text{div}(b(t,Du)) - \Delta(u') + \overline{G}(t,u') = f \ \text{ in } \ Q, \\ u \in \mathcal{Z}_0, \end{cases}$$

*where the homogenized map* $b$ *is defined as in* (10.1) *and where*

$$\overline{G}(t,u') = \langle \rho(t) \rangle |u'|^\gamma sgn(u'),$$

*where* $\langle \rho(t) \rangle$ *is the expectation with respect to* $\mu_3$ *of the realization* $\rho(T_3(\frac{x}{\epsilon_3}), t)$ *of the process* $\rho$.

*Proof.* For fixed $\omega$, the realization (10.2) possess a solution $u_\epsilon^\omega \in \mathcal{Z}_0$ for every $\epsilon > 0$. By the general $G$-convergence Theorem 8.3 there exists a limit problem corresponding to (10.2) with the same qualitative behaviour. The identification of the terms in the limit problem is performed as follows. From Theorem 10.2 we obtain the explicit homogenized limit for the auxiliary elliptic problem. By the comparison Theorem 7.2 we obtain the explicit homogenized limit for the auxiliary parabolic problem. By the hyperbolic $G$-convergence Theorem 8.2 we obtain the explicit limit for the hyperbolic problem without the non-linear damping term. Using again Theorem 8.3 we finally conclude by the Birkhoff ergodic theorem that the nonlinear damping term

$$G_\epsilon^\omega(x,t,(u')_\epsilon^\omega) \rightharpoonup \ \overline{G}(t,u') = \langle \rho(t) \rangle |u'|^\gamma sgn(u') \ \ in \ \ \mathcal{U}',$$

where $\langle \rho(t) \rangle$ is the expectation with respect to $\mu_3$ of the realization $\rho(T_3(\frac{x}{\epsilon_3}), t)$ of the process $\rho$. □

*Remark* 10.4. Theorem 10.3 remains valid also for random stationary oscillatory forcing or initial data. See [14] for details in the general $G$-convergence setting.

*Remark* 10.5. Theorem 10.3 remains valid also for non-homogeneous boundary data. This means that we can impose random oscillatory boundary data. See [14] for the general $G$-convergence setting.

*Remark* 10.6. The result of Theorem 10.3 can easily be extended to any number of well separated scales.

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