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MATRIX ORDER IN BOHR INEQUALITY FOR OPERATORS

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This paper is dedicated to Professor Lars-Erik Persson

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Abstract. The classical Bohr inequality says that $|a+b|^2 \le p|a|^2+q|b|^2$ for all scalars a, b and p, $q > 0$ with $\frac{1}{p}+\frac{1}{q}=1$. The equality holds if and only if $(p-1)a=b$. Several authors discussed operator version of Bohr inequality. In this paper, we give a unified proof to operator generalizations of Bohr inequality. One viewpoint of ours is a matrix inequality, and the other is a generalized parallelogram law for absolute value of operators, i.e., for operators A and B on a Hilbert space and $t \ne 0$,

$$|A-B|^2 + \frac{1}{t}|tA+B|^2 = (1+t)|A|^2 + (1+\frac{1}{t})|B|^2.$$

1. Introduction

Let $\mathscr{H}$ be a complex separable Hilbert space and $\mathbb{B}(\mathscr{H})$ the algebra of all bounded operators on $\mathscr{H}$. We say that $A \in \mathbb{B}(\mathscr{H})$ is a positive operator if $(Ax,x) \ge 0$ for all $x \in \mathscr{H}$, denoted by $A \ge 0$. The absolute value of $A \in \mathbb{B}(\mathscr{H})$ is denoted by $|A| = (A^*A)^{1/2}$.

The classical Bohr inequality [2] says that

$$|a+b|^2 \le p|a|^2 + q|b|^2$$

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for all scalars $a,\ b$ and $p,\ q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. The equality holds if and only if $(p-1)a = b$.

For this, Hirzallah [4] proposed an operator version of Bohr inequality:

If A and B are operators on a Hilbert space, and $q \geq p > 0$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then

$$|A-B|^2 + |(p-1)A+B|^2 \leq p|A|^2 + q|B|^2.$$

Afterwards, several authors have presented generalizations of Bohr inequality, [3, 7].

In this note, we approach to Bohr inequality from the viewpoint of the matrix order preserving map. We propose the following general theorem: For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we define an $n \times n$ matrix $\Lambda(x) = x^*x = (x_i x_j)$ and $D(x) = \mathrm{diag}(x_1, \cdots, x_n)$.

If $\Lambda(a) + \Lambda(b) \leq D(c)$ for $a, b, c \in \mathbb{R}^n$, then

$$|\sum_{i=1}^n a_i A_i|^2 + |\sum_{i=1}^n b_i A_i|^2 \leq \sum_{i=1}^n c_i |A_i|^2$$

for arbitrary n-tuple (A_i) in $\mathbb{B}(\mathscr{H})$.

We show that generalized Bohr inequalities are covered by this theorem.

On the other hand, a generalized parallelogram law also implies generalized Bohr inequalities obtained in this paper. It is essentially same as the discussion in [1].

2. Generalized Bohr inequality

First of all, we cite Bohr type inequalities obtained in [3, 4].

Theorem 2.1. *If $A, B \in \mathbb{B}(\mathscr{H})$, $\frac{1}{p} + \frac{1}{q} = 1$, and $1 < p \leq 2$, i.e., $q \geq p > 1$, then*

(i) $|A-B|^2 + |(p-1)A+B|^2 \leq p|A|^2 + q|B|^2$,

(ii) $|A-B|^2 + |A+(q-1)B|^2 \geq p|A|^2 + q|B|^2$.

On the other hand, if either $p < 1$ or $p \geq 2$, then

(iii) $|A-B|^2 + |(p-1)A+B|^2 \geq p|A|^2 + q|B|^2$.

Next we point out that [3, Theorem 3] is unified as follows:

Theorem 2.2. *If $A, B \in \mathbb{B}(\mathscr{H})$ and $\alpha \geq \beta > 0$, then*

$$|A-B|^2 + \frac{1}{\alpha^2}|\beta A + \alpha B|^2 \leq (1+\frac{\beta}{\alpha})|A|^2 + (1+\frac{\alpha}{\beta})|B|^2.$$

We here explain the relation between Theorem 2.2 and (a), (b) in [3, Theorem 3]: The former (a) is contained in Theorem 2.2. The latter (b) can be expressed as follows:

(i) If $\alpha \geq -\beta > 0$, then

$$|A-B|^2 + |\frac{|\beta|}{\alpha}A + B|^2 \leq (1+\frac{|\beta|}{\alpha})|A|^2 + (1+\frac{\alpha}{|\beta|})|B|^2.$$

(ii) If $0 < \alpha \leq -\beta$, then

$$|A-B|^2 + |\frac{\alpha}{|\beta|}A + B|^2 \leq (1+\frac{\alpha}{|\beta|})|A|^2 + (1+\frac{|\beta|}{\alpha})|B|^2.$$

Next we discuss Bohr inequalities for multi-operators. So we introduce the following result [7, Theorem 7].

Theorem 2.3. *Suppose that $A_i \in \mathbb{B}(\mathscr{H})$, and $r_i \geq 1$ for $i = 1, 2, \cdots, n$ with $\sum_{i=1}^{n} \frac{1}{r_i} = 1$. Then*

$$|\sum_{i=1}^{n} A_i|^2 \leq \sum_{i=1}^{n} r_i |A_i|^2.$$

In other words, it says that $K(z) = |z|^2$ satisfies the (operator) Jensen inequality:

$$K(\sum_{i=1}^{n} t_i A_i) \leq \sum_{i=1}^{n} t_i K(A_i)$$

holds for $t_1, \cdots, t_n > 0$ with $\sum_{i=1}^{n} t_i = 1$.

3. Matrix approach to Bohr inequalities

In this section, we present an approach to Bohr inequalities by the use of the matrix order.

For this, we introduce two notations: For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we define an $n \times n$ matrix $\Lambda(x) = x^* x = (x_i x_j)$ and $D(x) = \mathrm{diag}(x_1, \cdots, x_n)$.

Theorem 3.1. *If $\Lambda(a) + \Lambda(b) \leq D(c)$ for $a, b, c \in \mathbb{R}^n$, then*

$$|\sum_{i=1}^{n} a_i A_i|^2 + |\sum_{i=1}^{n} b_i A_i|^2 \leq \sum_{i=1}^{n} c_i |A_i|^2$$

for arbitrary n-tuple (A_i) in $\mathbb{B}(\mathscr{H})$. Incidentally, if $\Lambda(a) + \Lambda(b) \geq D(c)$ for $a, b, c \in \mathbb{R}^n$, then

$$|\sum_{i=1}^{n} a_i A_i|^2 + |\sum_{i=1}^{n} b_i A_i|^2 \geq \sum_{i=1}^{n} c_i |A_i|^2$$

for arbitrary n-tuple (A_i) in $\mathbb{B}(\mathscr{H})$.

Proof. We define a positive map Φ of $\mathbb{B}(\mathbb{R}^n)$ to $\mathbb{B}(\mathscr{H})$ by

$$\Phi(X) = (A_1^* \cdots A_n^*) X \ (A_1 \cdots A_n)^t.$$

Since $\Lambda(a) = \ (a_1, \cdots, a_n)^t \ (a_1, \cdots, a_n)$, we have

$$\Phi(\Lambda(a)) = (\sum_{i=1}^{n} a_i A_i)^* (\sum_{i=1}^{n} a_i A_i) = |\sum_{i=1}^{n} a_i A_i|^2,$$

so that

$$|\sum_{i=1}^{n} a_i A_i|^2 + |\sum_{i=1}^{n} b_i A_i|^2 = \Phi(\Lambda(a) + \Lambda(b)) \leq \Phi(D(c)) = \sum_{i=1}^{n} c_i |A_i|^2.$$

The additional part is easily shown by the same way. □

The meaning of Theorem 3.1 will be understood in the proof of the following theorem well. More precisely, the essence of the proof is to check a matrix inequality.

Theorem 3.2. (i) *If* $0 < t \le 1$, *then*
$|A \mp B|^2 + |tA \pm B|^2 \le (1+t)|A|^2 + (1+\frac{1}{t})|B|^2$;
(ii) *If either* $t \ge 1$ *or* $t < 0$, *then*
$|A \mp B|^2 + |tA \pm B|^2 \ge (1+t)|A|^2 + (1+\frac{1}{t})|B|^2$.

Proof. We apply Theorem 3.1 to $a = (1, -1)$, $b = (t, 1)$ and $c = (1+t, 1+\frac{1}{t})$. We consider the order between corresponding matrices $\Lambda(a) + \Lambda(b)$ and $D(c)$:

$$\begin{aligned} T &= D(c) - \Lambda(a) - \Lambda(b) \\ &= \begin{pmatrix} 1+t & 0 \\ 0 & 1+\frac{1}{t} \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} t^2 & t \\ t & 1 \end{pmatrix} = (1-t)\begin{pmatrix} t & 1 \\ 1 & \frac{1}{t} \end{pmatrix}. \end{aligned}$$

Since $\det(T) = 0$, T is positive (resp. negative) if $0 < t < 1$ (resp. $t > 1$ or $t < 0$). □

Remark. We note that Theorem 3.2 implies Theorem 2.1 easily: Actually, for (i) and (iii) of Theorem 2.1, we take $t = p - 1$ in Theorem 3.2. For (ii), we take $t = q - 1$ and permute A and B. Also Theorem 3.2 implies Theorem 2.2 by taking $t = \frac{\beta}{\alpha}$ only.

As another application of Theorem 3.1, we give a proof of Theorem 2.3:

Proof of Theorem 2.3. We check the order between the corresponding matrices $D = \mathrm{diag}(r_1, \cdots, r_n)$ and $C = (c_{ij})$ where $c_{ij} = 1$. For natural numbers $k \le n$, put $D_k = \mathrm{diag}(r_{i(1)}, \cdots, r_{i(k)})$, $C_k = (c_{ij})$ with $c_{ij} = 1$ for $i, j = 1, \cdots, k$ and $s_k = \sum_{j=1}^{k} r_{i(j)}{}^{-1}$ for $1 \le i(1) < \cdots < i(k) \le n$. Noting that

$$\det(D_k - C_k) = (r_{i(1)} \cdots r_{i(k)})(1 - s_k) \ge 0 \tag{3.1}$$

for arbitrary $k \le n$, the determinants of all $k \times k$ submatrix of $D - C$ are nonnegative, so that $C \le D$. Hence we have Theorem 2.3 by Theorem 3.1. For convenience, we give a proof of (3.1) for $i(j) = j$ simply: It is done by the induction. The case $n = 2$ is trivial. Suppose that it is true for $n = k$, i.e., $|E_k| = r_1 \cdots r_k(1 - s_k)$, where $E_j = D_j - C_j$. Then we have

$$\begin{aligned} |E_{k+1}| &= \begin{vmatrix} E_k & 0 \\ -1 & r_{k+1} \end{vmatrix} - \begin{vmatrix} E_k & 1 \\ -1 & 1 \end{vmatrix} = r_{k+1}|E_k| - \begin{vmatrix} D_k & 1 \\ 0 & 1 \end{vmatrix} \\ &= r_{k+1} r_1 \cdots r_k (1 - s_k) - r_1 \cdots r_k \\ &= r_1 \cdots r_{k+1}(1 - s_k - \frac{1}{r_{k+1}}) \\ &= r_1 \cdots r_{k+1}(1 - s_{k+1}). \end{aligned}$$

We state Zhang's result [7, Theorem 6] that if $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $p = (p_1, p_2)$ satisfy

$$p_1 \ge a_1^2 + b_1^2, p_2 \ge a_2^2 + b_2^2, (p_1 - (a_1^2 + b_1^2))(p_2 - (a_2^2 + b_2^2)) \ge (a_1 a_2 + b_1 b_2)^2,$$

then

$$|a_1A + a_2B|^2 + |b_1A + b_2B|^2 \le p_1|A|^2 + p_2|B|^2$$

holds for $A, B \in \mathbb{B}(\mathscr{H})$.

Since the assumption of the above is nothing but the matrix inequality $\Lambda(a) + \Lambda(b) \le D(p)$, Theorem 3.1 implies the conclusion.

Concluding this section, we remark the monotonicity of the operator function $F(a) = |\sum_{i=1}^{n} a_iA_i|^2$. It is proved by $F(a) = \Phi(a^*a)$, where Φ is as in the proof of Theorem 3.1.

Corollary 3.3. *For a fixed n-tuple (A_i) in $\mathbb{B}(\mathscr{H})$, the operator function $F(a) = |\sum_{i=1}^{n} a_iA_i|^2$ for $a = (a_1, \cdots, a_n)$ is order preserving, that is, if $\Lambda(a) \le \Lambda(b)$, then $F(a) \le F(b)$.*

The following corollary is a 3-dimensional version of the 2-dimensional one in [7, Lemma 2].

Corollary 3.4. *If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ satisfy $|a_i| \le |b_i|$ for $i = 1, 2, 3$ and $a_ib_j = a_jb_i$ for $i \ne j$, then $F(a) \le F(b)$.*

Proof. It suffices to check that $\Lambda(a) \le \Lambda(b)$ by the preceding corollary. First of all, it follows from the assumption that if $i \ne l$ and $j \ne k$, then

$$\begin{vmatrix} a_ia_j - b_ib_j & a_ia_k - b_ib_k \\ a_la_j - b_lb_j & a_la_k - b_lb_k \end{vmatrix} = a_kb_j(a_ib_l - b_ia_l) + a_jb_k(b_ia_l - a_ib_l) = 0.$$

This means that the determinants of all 2×2 submatrix of $\Lambda(b) - \Lambda(a)$ are zero. Moreover it implies that $\det(\Lambda(b) - \Lambda(a)) = 0$ by the use of the expansion of determinants. Since the diagonal elements satisfy $|a_i| \le |b_i|$ for $i = 1, 2, 3$, we have the desired matrix inequality $\Lambda(a) \le \Lambda(b)$. □

4. Generalized parallelogram law for operators

Finally we mention another approach to Bohr inequality, whose idea is essentially same as that of Abramovich, Barić and Pečarić [1]. In our frame, the following generalization of the parallelogram law easily implies Theorem 3.2 which covers many previous results as discussed in the preceding section.

Theorem 4.1. *If A and B are operators on a Hilbert space and $t \ne 0$, then*

$$|A + B|^2 + \frac{1}{t}|tA - B|^2 = (1 + t)|A|^2 + (1 + \frac{1}{t})|B|^2.$$

Proof. It is easily checked that

$$\begin{aligned} & |A + B|^2 + \frac{1}{t}|tA - B|^2 \\ = \; & |A|^2 + |B|^2 + A^*B + B^*A + t|A|^2 + \frac{1}{t}|B|^2 - A^*B - B^*A \\ = \; & (1 + t)|A|^2 + (1 + \frac{1}{t})|B|^2. \end{aligned}$$

□

Remark. We immediately obtain Theorem 3.2 from Theorem 4.1 by considering the condition of t in it; that is, if $0 < t \le 1$, then $\frac{1}{t} \ge 1$, so that the second term $\frac{1}{t}|tA - B|^2$ of the left hand side in Theorem 4.1 is greater than $|tA - B|^2$. Hence we have (i) in Theorem 3.2. Similarly (ii) in Theorem 3.2 is obtained. Consequently Theorem 4.1 also implies Theorems 2.1 and 2.2 stated in Section 2.

Next we extend Theorem 4.1 for multi-operators. As an easy consequence, we have Theorem 2.3 [7, Theorem 7].

Theorem 4.2. *Suppose that* $A_i \in \mathbb{B}(\mathscr{H})$ *and* $r_i \ge 1$ *with* $\sum_{i=1}^{n} \frac{1}{r_i} = 1$ *for* $i = 1, 2, ..., n$. *Then*

$$\sum_{i=1}^{n} r_i |A_i|^2 - |\sum_{i=1}^{n} A_i|^2 = \sum_{1 \le i < j \le n} \left| \sqrt{\frac{r_i}{r_j}} A_i - \sqrt{\frac{r_j}{r_i}} A_j \right|^2.$$

Proof. We show it by the induction on n. Note that it is true for $n = 2$ by Theorem 4.1. Because it is expressed as follows: Let $A_i \in \mathbb{B}(\mathscr{H})$ and $r_i \ge 1$ for $i = 1, 2$ satisfying $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then

$$r_1|A_1|^2 + r_2|A_2|^2 - |A_1 + A_2|^2 = |\sqrt{\frac{r_1}{r_2}} A_1 - \sqrt{\frac{r_2}{r_1}} A_2|^2.$$

Now suppose that it is true for $n = k$, then we take $A_1, \cdots, A_{k+1} \in \mathbb{B}(\mathscr{H})$ and $r_1, \cdots, r_{k+1} > 1$ satisfying $\sum_{i=1}^{k+1} \frac{1}{r_i} = 1$. We here put $r_i' = r_i(1 - \frac{1}{r_{k+1}})$ for $i = 1, \cdots, k$ and $B = \sum_{i=1}^{k} A_i$ for convenience, then $r_i' > 1$ and $\sum_{i=1}^{k} \frac{1}{r_{i'}} = 1$. Hence we have

$$\sum_{i=1}^{k+1} r_i |A_i|^2 - |\sum_{i=1}^{k+1} A_i|^2 = \sum_{i=1}^{k} r_i |A_i|^2 + r_{k+1}|A_{k+1}|^2 - |\sum_{i=1}^{k} A_i + A_{k+1}|^2$$

$$= (1 - \frac{1}{r_{k+1}}) \sum_{i=1}^{k} r_i |A_i|^2 - |B|^2 + (r_{k+1} - 1)|A_{k+1}|^2 + \frac{1}{r_{k+1}} \sum_{i=1}^{k} r_i |A_i|^2 - B^* A_{k+1} - A_{k+1}^* B$$

$$= \left(\sum_{i=1}^{k} r_i' |A_i|^2 - |B|^2 \right) + \sum_{i=1}^{k} \frac{r_i}{r_{k+1}} |A_i|^2 - B^* A_{k+1} - A_{k+1}^* B + (r_{k+1} - 1)|A_{k+1}|^2$$

$$= \sum_{1 \le i < j \le k} \left| \sqrt{\frac{r_i}{r_j}} A_i - \sqrt{\frac{r_j}{r_i}} A_j \right|^2 + \sum_{i=1}^{k} \frac{r_i}{r_{k+1}} |A_i|^2 - B^* A_{k+1} - A_{k+1}^* B + \sum_{i=1}^{k} \frac{r_{k+1}}{r_i} |A_{k+1}|^2$$

$$= \sum_{1 \le i < j \le k} \left| \sqrt{\frac{r_i}{r_j}} A_i - \sqrt{\frac{r_j}{r_i}} A_j \right|^2 + \sum_{i=1}^{k+1} \left| \sqrt{\frac{r_i}{r_{k+1}}} A_i - \sqrt{\frac{r_{k+1}}{r_i}} A_{k+1} \right|^2$$

$$= \sum_{1 \le i < j \le k+1} \left| \sqrt{\frac{r_i}{r_j}} A_i - \sqrt{\frac{r_j}{r_i}} A_j \right|^2.$$

Therefore, the desired equality holds for all $n \in \mathbb{N}$. That is, Theorem 4.2 is proved. □

We note that the condition $r_i \geq 1$ in Theorem 4.2 is not necessary. As a matter of fact, we can weaken it to $r_i \neq 0$ by the adoption of the following expression:

Theorem 4.3. *Let $A_i \in \mathbb{B}(\mathscr{H})$ and $r_i \neq 0$ for $i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \frac{1}{r_i} = 1$. Then*

$$\sum_{i=1}^{n} r_i |A_i|^2 - |\sum_{i=1}^{n} A_i|^2 = \sum_{1 \leq i \leq j \leq n} \frac{r_j}{r_i} \left| \frac{r_i}{r_j} A_i - A_j \right|^2.$$

Further development of operator Bohr inequalities are appeared in [5, 6].

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