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ON A REVERSE OF ANDO–HIAI INEQUALITY

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This paper is dedicated to Professor Lars-Erik Persson

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ABSTRACT. In this paper, we show a complement of Ando–Hiai inequality: Let A and B be positive invertible operators on a Hilbert space H and $\alpha \in [0,1]$. If $A \sharp_\alpha B \le I$, then

$$A^r \sharp_\alpha B^r \le \|(A \sharp_\alpha B)^{-1}\|^{1-r} I \qquad \text{for all } 0 < r \le 1,$$

where I is the identity operator and the symbol $\|\cdot\|$ stands for the operator norm.

1. INTRODUCTION

A (bounded linear) operator A on a Hilbert space H is said to be positive (in symbol: $A \ge 0$) if $(Ax, x) \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For some scalars m and M, we write $mI \le A \le MI$ if $m(x,x) \le (Ax,x) \le M(x,x)$ for all $x \in H$. The symbol $\|\cdot\|$ stands for the operator norm. Let A and B be two positive operators on a Hilbert space H. For each $\alpha \in [0,1]$, the weighted geometric mean $A \sharp_\alpha B$ of A and B in the sense of Kubo-Ando [6] is defined by

$$A \sharp_\alpha B \;=\; A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\alpha} A^{\frac{1}{2}}$$

if A is invertible. In fact, the geometric mean $A \sharp_{\frac{1}{2}} B$ is a unique positive solution of $XA^{-1}X = B$.

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To study the Golden-Thompson inequality, Ando–Hiai in [1] developed the following inequality, which is called Ando–Hiai inequality: Let A and B be positive invertible operators on a Hilbert space H and $\alpha \in [0,1]$. Then

$$A \sharp_\alpha B \le I \qquad \Longrightarrow \qquad A^r \sharp_\alpha B^r \le I \qquad \text{for all } r \ge 1, \tag{AH}$$

or equivalently

$$\|A^r \sharp_\alpha B^r\| \le \|A \sharp_\alpha B\|^r \qquad \text{for all } r \ge 1.$$

Löwner–Heinz inequality asserts that $A \ge B \ge 0$ implies $A^r \ge B^r$ for all $0 \le r \le 1$. As compared with Löwner–Heinz inequality, Ando–Hiai inequality is rephased as follows: For each $\alpha \in [0,1]$

$$\left(A^{r/2}B^rA^{r/2}\right)^\alpha \le A^r \quad \Longrightarrow \quad \left(A^{1/2}BA^{1/2}\right)^\alpha \le A \quad \text{for all } 0 < r \le 1. \tag{1.1}$$

Now, Ando–Hiai inequality does not hold for $0 < r \le 1$ in general. In fact, put $r = 1/2, \alpha = 1/3$ and

$$A = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \qquad \text{and} \qquad B = \frac{1}{25}\begin{pmatrix} 45+14\sqrt{5} & -5-7\sqrt{5} \\ -5-7\sqrt{5} & 50-14\sqrt{5} \end{pmatrix}.$$

Then we have

$$A \sharp_{\frac{1}{3}} B = \frac{1}{25}\begin{pmatrix} 15+2\sqrt{5} & -5-\sqrt{5} \\ -5-\sqrt{5} & 20-2\sqrt{5} \end{pmatrix} \le I$$

since $\sigma(A \sharp_{\frac{1}{3}} B) = \{1, 0.4\}$. On the other hand, since

$$A^{\frac{1}{2}} \sharp_{\frac{1}{3}} B^{\frac{1}{2}} = \begin{pmatrix} 0.866032 & -0.187030 \\ -0.187030 & 0.770683 \end{pmatrix} \quad \text{and} \quad \sigma(A^{\frac{1}{2}} \sharp_{\frac{1}{3}} B^{\frac{1}{2}}) = \{1.01137, 0.625347\},$$

we have $A^{\frac{1}{2}} \sharp_{\frac{1}{3}} B^{\frac{1}{2}} \not\le I$.

Thus, in [7], Nakamoto and Seo showed the following complement of Ando–Hiai inequality (AH):

Theorem A. *Let A and B be positive operators such that $mI \le A, B \le MI$ for some scalars $0 < m < M$, $h = \frac{M}{m}$ and $\alpha \in [0,1]$. Then*

$$A \sharp_\alpha B \le I \qquad \Longrightarrow \qquad A^r \sharp_\alpha B^r \le K(h^2,\alpha)^{-r} I \qquad \textit{for all } 0 < r \le 1,$$

where the generalized Kantorovich constant $K(h,\alpha)$ is defined by

$$K(h,p) = \frac{h^p - h}{(p-1)(h-1)}\left(\frac{p-1}{p}\frac{h^p-1}{h^p-h}\right)^p \qquad \textit{for all } p \in \mathbb{R},$$

see [5, (2.79)].

We remark that $K(h^2,\alpha)^{-r} \ne 1$ in the case of $r = 1$, though $K(h^2,\alpha)^{-r} = 1$ in the case of $\alpha = 0, 1$ in Theorem A. Thereby, in this paper, we consider another complement of Ando–Hiai inequality (AH) which differ from Theorem A.

2. Main results

First of all, we state the main result:

Theorem 2.1. *Let A and B be positive invertible operators and $\alpha \in [0,1]$. Then*

$$A \sharp_\alpha B \leq I \qquad \Longrightarrow \qquad A^r \sharp_\alpha B^r \leq \|A^{-1} \sharp_\alpha B^{-1}\|^{1-r} I \qquad \text{for all } 0 < r \leq 1,$$

or equivalently

$$\|A^r \sharp_\alpha B^r\| \leq \|A^{-1} \sharp_\alpha B^{-1}\|^{1-r} \|A \sharp_\alpha B\|^r \quad \text{for all } 0 < r \leq 1.$$

We remark that $\|A^{-1} \sharp_\alpha B^{-1}\|^{1-r} = 1$ in the case of $r = 1$.

We need the following lemmas to give a proof of Theorem 2.1. Lemma 2.2 is regarded as a reversal of Löwner–Heinz inequality:

Lemma 2.2. *Let A and B be positive invertible operators. Then*

$$A \geq B \qquad \Longrightarrow \qquad \|A^{\frac{p}{2}} B^{-p} A^{\frac{p}{2}}\| B^p \geq A^p \qquad \text{for all } 0 < p \leq 1.$$

Proof. This lemma follows from Löwner–Heinz inequality. In fact, $A \geq B$ implies $A^p \geq B^p$ for all $0 < p \leq 1$ and then

$$I \geq A^{-\frac{p}{2}} B^p A^{-\frac{p}{2}} \geq \|A^{\frac{p}{2}} B^{-p} A^{\frac{p}{2}}\|^{-1}.$$

□

Lemma 2.3 ([3]). *Let A be a positive invertible operator and B an invertible operator. For each real numbers r*

$$(BAB^*)^r = BA^{\frac{1}{2}} (A^{\frac{1}{2}} B^* B A^{\frac{1}{2}})^{r-1} A^{\frac{1}{2}} B^*.$$

Proof of Theorem 2.1. If we put $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$, then the assumption implies $A^{-1} \geq C^\alpha$. By Lemma 2.2 and $0 < 1 - r < 1$, we have

$$A^r = A^{\frac{1}{2}} A^{r-1} A^{\frac{1}{2}} \leq \|A^{\frac{r-1}{2}} C^{\alpha(r-1)} A^{\frac{r-1}{2}}\| A^{\frac{1}{2}} C^{\alpha(1-r)} A^{\frac{1}{2}}.$$

On the other hand, it follows that $A \leq C^{-\alpha}$ implies $C^{\alpha-1} \leq (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{-1}$. By Lemma 2.2, we have

$$\|(C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{r-1}{2}} C^{(\alpha-1)(r-1)} (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{r-1}{2}}\| C^{(\alpha-1)(1-r)} \geq (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{r-1}.$$

Furthermore, by Lemma 2.3, we have

$$\begin{aligned} B^r &= (A^{\frac{1}{2}} C A^{\frac{1}{2}})^r = A^{\frac{1}{2}} C^{\frac{1}{2}} (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{r-1} C^{\frac{1}{2}} A^{\frac{1}{2}} \\ &\leq \|(C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{r-1}{2}} C^{(\alpha-1)(r-1)} (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{r-1}{2}}\| A^{\frac{1}{2}} C^{\frac{1}{2}} C^{(\alpha-1)(1-r)} C^{\frac{1}{2}} A^{\frac{1}{2}}. \end{aligned}$$

Hence, by Araki-Cordes inequality [2, Theorem IX.2.10], we have

$$\|(C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{r-1}{2}} C^{(\alpha-1)(r-1)} (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{r-1}{2}}\| \leq \|(C^{\frac{1}{2}} A C^{\frac{1}{2}})^{-\frac{1}{2}} C^{1-\alpha} (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{-\frac{1}{2}}\|^{1-r}$$

since $0 < 1 - r < 1$. Let $r(A)$ be the spectral radius of A. Then we have

$$\begin{aligned}\|(C^{\frac{1}{2}}AC^{\frac{1}{2}})^{-\frac{1}{2}}C^{1-\alpha}(C^{\frac{1}{2}}AC^{\frac{1}{2}})^{-\frac{1}{2}}\| &= r((C^{\frac{1}{2}}AC^{\frac{1}{2}})^{-\frac{1}{2}}C^{1-\alpha}(C^{\frac{1}{2}}AC^{\frac{1}{2}})^{-\frac{1}{2}}) \\ &= r((C^{-\frac{1}{2}}AC^{-\frac{1}{2}})^{-1}C^{1-\alpha}) \\ &= r(A^{-1}C^{-\alpha}) \\ &= r(A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}}) \\ &\le \|A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}}\|.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&A^r \natural_\alpha B^r \\ &\le \|A^{\frac{r-1}{2}}C^{\alpha(r-1)}A^{\frac{r-1}{2}}\|^{1-\alpha}\|(C^{\frac{1}{2}}AC^{\frac{1}{2}})^{\frac{r-1}{2}}C^{(\alpha-1)(r-1)}(C^{\frac{1}{2}}AC^{\frac{1}{2}})^{\frac{r-1}{2}}\|^{\alpha} \\ &\qquad \times \left(A^{\frac{1}{2}}C^{(1-r)\alpha}A^{\frac{1}{2}} \natural_\alpha A^{\frac{1}{2}}C^{(\alpha-1)(1-r)+1}A^{\frac{1}{2}}\right) \\ &\le \|A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}}\|^{(1-r)(1-\alpha)}\|(C^{\frac{1}{2}}AC^{\frac{1}{2}})^{-\frac{1}{2}}C^{1-\alpha}(C^{\frac{1}{2}}AC^{\frac{1}{2}})^{-\frac{1}{2}}\|^{(1-r)\alpha} \\ &\qquad \times A^{\frac{1}{2}}(C^{(1-r)\alpha} \natural_\alpha C^{(\alpha-1)(1-r)+1})A^{\frac{1}{2}} \\ &= \|A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}}\|^{1-r}A \natural_\alpha B \le \|(A \natural_\alpha B)^{-1}\|^{1-r}I\end{aligned}$$

by $C^{(1-r)\alpha} \natural_\alpha C^{(\alpha-1)(1-r)+1} = C^\alpha$ and the assumption of $A \natural_\alpha B \le I$. Hence the proof is complete. □

By Theorem 2.1, we immediately have the following corollary in the case of $r \ge 1$.

Corollary 2.4. *Let A and B be positive invertible operators on H. Then*

$$\|A^{-r} \natural_\alpha B^{-r}\|^{1-r}\|A \natural_\alpha B\|^r \le \|A^r \natural_\alpha B^r\| \quad \textit{for all } r \ge 1.$$

Finally, Furuta [4] showed the following Knatorovich type operator inequality in terms of the condition number: Let A and B be positive invertible operators. Then

$$B \le A \qquad \Longrightarrow \qquad B^r \le \left(\|B\|\|B^{-1}\|\right)^{r-1} A^r \quad \text{for all } r \ge 1. \tag{2.1}$$

By Theorem 2.1, we have the following Kantorovich type inequality of (1.1) which corresponds to (2.1):

Theorem 2.5. *Let A and B be positive invertible operators and $\alpha \in [0,1]$. Then*

$$\left(A^{\frac{r}{2}}B^rA^{\frac{r}{2}}\right)^\alpha \le A^r \qquad \Longrightarrow \qquad \left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)^\alpha \le \|A^r \natural_\alpha B^{-r}\|^{1-\frac{1}{r}}A$$

for all $r \ge 1$.

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