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ON A NEW CLASS OF REFINED DISCRETE HARDY-TYPE INEQUALITIES

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Dedicated to Professor Lars-Erik Persson for his 65th birthday

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Abstract. In this paper, we state, prove and discuss a new refined general weighted discrete Hardy-type inequality with a non-negative kernel, related to an arbitrary non-negative convex (or positive concave) function on a real interval and to a positive real parameter. As its consequences, obtained by rewriting it for various suitably chosen parameters, kernels, weights and convex (or concave) functions, we derive new weighted and unweighted generalizations and refinements of some well-known inequalities such as Carleman's inequality and the so-called Godunova's inequality. Finally, by employing exponential and logarithmic convexity, as special cases of the usual convexity, we obtain some further refinements of the inequalities mentioned above.

1. Introduction

Generalizing certain results of Godunova, [5] (see also [8, Chapter IV, p. 152]), Vasić and Pečarić in [12] proved that the Hardy-type inequality

$$\sum_{m=1}^{\infty} u_m \Phi\left(\sum_{n=1}^{m} k_{mn} a_n\right) \leq \sum_{n=1}^{\infty} v_n \Phi(a_n) \tag{1.1}$$

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holds for all non-negative convex functions Φ on an interval $I \subseteq \mathbb{R}$, sequences $(a_n)_{n\in\mathbb{N}}$ in I, sequences $(u_n)_{n\in\mathbb{N}}$ of positive real numbers and positive real numbers k_{mn}, $m \in \mathbb{N}$, $n = 1, \dots, m$ such that

$$\sum_{n=1}^{m} k_{mn} = 1,\ m \in \mathbb{N}, \quad \text{and} \quad \sum_{m=n}^{\infty} u_m k_{mn} \leq v_n,\ n \in \mathbb{N}. \tag{1.2}$$

Moreover, if the function Φ is concave and the sign of inequality in (1.2) is reversed, then (1.1) holds with the reversed sign of inequality.

As special cases of (1.1) for sequences of positive real numbers $(a_n)_{n\in\mathbb{N}}$, we get the so-called Godunova's inequality

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{1}{n} \sum_{m=1}^{n} a_m \right)^p < \sum_{n=1}^{\infty} \frac{a_n^p}{n}, \tag{1.3}$$

where $p \in \mathbb{R}$, $p > 1$ and Akerberg's inequality

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left(n! \prod_{m=1}^{n} a_m \right)^{\frac{1}{n}} < \sum_{n=1}^{\infty} a_n, \tag{1.4}$$

obtained by Akerberg in [1]. It can be shown that inequality (1.4) implies the well-known Carleman's inequality

$$\sum_{n=1}^{\infty} \left(\prod_{m=1}^{n} a_m \right)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} a_n, \tag{1.5}$$

with the best possible constant e, proved by Carleman in [3].

Motivated by these results, in this paper we obtain a generalization and a refinement of (1.1) by proving a new refined general weighted discrete Hardy-type inequality with a positive real parameter. As its consequences, obtained by rewriting it for various parameters, kernels, weights and convex (or concave) functions, we derive new weighted and unweighted generalizations and refinements of inequalities (1.3)–(1.5). Finally, by introducing the notion of exponential and logarithmic convexity, as special cases of the usual convexity, we obtain some further refinements of the inequalities mentioned above.

The paper is organized in the following way. After this Introduction, in Section 2 we introduce some necessary notation, recall some basic facts about convex and concave functions and state, prove and discuss our main result in this paper: a new general refined discrete Hardy-type inequality with a non-negative kernel, related to an arbitrary non-negative convex (or positive concave) function on a real interval and to a positive real parameter. This result is given in Theorem 2.1. The rest of the paper is mainly dedicated to a deeper analysis of particularly interesting special cases of the inequality obtained. Namely, in Section 3 we obtain a refined discrete Jensen's inequality and refine and even generalize the Vasić-Pečarić inequality (1.1). As its special cases, we derive a new refined weighted version of Godunova's inequality (1.3) and of inequality (1.4). Moreover, we show that our result improves and generalizes Carleman's inequality (1.5), that is, we get a new refined weighted strengthened Carleman's inequality.

In the concluding Section 4 we make a further step in applications of Theorem 2.1 to some suitably chosen convex functions and parameters. By employing the concepts of exponential and logarithmic convexity, we obtain upper and lower bounds for the left-hand sides of some refined Hardy-type inequalities from the previous section. In particular, we derive both-hand side bounds for the left-hand side of the weighted Godunova's inequality, as well as of the strengthened weighted Carleman's inequality.

Conventions. Throughout this paper, by an interval I in $\mathbb{R}$ we mean any convex set in $\mathbb{R}$, while $\operatorname{Int} I$ denotes its interior. Further, we set $\mathbb{N}_k = \{1, 2, \ldots, k\}$ for $k \in \mathbb{N}$. Moreover, all expressions of the form 0^0, $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{a}{\infty}$, where $a \in \mathbb{R}$, are taken to be equal to zero. Finally, inequalities like (2.5) are interpreted to mean that if the left-hand side is finite, so is the right-hand side and the inequality holds.

2. New refined discrete Hardy-type inequalities

To start with, we introduce some necessary notation and recall basic facts about convex and concave functions. Suppose I is an interval in $\mathbb{R}$ and $\Phi : I \to \mathbb{R}$ is a convex function. By $\partial\Phi(x)$ we denote the subdifferential of Φ at $x \in I$, that is, the set $\partial\Phi(x) = \{\alpha \in \mathbb{R} : \Phi(y) - \Phi(x) - \alpha(y - x) \geq 0,\ y \in I\}$. It is well-known that $\partial\Phi(x) \neq \emptyset$ for all $x \in \operatorname{Int} I$. More precisely, at each point $x \in \operatorname{Int} I$ we have $-\infty < \Phi'_-(x) \leq \Phi'_+(x) < \infty$ and $\partial\Phi(x) = [\Phi'_-(x), \Phi'_+(x)]$, while the set on which Φ is not differentiable is at most countable. Moreover, each function $\varphi : I \to \mathbb{R}$ satisfying $\varphi(x) \in \partial\Phi(x)$, whenever $x \in \operatorname{Int} I$, is increasing on $\operatorname{Int} I$. For any such function φ and arbitrary $x \in \operatorname{Int} I$, $y \in I$ we have

$$\Phi(y) - \Phi(x) - \varphi(x)(y - x) \geq 0$$

and further

$$\begin{aligned}\Phi(y) - \Phi(x) - \varphi(x)(y - x) &= |\Phi(y) - \Phi(x) - \varphi(x)(y - x)| \\ &\geq \left|\,|\Phi(y) - \Phi(x)| - |\varphi(x)| \cdot |y - x|\,\right|. \qquad (2.1)\end{aligned}$$

On the other hand, if $\Phi : I \to \mathbb{R}$ is a concave function, that is, $-\Phi$ is convex, then $\partial\Phi(x) = \{\alpha \in \mathbb{R} : \Phi(x) - \Phi(y) - \alpha(x - y) \geq 0,\ y \in I\}$ denotes the superdifferential of Φ at the point $x \in I$. For all $x \in \operatorname{Int} I$, in this setting we have $-\infty < \Phi'_+(x) \leq \Phi'_-(x) < \infty$ and $\partial\Phi(x) = [\Phi'_+(x), \Phi'_-(x)] \neq \emptyset$. Hence, the inequality

$$\Phi(x) - \Phi(y) - \varphi(x)(x - y) \geq 0$$

holds for all $x \in \operatorname{Int} I$, $y \in I$ and all real functions φ on I such that $\varphi(z) \in \partial\Phi(z)$, $z \in \operatorname{Int} I$. Finally, we get

$$\begin{aligned}\Phi(x) - \Phi(y) - \varphi(x)(x - y) &= |\Phi(x) - \Phi(y) - \varphi(x)(x - y)| \\ &\geq \left|\,|\Phi(y) - \Phi(x)| - |\varphi(x)| \cdot |y - x|\,\right|. \qquad (2.2)\end{aligned}$$

Note that, although the symbol $\partial\Phi(x)$ has two different notions, it will be clear from the context whether it applies to a convex or to a concave function Φ. Many further information on convex and concave functions can be found e.g. in the monographs [9] and [10] and in references cited therein.

Now, we are ready to state and prove a new general refined discrete Hardy-type inequality with a kernel, related to arbitrary non-negative convex functions on real intervals.

Theorem 2.1. *Let $t \in \mathbb{R}^+$, $M, N \in \mathbb{N}$ and let non-negative real numbers u_m, v_n, k_{mn}, where $m \in \mathbb{N}_M$, $n \in \mathbb{N}_N$, be such that*

$$K_m = \sum_{n=1}^{N} k_{mn} > 0, \;\; m \in \mathbb{N}_M, \tag{2.3}$$

and

$$v_n = \left[\sum_{m=1}^{M} u_m \left(\frac{k_{mn}}{K_m} \right)^t \right]^{\frac{1}{t}}, \;\; n \in \mathbb{N}_N. \tag{2.4}$$

Let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ be any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$. Then the inequality

$$\left(\sum_{n=1}^{N} v_n \Phi(a_n) \right)^t - \sum_{m=1}^{M} u_m \Phi^t(A_m) \geq t \sum_{m=1}^{M} u_m \frac{\Phi^{t-1}(A_m)}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn} \tag{2.5}$$

holds for all $t \geq 1$ and real numbers $a_n \in I$, for $n \in \mathbb{N}_N$, where

$$A_m = \frac{1}{K_m} \sum_{n=1}^{N} k_{mn} a_n \tag{2.6}$$

and

$$r_{mn} = \big| \left| \Phi(a_n) - \Phi(A_m) \right| - \left| \varphi(A_m) \right| \cdot \left| a_n - A_m \right| \big|, \tag{2.7}$$

for $m \in \mathbb{N}_M$, $n \in \mathbb{N}_N$. If $t \in \langle 0, 1]$ and the function $\Phi : I \to \mathbb{R}$ is positive and concave, then the order of terms on the left-hand side of (2.5) is reversed, that is, the inequality

$$\sum_{m=1}^{M} u_m \Phi^t(A_m) - \left(\sum_{n=1}^{N} v_n \Phi(a_n) \right)^t \geq t \sum_{m=1}^{M} u_m \frac{\Phi^{t-1}(A_m)}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn} \tag{2.8}$$

holds for all $t \in \langle 0, 1]$.

Proof. First, note that

$$\sum_{n=1}^{N} k_{mn}(a_n - A_m) = \sum_{n=1}^{N} k_{mn} a_n - A_m \sum_{n=1}^{N} k_{mn} = K_m A_m - A_m K_m = 0 \tag{2.9}$$

holds for all $m \in \mathbb{N}_M$. Further, since $\min_{n \in \mathbb{N}_N} a_n \in I$, $\max_{n \in \mathbb{N}_N} a_n \in I$ and

$$\min_{n \in \mathbb{N}_N} a_n \leq a_n \leq \max_{n \in \mathbb{N}_N} a_n, \;\; n \in \mathbb{N}_N,$$

we easily get

$$\min_{n \in \mathbb{N}_N} a_n \leq \frac{1}{K_m} \sum_{n=1}^{N} k_{mn} a_n \leq \max_{n \in \mathbb{N}_N} a_n.$$

Therefore, $A_m \in I$ for all $m \in \mathbb{N}_M$. Moreover, if for all $n \in \mathbb{N}_N$ we have $a_n \in \operatorname{Int} I$, then $A_m \in \operatorname{Int} I$ for all $m \in \mathbb{N}_M$, as well.

Now, we are ready to prove (2.5), so suppose that the function Φ is convex and $t \geq 1$. Fix $m \in \mathbb{N}_M$ and $n \in \mathbb{N}_N$. If $A_m \in \operatorname{Int} I$, then substituting $x = A_m$ and $y = a_n$ in (2.1) yields

$$\Phi(a_n) - \Phi(A_m) - \varphi(A_m)(a_n - A_m) \geq \big|\,|\Phi(a_n) - \Phi(A_m)| - |\varphi(A_m)| \cdot |a_n - A_m|\,\big|$$

and therefrom

$$\frac{k_{mn}}{K_m}\left[\Phi(a_n) - \Phi(A_m) - \varphi(A_m)(a_n - A_m)\right] \geq \frac{k_{mn}}{K_m} r_{mn}. \tag{2.10}$$

Observe that (2.10) holds trivially also if $k_{mn} = 0$ and A_m is an endpoint of I (if I is not an open interval). Hence, it is only left to analyze the case when A_m is an endpoint of I and $k_{mn} > 0$ (from the condition (2.3) we see that such n exists for every $m \in \mathbb{N}_M$). Without loss of generality, assume that A_m is the left endpoint of I, that is, $A_m = \min I$. Then $a_l - A_m \geq 0$ for all $l \in \mathbb{N}_N$, so (2.9) implies that $k_{ml}(a_l - A_m) = 0$ for all $l \in \mathbb{N}_N$. In particular, from $k_{mn} > 0$ we get $a_n = A_m$, so both-hand sides of (2.10) are equal to 0. The case when $A_m = \max I$ is analogous. Thus, (2.10) holds for all $m \in \mathbb{N}_M$ and $n \in \mathbb{N}_N$. Summing it up over $n \in \mathbb{N}_N$ gives

$$\begin{aligned}\frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\Phi(a_n) - \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\Phi(A_m) - \frac{\varphi(A_m)}{K_m}\sum_{n=1}^{N} k_{mn}(a_n - A_m)\\ \geq \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn}\end{aligned}$$

and, by using (2.9), further

$$\Phi(A_m) + \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn} \leq \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\Phi(a_n). \tag{2.11}$$

Since the left-hand side of (2.11) is non-negative and the function $\alpha \mapsto \alpha^t$ is strictly increasing on $[0, \infty\rangle$ for $t \geq 1$, by applying Bernoulli's inequality we obtain

$$\begin{aligned}\Phi^t(A_m) + t\frac{\Phi^{t-1}(A_m)}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn} &\leq \left(\Phi(A_m) + \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn}\right)^t\\ &\leq \left(\frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\Phi(a_n)\right)^t.\end{aligned} \tag{2.12}$$

Multiplying (2.12) by u_m, then summing up over $m \in \mathbb{N}_M$ and applying Minkowski's inequality to the right-hand side, we get

$$\begin{aligned}
&\sum_{m=1}^{M} u_m \Phi^t(A_m) + t \sum_{m=1}^{M} u_m \frac{\Phi^{t-1}(A_m)}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn} \\
&\quad \leq \sum_{m=1}^{M} u_m \left(\Phi(A_m) + \frac{1}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn} \right)^t \leq \sum_{m=1}^{M} u_m \left(\frac{1}{K_m} \sum_{n=1}^{N} k_{mn} \Phi(a_n) \right)^t \\
&\quad = \left\{ \left[\sum_{m=1}^{M} u_m \left(\frac{1}{K_m} \sum_{n=1}^{N} k_{mn} \Phi(a_n) \right)^t \right]^{\frac{1}{t}} \right\}^t \\
&\quad \leq \left\{ \sum_{n=1}^{N} \Phi(a_n) \left[\sum_{m=1}^{M} u_m \left(\frac{k_{mn}}{K_m} \right)^t \right]^{\frac{1}{t}} \right\}^t = \left(\sum_{n=1}^{N} v_n \Phi(a_n) \right)^t,
\end{aligned}$$

so (2.5) holds. The proof for a concave function Φ and $t \in \langle 0, 1]$ is similar. Namely, by the same arguments as for convex functions, from (2.2) we first obtain

$$\frac{k_{mn}}{K_m} \left[\Phi(A_m) - \Phi(a_n) - \varphi(A_m)(A_m - a_n) \right] \geq \frac{k_{mn}}{K_m} r_{mn}, \; m \in \mathbb{N}_M, \; n \in \mathbb{N}_N,$$

then

$$\begin{aligned}
\Phi^t(A_m) - t \frac{\Phi^{t-1}(A_m)}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn} &\geq \left(\Phi(A_m) - \frac{1}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn} \right)^t \\
&\geq \left(\frac{1}{K_m} \sum_{n=1}^{N} k_{mn} \Phi(a_n) \right)^t, \; m \in \mathbb{N}_M,
\end{aligned}$$

and finally

$$\begin{aligned}
&\sum_{m=1}^{M} u_m \Phi^t(A_m) - t \sum_{m=1}^{M} u_m \frac{\Phi^{t-1}(A_m)}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn} \\
&\quad \geq \sum_{m=1}^{M} u_m \left(\Phi(A_m) - \frac{1}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn} \right)^t \geq \left(\sum_{n=1}^{N} v_n \Phi(a_n) \right)^t,
\end{aligned}$$

that is, we get (2.8).

Remark 2.2. In particular, for $t = 1$ inequality (2.5) reduces to

$$\sum_{n=1}^{N} v_n \Phi(a_n) - \sum_{m=1}^{M} u_m \Phi(A_m) \geq \sum_{m=1}^{M} \frac{u_m}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn}, \tag{2.13}$$

where in this setting we have

$$v_n = \sum_{m=1}^{M} u_m \frac{k_{mn}}{K_m}, \; m \in \mathbb{N}_M. \tag{2.14}$$

Moreover, by analyzing the proof of Theorem 2.1, we see that (2.13) holds for all convex functions $\Phi : I \to \mathbb{R}$, that is, Φ does not need to be non-negative. Similarly, if Φ is any real concave function on I (not necessarily positive), then (2.13) holds with the reversed order of terms on its left-hand side.

Remark 2.3. Rewriting (2.5) with $t = \frac{q}{p} \geq 1$, that is, for $0 < p \leq q < \infty$ or $-\infty < q \leq p < 0$ and with an arbitrary non-negative convex function Φ, we obtain

$$\left(\sum_{n=1}^{N} v_n \Phi(a_n)\right)^{\frac{q}{p}} - \sum_{m=1}^{M} u_m \Phi^{\frac{q}{p}}(A_m) \geq \frac{q}{p} \sum_{m=1}^{M} u_m \frac{\Phi^{\frac{q}{p}-1}(A_m)}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn}, \quad (2.15)$$

where

$$v_n = \left[\sum_{m=1}^{M} u_m \left(\frac{k_{mn}}{K_m}\right)^{\frac{q}{p}}\right]^{\frac{p}{q}}, \ n \in \mathbb{N}_N.$$

Especially, if $p \geq 1$ or $p < 0$ (in that case Φ should be positive), then the function Φ^p is convex as well, so by replacing Φ with Φ^p relation (2.15) becomes

$$\left(\sum_{n=1}^{N} v_n \Phi^p(a_n)\right)^{\frac{q}{p}} - \sum_{m=1}^{M} u_m \Phi^q(A_m) \geq \frac{q}{p} \sum_{m=1}^{M} u_m \frac{\Phi^{q-p}(A_m)}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn}. \quad (2.16)$$

On the other hand, if Φ is a positive concave function and $t = \frac{q}{p} \in \langle 0, 1]$, that is, $0 < q \leq p < \infty$ or $-\infty < p \leq q < 0$, then (2.15) holds with the reversed order of terms on its left-hand side. Moreover, if $p \in \langle 0, 1]$, then the function Φ^p is concave, so the order of terms on the left-hand side of (2.16) is reversed.

Theorem 2.1 holds even if $M = N = \infty$. More precisely, following a similar procedure as in the proof of Theorem 2.1, we get the following corollary.

Corollary 2.4. *Suppose $t \in \mathbb{R}^+$ and non-negative numbers u_m, v_n, k_{mn}, for $m, n \in \mathbb{N}$, are such that*

$$K_m = \sum_{n=1}^{\infty} k_{mn} \in \mathbb{R}^+, \ m \in \mathbb{N}, \quad \text{and} \quad v_n = \left[\sum_{m=1}^{\infty} u_m \left(\frac{k_{mn}}{K_m}\right)^t\right]^{\frac{1}{t}} < \infty, \ n \in \mathbb{N}.$$

If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\left(\sum_{n=1}^{\infty} v_n \Phi(a_n)\right)^t - \sum_{m=1}^{\infty} u_m \Phi^t(A_m) \geq t \sum_{m=1}^{\infty} u_m \frac{\Phi^{t-1}(A_m)}{K_m} \sum_{n=1}^{\infty} k_{mn} r_{mn} \quad (2.17)$$

holds for all $t \geq 1$ and all real numbers $a_n \in I$, $n \in \mathbb{N}$ such that

$$A_m = \frac{1}{K_m} \sum_{n=1}^{\infty} k_{mn} a_n \in I, \ m \in \mathbb{N}, \quad (2.18)$$

where r_{mn} is defined by (2.7). If $t \in \langle 0, 1]$ and $\Phi : I \to \mathbb{R}$ is a positive concave function, then the order of terms on the left-hand side of (2.17) is reversed.

Remark 2.5. If I is a segment, that is, a closed subset of $\mathbb{R}$, condition (2.18) is fulfilled automatically since the series defining K_m converge for all $m \in \mathbb{N}$. Note that this condition cannot be omitted in any other general case. Further, according to Remark 2.2, in the case when $t = 1$ the function Φ from Corollary 2.4 needs not to be non-negative (or positive if it is concave). Finally, under conditions of Corollary 2.4, Remark 2.3 holds also with $M = N = \infty$.

Since the right-hand sides of relations (2.5) and (2.8) are non-negative, the next general discrete Hardy-type inequality follows as a direct consequence of Theorem 2.1.

Corollary 2.6. *Let $t \in \mathbb{R}^+$, $M, N \in \mathbb{N}$ and let non-negative real numbers u_m, v_n, k_{mn}, for $m \in \mathbb{N}_M$, $n \in \mathbb{N}_N$, fulfill (2.3) and (2.4). If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$, then*

$$\sum_{m=1}^{M} u_m \Phi^t(A_m) \leq \left(\sum_{n=1}^{N} v_n \Phi(a_n) \right)^t \tag{2.19}$$

holds for all $t \geq 1$, real numbers $a_n \in I$, for $n \in \mathbb{N}_N$ and A_m defined by (2.6). If $t \in \langle 0, 1]$ and the function $\Phi : I \to \mathbb{R}$ is positive and concave, then the sign of inequality in (2.19) is reversed.

Remark 2.7. Observing that the right-hand side of (2.16) is non-negative, for $p \geq 1$ and a non-negative convex function Φ we get

$$\left(\sum_{m=1}^{M} u_m \Phi^q(A_m) \right)^{\frac{1}{q}} \leq \left(\sum_{n=1}^{N} v_n \Phi^p(a_n) \right)^{\frac{1}{p}}. \tag{2.20}$$

Obviously, similar arguments can be applied also to other cases analyzed in Remark 2.3. However, here we omit their further analysis since it reflects only to the sign of inequality in (2.20). On the other hand, if non-negative real numbers u_m, v_n, k_{mn}, where $m, n \in \mathbb{N}$, fulfill the conditions of Corollary 2.4, then Corollary 2.6 holds also with $M = N = \infty$.

3. Applications. A new refined Carleman's inequality

In this section we continue previous analysis by considering some interesting particular cases of Theorem 2.1 and of its consequences. Especially, we obtain a refined discrete Jensen's inequality and a refinement and a generalization of the Hardy-type inequality (1.1) from the Introduction. As a special case of the Hardy-type inequality obtained, we get a new refined weighted version of Godunova's inequality (1.3). Finally, as our most important result in this section, we state and prove a new refined weighted strengthened Carleman's inequality and show how it refines and generalizes inequality (1.4). More about history, proofs and new developments regarding Carleman's inequality can be found in [4], [6], [11] and in in the references cited in those papers.

First, as a consequence of Theorem 2.1 we obtain a general refined discrete Jensen's inequality.

Theorem 3.1. *Let $\Phi : I \to \mathbb{R}$ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ be such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$. Let $t \geq 1$ and $N \in \mathbb{N}$. Then the inequality*

$$\left(\frac{1}{W_N} \sum_{n=1}^{N} w_n \Phi(a_n) \right)^t - \Phi^t(A_N) \geq t \frac{\Phi^{t-1}(A_N)}{W_N} \sum_{n=1}^{N} w_n r_n \tag{3.1}$$

holds for all real numbers $a_n \in I$ and $w_n \geq 0$, $n \in \mathbb{N}_N$, where

$$W_N = \sum_{n=1}^{N} w_n > 0, \quad A_N = \frac{1}{W_N} \sum_{n=1}^{N} w_n a_n,$$

and

$$r_n = \left| \left| \Phi(a_n) - \Phi(A_N) \right| - \left| \varphi(A_N) \right| \cdot \left| a_n - A_N \right| \right|, \ n \in \mathbb{N}_N.$$

If Φ is a positive concave function and $t \in \langle 0, 1]$, then the order of terms on the left-hand side of (3.1) is reversed.

Proof. Follows directly from Theorem 2.1, by taking arbitrary $M \in \mathbb{N}$ and positive real numbers u_m and α_m for $m \in \mathbb{N}_M$. Substituting $k_{mn} = \alpha_m w_n$, for all $m \in \mathbb{N}_M$ we get $K_m = \alpha_m W_N$, $A_m = A_N$ and $r_{mn} = r_n$, while $v_n = \frac{w_n}{W_N} U_M^{\frac{1}{t}}$ holds for all $n \in \mathbb{N}_N$, where $U_M = \sum_{m=1}^{M} u_m$. Thus, (2.5) reduces to (3.1) and does not depend on M, u_m and α_m.

Remark 3.2. For $t = 1$ inequality (3.1) becomes the classical refined discrete Jensen's inequality

$$\frac{1}{W_N} \sum_{n=1}^{N} w_n \Phi(a_n) - \Phi(A_N) \geq \frac{1}{W_N} \sum_{n=1}^{N} w_n r_n \tag{3.2}$$

and the function Φ is not necessarily non-negative. Of course, if the function Φ is concave, relation (3.2) holds with the reversed order of terms on its left-hand side.

Observe that Theorem 2.1 and Corollary 2.4 can be easily rewritten with arbitrary $M, N \in \mathbb{N}$ and $K_m = 1$ for all $m \in \mathbb{N}_M$. Here, we emphasize just such case with $M = N = \infty$ since it provides a generalization and a refinement of the Hardy-type inequality (1.1).

Theorem 3.3. *Let I be an interval in $\mathbb{R}$, $\Phi : I \to \mathbb{R}$ be a non-negative convex function and $\varphi : I \to \mathbb{R}$ be such that $\varphi(x) \in \partial\Phi(x)$, $x \in \operatorname{Int} I$. Let $t \in \mathbb{R}^+$. If real numbers $u_m, v_n, k_{mn} \geq 0$, $m, n \in \mathbb{N}$, are such that*

$$\sum_{n=1}^{\infty} k_{mn} = 1, \ m \in \mathbb{N}, \quad and \quad v_n = \left(\sum_{m=1}^{\infty} u_m k_{mn}^t \right)^{\frac{1}{t}} < \infty, \ n \in \mathbb{N},$$

if real numbers $a_n \in I$, $n \in \mathbb{N}$, *fulfill* $A_m = \sum_{n=1}^{\infty} k_{mn} a_n \in I$, $m \in \mathbb{N}$ *and if* r_{mn} *is defined by* (2.7), *then the inequality*

$$\left(\sum_{n=1}^{\infty} v_n \Phi(a_n)\right)^t - \sum_{m=1}^{\infty} u_m \Phi^t(A_m) \geq t \sum_{m=1}^{\infty} u_m \Phi^{t-1}(A_m) \sum_{n=1}^{\infty} k_{mn} r_{mn} \tag{3.3}$$

holds for all $t \geq 1$. *If* $t \in \langle 0, 1]$ *and the function* Φ *is positive and concave, the order of terms on the left-hand side of* (3.3) *is reversed.*

Remark 3.4. Set $k_{mn} = 0$ for $m < n$ in Theorem 3.3. Then

$$\sum_{n=1}^{m} k_{mn} = 1, \ A_m = \sum_{n=1}^{m} k_{mn} a_n, \ m \in \mathbb{N}, \quad \text{and} \quad v_n = \left(\sum_{m=n}^{\infty} u_m k_{mn}^t\right)^{\frac{1}{t}}, \ n \in \mathbb{N}.$$

Therefore, in this setting (3.3) becomes

$$\begin{aligned} &\left(\sum_{n=1}^{\infty} v_n \Phi(a_n)\right)^t - \sum_{m=1}^{\infty} u_m \Phi^t\left(\sum_{n=1}^{m} k_{mn} a_n\right) \\ &\qquad \geq \ t \sum_{m=1}^{\infty} u_m \Phi^{t-1}\left(\sum_{n=1}^{m} k_{mn} a_n\right) \sum_{n=1}^{m} k_{mn} r_{mn}. \end{aligned} \tag{3.4}$$

In particular, for $t = 1$ we get $v_n = \sum_{m=n}^{\infty} u_m k_{mn}$ and

$$\sum_{n=1}^{\infty} v_n \Phi(a_n) - \sum_{m=1}^{\infty} u_m \Phi\left(\sum_{n=1}^{m} k_{mn} a_n\right) \geq \sum_{m=1}^{\infty} u_m \sum_{n=1}^{m} k_{mn} r_{mn}, \tag{3.5}$$

so (3.3), (3.4) and (3.5) can be respectively regarded as two generalizations and a refinement of the Vasić–Pečarić relation (1.1). As in Theorem 3.3, for $t \in \langle 0, 1]$ and a positive convex function Φ, inequality (3.4) holds with the reversed order of terms on its left-hand side. The same goes also for (3.5), although in this case Φ does not have to be non-negative (or positive, if it is concave).

Now, we consider some particular functions Φ and non-negative real numbers u_m and k_{mn}. The following result provides a new weighted refinement of Godunova's inequality (1.3). Here we make use of the function $\Phi : \mathbb{R}^+ \to \mathbb{R}$, $\Phi(x) = x^p$, where $p \in \mathbb{R}$, $p \neq 0$. For $p \geq 1$ and $p < 0$ this function is convex, while it is concave for $p \in \langle 0, 1]$. In both cases we have $\varphi(x) = p x^{p-1}$, $x \in \mathbb{R}^+$.

Theorem 3.5. *Let* $N \in \mathbb{N}$, $t \in \mathbb{R}^+$ *and* $p \in \mathbb{R}$, $p \neq 0$. *Let* $(w_n)_{n \in \mathbb{N}}$ *be a sequence of non-negative real numbers such that* $w_1 > 0$ *and let*

$$W_n = \sum_{m=1}^{n} w_m, \ n \in \mathbb{N}. \tag{3.6}$$

If $t \geq 1$ and $p \in \mathbb{R} \setminus [0,1\rangle$, then the inequality

$$\left[\sum_{n=1}^{N} w_n \left(\sum_{m=n}^{N} \frac{w_{m+1}}{W_m^t W_{m+1}}\right)^{\frac{1}{t}} a_n^p\right]^t - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} A_m^{pt}$$
$$\geq t \sum_{m=1}^{N} \frac{w_{m+1}}{W_m W_{m+1}} A_m^{p(t-1)} \sum_{n=1}^{m} r_{mn} w_n \tag{3.7}$$

holds for all sequences $(a_n)_{n\in\mathbb{N}}$ of positive real numbers, where

$$A_m = \frac{1}{W_m} \sum_{n=1}^{m} w_n a_n \quad \text{and} \quad r_{mn} = \left| |a_n^p - A_m^p| - |p| \cdot |A_m|^{p-1} \cdot |a_n - A_m| \right|, \tag{3.8}$$

for $m, n \in \mathbb{N}$. If $t, p \in \langle 0, 1]$, then the order of terms on the left-hand side of (3.7) is reversed.

Proof. Note that $w_1 > 0$ implies $W_n > 0$ for all $n \in \mathbb{N}$. In Theorem 2.1, set $\Phi : \mathbb{R}^+ \to \mathbb{R}$, $\Phi(x) = x^p$, $M = N$, $u_m = \dfrac{w_{m+1}}{W_{m+1}}$ and

$$k_{mn} = \begin{cases} \dfrac{w_n}{W_m}, & m \geq n, \\ 0, & \text{otherwise}, \end{cases}$$

for $m, n \in \mathbb{N}_N$. Then we have $K_m = \displaystyle\sum_{n=1}^{m} \frac{w_n}{W_m} = 1$, $m \in \mathbb{N}_N$ and

$$v_n = w_n \left(\sum_{m=n}^{N} \frac{w_{m+1}}{W_m^t W_{m+1}}\right)^{\frac{1}{t}}, \quad n \in \mathbb{N}_N,$$

so (3.7) holds.

According to Theorem 3.3 and Remark 3.4, Theorem 3.5 can be easily extended to $N = \infty$.

Corollary 3.6. *Let $t \in \mathbb{R}^+$ and $p \in \mathbb{R}$, $p \neq 0$. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers and the sequence $(W_n)_{n\in\mathbb{N}}$ be defined by (3.6). Let $w_1 > 0$ and $\displaystyle\sum_{m=1}^{\infty} \frac{w_{m+1}}{W_m^t W_{m+1}} < \infty$. If $t \geq 1$ and $p \in \mathbb{R} \setminus [0,1\rangle$, then the inequality*

$$\left[\sum_{n=1}^{\infty} w_n \left(\sum_{m=n}^{\infty} \frac{w_{m+1}}{W_m^t W_{m+1}}\right)^{\frac{1}{t}} a_n^p\right]^t - \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_{m+1}} A_m^{pt}$$
$$\geq t \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_m W_{m+1}} A_m^{p(t-1)} \sum_{n=1}^{m} r_{mn} w_n \tag{3.9}$$

holds for all sequences $(a_n)_{n\in\mathbb{N}}$ of positive real numbers and A_m, r_{mn} defined by (3.8) for $m, n \in \mathbb{N}$. If $t, p \in \langle 0, 1]$, then (3.9) holds with the reversed order of terms on its left-hand side.

Remark 3.7. Rewrite Theorem 3.5 with $t = 1$. Then we have

$$v_n = w_n \sum_{m=n}^{N} \frac{w_{m+1}}{W_m W_{m+1}} = \frac{w_n}{W_n} \left(1 - \frac{W_n}{W_{N+1}} \right), \tag{3.10}$$

so for $p \in \mathbb{R} \setminus [0,1]$ we get the inequality

$$\begin{aligned} \sum_{n=1}^{N} \left(1 - \frac{W_n}{W_{N+1}} \right) \frac{w_n}{W_n} a_n^p - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} \left(\frac{1}{W_m} \sum_{n=1}^{m} w_n a_n \right)^p \\ \geq \sum_{m=1}^{N} \frac{w_{m+1}}{W_m W_{m+1}} \sum_{n=1}^{m} r_{mn} w_n, \end{aligned} \tag{3.11}$$

while for $p \in \langle 0, 1 \rangle$ terms on the left-hand side of (3.11) swap their positions. If $p = 1$, (3.11) holds trivially with both-hand sides equal to 0. On the other hand, denote

$$W_\infty = \sum_{n=1}^{\infty} w_n \tag{3.12}$$

and set $t = 1$ in Corollary 3.6. By using (3.10) and that $0 < W_n \leq W_{n+1} \leq W_\infty$, that is, $0 \leq 1 - \dfrac{W_n}{W_\infty} \leq 1$ for all $n \in \mathbb{N}$, relation (3.9) becomes

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{w_n}{W_n} a_n^p - \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_{m+1}} \left(\frac{1}{W_m} \sum_{n=1}^{m} w_n a_n \right)^p \\ &\quad \geq \sum_{n=1}^{\infty} \left(1 - \frac{W_n}{W_\infty} \right) \frac{w_n}{W_n} a_n^p - \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_{m+1}} \left(\frac{1}{W_m} \sum_{n=1}^{m} w_n a_n \right)^p \\ &\quad \geq \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_m W_{m+1}} \sum_{n=1}^{m} r_{mn} w_n \geq 0. \end{aligned}$$

Here we also covered the case when $W_\infty = \infty$.

Remark 3.8. Theorem 3.5 can be considered in the unweighted case, that is, for $w_n = 1$, $n \in \mathbb{N}$. Then $A_m = \dfrac{1}{m} \displaystyle\sum_{n=1}^{m} a_n$, $m \in \mathbb{N}$, so relation (3.7) reduces to

$$\left[\sum_{n=1}^{N} \left(\sum_{m=n}^{N} \frac{m^{-t}}{m+1} \right)^{\frac{1}{t}} a_n^p \right]^t - \sum_{m=1}^{N} \frac{1}{m+1} A_m^{pt} \geq t \sum_{m=1}^{N} \frac{1}{m(m+1)} A_m^{p(t-1)} \sum_{n=1}^{m} r_{mn}.$$

Moreover, for $t = 1$ and $p \in \mathbb{R} \setminus [0, 1\rangle$ we have

$$\begin{aligned} &\sum_{n=1}^{N} \frac{a_n^p}{n} - \sum_{m=1}^{N} \frac{1}{m+1} \left(\frac{1}{m} \sum_{n=1}^{m} a_n \right)^p \\ &\geq \sum_{n=1}^{N} \left(1 - \frac{n}{N+1} \right) \frac{a_n^p}{n} - \sum_{m=1}^{N} \frac{1}{m+1} \left(\frac{1}{m} \sum_{n=1}^{m} a_n \right)^p \\ &\geq \sum_{m=1}^{N} \frac{1}{m(m+1)} \sum_{n=1}^{m} r_{mn} \geq 0. \end{aligned} \tag{3.13}$$

Finally, for $N = \infty$ inequality (3.7) becomes

$$\sum_{n=1}^{\infty} \frac{a_n^p}{n} - \sum_{m=1}^{\infty} \frac{1}{m+1} \left(\frac{1}{m} \sum_{n=1}^{m} a_n \right)^p \geq \sum_{m=1}^{\infty} \frac{1}{m(m+1)} \sum_{n=1}^{m} r_{mn} \geq 0, \tag{3.14}$$

so (3.13) and (3.14) respectively provide a finite section and a refinement of Godunova's inequality (1.3). Therefore, Theorem 3.5 can be regarded as a weighted finite section of (1.3), while Corollary 3.6 gives a weighted generalization of Godunova's inequality.

As the last result in this section, applying Theorem 2.1 to the convex function $\Phi : \mathbb{R} \to \mathbb{R}^+$, $\Phi(x) = e^x$, we obtain a new strengthened weighted Carleman's inequality. Here we have $\varphi = \Phi$. The following theorem provides our first result in that direction.

Theorem 3.9. *Let $N \in \mathbb{N}$ and $t \in [1, \infty\rangle$. If $(w_n)_{n \in \mathbb{N}}$ is a sequence of nonnegative real numbers such that $w_1 > 0$ and the sequence $(W_n)_{n \in \mathbb{N}}$ is defined as in (3.6), then the inequality*

$$\begin{aligned} &\left[\sum_{n=1}^{N} w_n W_n \left(\sum_{m=n}^{N} \frac{w_{m+1}}{W_m^t W_{m+1}} \right)^{\frac{1}{t}} a_n \right]^t - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} G_m^t \\ &\qquad \geq t \sum_{m=1}^{N} \frac{w_{m+1}}{W_m W_{m+1}} G_m^{t-1} \sum_{n=1}^{m} r_{mn} w_n \end{aligned} \tag{3.15}$$

holds for all sequences $(a_n)_{n \in \mathbb{N}}$ of positive real numbers, where

$$G_m = \left[\prod_{n=1}^{m} (W_n a_n)^{w_n} \right]^{\frac{1}{W_m}}, \quad m \in \mathbb{N}, \tag{3.16}$$

and

$$r_{mn} = \left| |W_n a_n - G_m| - G_m \left| \log \frac{W_n a_n}{G_m} \right| \right|, \quad m, n \in \mathbb{N}. \tag{3.17}$$

In particular, for $t = 1$ relation (3.15) reduces to

$$\sum_{n=1}^{N}\left(1-\frac{W_n}{W_{N+1}}\right)w_n a_n - \sum_{m=1}^{N}\frac{w_{m+1}}{W_{m+1}}\left(\prod_{n=1}^{m}W_n^{w_n}\right)^{\frac{1}{W_m}}\left(\prod_{n=1}^{m}a_n^{w_n}\right)^{\frac{1}{W_m}} \geq \sum_{m=1}^{N}\frac{w_{m+1}}{W_m W_{m+1}}\sum_{n=1}^{m} r_{mn} w_n. \tag{3.18}$$

Proof. Follows immediately by rewriting Theorem 2.1 with $M = N$, $\Phi : \mathbb{R} \to \mathbb{R}^+$, $\Phi(x) = e^x$, parameters u_m and k_{mn} as in the proof of Theorem 3.5 and with the sequence $(\log(W_n a_n))_{n\in\mathbb{N}}$ instead of $(a_n)_{n\in\mathbb{N}}$. Then we have $A_m = \log G_m$, $m \in \mathbb{N}$, so (3.15) and (3.18) hold.

Reformulating Theorem 3.9 for $N = \infty$, as in Theorem 3.3 and Remark 3.4 we get the following corollary.

Corollary 3.10. *Suppose $t \in [1,\infty\rangle$, $(w_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers and the sequence $(W_n)_{n\in\mathbb{N}}$ is defined by (3.6). If $w_1 > 0$ and $\sum_{m=1}^{\infty}\frac{w_{m+1}}{W_m^t W_{m+1}} < \infty$, then*

$$\left[\sum_{n=1}^{\infty} w_n W_n \left(\sum_{m=n}^{\infty}\frac{w_{m+1}}{W_m^t W_{m+1}}\right)^{\frac{1}{t}} a_n\right]^t - \sum_{m=1}^{\infty}\frac{w_{m+1}}{W_{m+1}}G_m^t \geq t\sum_{m=1}^{\infty}\frac{w_{m+1}}{W_m W_{m+1}}G_m^{t-1}\sum_{n=1}^{m} r_{mn} w_n$$

holds for all sequences $(a_n)_{n\in\mathbb{N}}$ of positive real numbers and G_m, r_{mn} respectively defined by (3.16) and (3.17). In particular, for $t = 1$ and W_∞ defined by (3.12), we get

$$\sum_{n=1}^{\infty}\left(1-\frac{W_n}{W_\infty}\right)w_n a_n - \sum_{m=1}^{\infty}\frac{w_{m+1}}{W_{m+1}}\left(\prod_{n=1}^{m}W_n^{w_n}\right)^{\frac{1}{W_m}}\left(\prod_{n=1}^{m}a_n^{w_n}\right)^{\frac{1}{W_m}} \geq \sum_{m=1}^{\infty}\frac{w_{m+1}}{W_m W_{m+1}}\sum_{n=1}^{m} r_{mn} w_n.$$

Under some additional conditions on weights w_n, the inequalities obtained in Theorem 3.9 and Corollary 3.10 can be seen as finite sections and refinements of the classical weighted Carleman's inequality. One possible such conditions are given in the next lemma, interesting in its own right.

Lemma 3.11. *Suppose $(w_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers such that $w_1 > 0$ and $w_1 \geq w_n$, for $n = 2, 3, \ldots$. If the sequence $(W_n)_{n\in\mathbb{N}}$ is defined by (3.6), then*

$$\frac{1}{W_{m+1}}\left(\prod_{n=1}^{m}W_n^{w_n}\right)^{\frac{1}{W_m}} > \frac{1}{e}, \quad m \in \mathbb{N}. \tag{3.19}$$

Proof. Since the mapping $x \mapsto \log x$ is strictly increasing on $\mathbb{R}^+$, for arbitrary $0 < a \le b < \infty$ we have

$$(b-a)\log b \ge \int_a^b \log x\,dx,$$

with the strict inequality if $a < b$. In particular, by substituting $a = W_{n-1}$ and $b = W_n$, we get $b - a = w_n$ and

$$w_n \log W_n \ge \int_{W_{n-1}}^{W_n} \log x\,dx,\ \ n = 2,\,3,\,\ldots\,. \tag{3.20}$$

Hence

$$\begin{aligned}\sum_{n=2}^{m+1} w_n \log W_n &\ge \int_{W_1}^{W_{m+1}} \log x\,dx \\ &= W_{m+1}\log W_{m+1} - W_{m+1} - w_1 \log W_1 + w_1\end{aligned}$$

holds for an arbitrary $m \in \mathbb{N}$. Therefrom

$$\sum_{n=1}^{m} w_n \log W_n \ge W_m \log W_{m+1} - W_{m+1} + w_1 \ge W_m \log W_{m+1} - W_m, \tag{3.21}$$

where we used the condition $w_1 \ge w_{m+1}$. Observe that at least one of inequalities in (3.21) is strict. Namely, if there exists $n \in \{2,\,3,\,\ldots,\,m+1\}$ such that $w_n > 0$, then the sign of inequality in (3.20) is strict and so is the first inequality in (3.21). Otherwise, we have $w_1 > 0 = w_{m+1}$ and the second inequality in (3.21) is strict. Finally,

$$\log\left(\prod_{n=1}^{m} W_n^{w_n}\right) > W_m \log \frac{W_{m+1}}{e},$$

so we get (3.19).

Remark 3.12. If $w_n = 1$, $n \in \mathbb{N}$, then (3.19) becomes

$$\frac{1}{m+1}\sqrt[m]{m!} > \frac{1}{e},\ \ m \in \mathbb{N},$$

that is,

$$\frac{m+1}{\sqrt[m]{m!}} < e,\ \ m \in \mathbb{N}.$$

Thus, Lemma 3.11 provides a class of lower bounds for the constant e.

Using Lemma 3.11 in Theorem 3.9 and Corollary 3.10, we obtain a new strengthened weighted Carleman's inequality and its finite sections. Here we emphasize only the most important case, that is, the case with $t = 1$. Since the general case can be derived analogously, it is omitted.

Corollary 3.13. *Under the conditions of Theorem 3.9 and Lemma 3.11, the left-hand side of (3.18) is strictly less than*

$$\sum_{n=1}^{N}\left(1 - \frac{W_n}{W_{N+1}}\right) w_n a_n - \frac{1}{e}\sum_{m=1}^{N} w_{m+1}\left(\prod_{n=1}^{m} a_n^{w_n}\right)^{\frac{1}{W_m}}.$$

Especially, if $N=\infty$, then the inequalities

$$\begin{aligned}
&\sum_{n=1}^{\infty} w_n a_n - \frac{1}{e}\sum_{m=1}^{\infty} w_{m+1}\left(\prod_{n=1}^{m} a_n^{w_n}\right)^{\frac{1}{W_m}} \\
&\quad\geq \sum_{n=1}^{\infty}\left(1-\frac{W_n}{W_\infty}\right) w_n a_n - \frac{1}{e}\sum_{m=1}^{\infty} w_{m+1}\left(\prod_{n=1}^{m} a_n^{w_n}\right)^{\frac{1}{W_m}} \\
&\quad> \sum_{n=1}^{\infty}\left(1-\frac{W_n}{W_\infty}\right) w_n a_n - \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_{m+1}}\left(\prod_{n=1}^{m} W_n^{w_n}\right)^{\frac{1}{W_m}}\left(\prod_{n=1}^{m} a_n^{w_n}\right)^{\frac{1}{W_m}} \\
&\quad\geq \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_m W_{m+1}}\sum_{n=1}^{m} r_{mn} w_n \geq 0
\end{aligned}$$

hold, where the case when $W_\infty=\infty$ is included as well.

Remark 3.14. For $w_n=1$, $n\in\mathbb{N}$, relation (3.15) reduces to

$$\left[\sum_{n=1}^{N} n\left(\sum_{m=n}^{N}\frac{m^{-t}}{m+1}\right)^{\frac{1}{t}} a_n\right]^{t} - \sum_{m=1}^{N}\frac{H_m^t}{m+1} \geq t\sum_{m=1}^{N}\frac{H_m^{t-1}}{m(m+1)}\sum_{n=1}^{m} r_{mn}, \tag{3.22}$$

where

$$H_m = \left(m!\prod_{n=1}^{m} a_n\right)^{\frac{1}{m}} \quad \text{and} \quad r_{mn} = \left|\,|na_n - H_m| - H_m\left|\log\frac{na_n}{H_m}\right|\,\right|, \quad m,n\in\mathbb{N}.$$

Since $\sum_{m=1}^{\infty}\frac{m^{-t}}{m+1}<\infty$ for all $t\in[1,\infty\rangle$, note that inequality (3.22) covers also the case when $N=\infty$. On the other hand, Corollary 3.13 and Remark 3.12 imply that

$$\begin{aligned}
&\sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right) a_n - \frac{1}{e}\sum_{m=1}^{N}\left(\prod_{n=1}^{m} a_n\right)^{\frac{1}{m}} \\
&\quad> \sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right) a_n - \sum_{m=1}^{N}\frac{1}{m+1}H_m \geq \sum_{m=1}^{N}\frac{1}{m(m+1)}\sum_{n=1}^{m} r_{mn} \geq 0
\end{aligned}$$

holds for all $N\in\mathbb{N}$, while for $N=\infty$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n - \frac{1}{e}\sum_{m=1}^{\infty}\left(\prod_{n=1}^{m} a_n\right)^{\frac{1}{m}} &> \sum_{n=1}^{\infty} a_n - \sum_{m=1}^{\infty}\frac{1}{m+1}H_m \\
&\geq \sum_{m=1}^{\infty}\frac{1}{m(m+1)}\sum_{n=1}^{m} r_{mn} \geq 0.
\end{aligned}$$

Therefore, our results refine and generalize relation (1.4) and Carleman's inequality (1.5). We take an opportunity to mention that another strengthened weighted

Carleman's inequality was obtained by Čižmešija et al. in [4], but that result can be hardly comparable with the inequalities derived in this section.

4. Exponential convexity and Hardy-type inequalities

By employing the concept of logarithmic and exponential convexity, here we obtain upper bounds and some further lower bounds for the left-hand sides of the Hardy-type inequalities from previous two sections, in settings with suitably chosen convex functions Φ and $t = 1$. Before presenting our ideas and results, we recall basic facts about log-convex and exponentially convex functions.

Let $I \subseteq \mathbb{R}$ be an interval. A positive function $\Phi : I \to \mathbb{R}$ is said to be logarithmically convex, or log-convex, if the function $\log \Phi$ is convex. It is well-known that each log-convex function is convex and that

$$\Phi(x_2)^{x_3-x_1} \leq \Phi(x_1)^{x_3-x_2}\Phi(x_3)^{x_2-x_1} \tag{4.1}$$

holds for all such functions Φ and all x_1, x_2, $x_3 \in I$ such that $x_1 < x_2 < x_3$. On the other hand, an exponentially convex function on I is any continuous function $\Phi : I \to \mathbb{R}$ satisfying

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j\Phi(x_i + x_j) \geq 0 \tag{4.2}$$

for all $k \in \mathbb{N}$ and all sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of real numbers such that $x_i + x_j \in I$, $i, j \in \mathbb{N}$. It can be proved that every exponentially convex function is log-convex and thus convex. Moreover, the condition (4.2) can be replaced with a more suitable condition

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j\Phi\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{4.3}$$

which has to hold for all $k \in \mathbb{N}$, all sequences $(\alpha_n)_{n\in\mathbb{N}}$ of real numbers and all sequences $(x_n)_{n\in\mathbb{N}}$ in I. More precisely, a function $\Phi : I \to \mathbb{R}$ is exponentially convex if and only if it is continuous and fulfills (4.3). Further information about log-convex and exponentially convex functions can be found in [2] and [7], as well as in the references given in those monographs.

Our analysis now continues by making use of two suitably chosen families of convex functions dependent on a real parameter. We need the following lemma.

Lemma 4.1. *Let $s \in \mathbb{R}$ and the functions $\Phi_s : \mathbb{R}^+ \to \mathbb{R}$ and $\Psi_s : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\Phi_s(x) = \begin{cases} \dfrac{x^s}{s(s-1)}, & s \neq 0,\ 1, \\[2ex] -\log x, & s = 0, \\[2ex] x \log x, & s = 1, \end{cases} \tag{4.4}$$

and

$$\Psi_s(x) = \begin{cases} \dfrac{1}{s^2} e^{sx}, & s \neq 0, \\ \\ \dfrac{1}{2} x^2, & s = 0. \end{cases} \tag{4.5}$$

For all $s \in \mathbb{R}$*, the functions* Φ_s *and* Ψ_s *are convex and differentiable on their respective domains. Moreover,* $\Phi_s''(x) = x^{s-2}$ *and* $\Psi_s''(x) = e^{sx}$*.*

Proof. Fix $s \in \mathbb{R}$. Since $\Phi_s''(x) = x^{s-2} > 0$, $x \in \mathbb{R}^+$ and $\Psi_s''(x) = e^{sx} > 0$, $x \in \mathbb{R}$, we easily conclude that both functions Φ_s and Ψ_s are convex.

According to Lemma 4.1, all the results from Section 2 can be rewritten with convex functions Φ_s and Ψ_s, $s \in \mathbb{R}$. In particular, observing that the right-hand side of (2.13) is non-negative, from Remark 2.2 we get

$$\sum_{n=1}^{N} v_n \Phi_s(a_n) - \sum_{m=1}^{M} u_m \Phi_s(A_m) \geq 0$$

and

$$\sum_{n=1}^{N} v_n \Psi_s(a_n) - \sum_{m=1}^{M} u_m \Psi_s(A_m) \geq 0,$$

where $M, N \in \mathbb{N}$ and u_m, k_{mn}, K_m, a_n and A_m are as in Theorem 2.1 ($a_n \in \mathbb{R}^+$ and $a_n \in \mathbb{R}$ in the cases with Φ_s and Ψ_s respectively), while v_n is defined by (2.14). Therefore, under assumptions of Theorem 2.1, the functions F, $G : \mathbb{R} \to \mathbb{R}$,

$$F(s) = \sum_{n=1}^{N} v_n \Phi_s(a_n) - \sum_{m=1}^{M} u_m \Phi_s(A_m) \tag{4.6}$$

and

$$G(s) = \sum_{n=1}^{N} v_n \Psi_s(a_n) - \sum_{m=1}^{M} u_m \Psi_s(A_m), \tag{4.7}$$

are well-defined and non-negative. By proving that they are log-convex, we provide upper bounds and some new lower bounds for the left-hand side of (2.13), in the setting with convex functions Φ_s and Ψ_s. In fact, in the sequel we prove a stronger result, that is, that F and G are exponentially convex functions.

Theorem 4.2. *Let* $M, N \in \mathbb{N}$*. For* $m \in \mathbb{N}_M$ *and* $n \in \mathbb{N}_N$*, let* $a_n \in \mathbb{R}^+$*,* u_m*,* k_{mn}*,* K_m *and* A_m *be as in Theorem 2.1 and let* v_n *be as in (2.14). Then the function* $F : \mathbb{R} \to \mathbb{R}$*, defined by (4.6), is exponentially convex and the inequality*

$$F(s_2)^{s_3 - s_1} \leq F(s_1)^{s_3 - s_2} F(s_3)^{s_2 - s_1} \tag{4.8}$$

holds for all s_1*,* s_2*,* $s_3 \in \mathbb{R}$ *such that* $s_1 < s_2 < s_3$*.*

Proof. The first step is to prove that F is continuous on $\mathbb{R}$. Since the mapping $s \mapsto \frac{a^s}{s(s-1)}$ is continuous on $\mathbb{R} \setminus \{0, 1\}$ for all $a \in \mathbb{R}^+$, we only need to prove the continuity of F in $s = 0$ and $s = 1$. Note that

$$\sum_{n=1}^{N} v_n - \sum_{m=1}^{M} u_m = \sum_{n=1}^{N}\sum_{m=1}^{M} u_m \frac{k_{mn}}{K_m} - \sum_{m=1}^{M} u_m$$
$$= \sum_{m=1}^{M} u_m \left(\frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\right) - \sum_{m=1}^{M} u_m = 0 \tag{4.9}$$

and

$$\sum_{n=1}^{N} v_n a_n - \sum_{m=1}^{M} u_m A_m = \sum_{n=1}^{N} a_n \sum_{m=1}^{M} u_m \frac{k_{mn}}{K_m} - \sum_{m=1}^{M} \frac{u_m}{K_m}\sum_{n=1}^{N} k_{mn} a_n = 0. \tag{4.10}$$

Applying the classical L'Hospital's rule, identity (4.9) and the definitions of the functions Φ_s and F, we have

$$\begin{aligned} \lim_{s\to 0} F(s) &= \lim_{s\to 0} \frac{\sum_{n=1}^{N} v_n a_n^s - \sum_{m=1}^{M} u_m A_m^s}{s(s-1)} \\ &= \lim_{s\to 0} \frac{\sum_{n=1}^{N} v_n a_n^s \log a_n - \sum_{m=1}^{M} u_m A_m^s \log A_m}{2s-1} \\ &= \sum_{m=1}^{M} u_m \log A_m - \sum_{n=1}^{N} v_n \log a_n = F(0) \end{aligned}$$

and similarly, by using (4.10),

$$\lim_{s\to 1} F(s) = \sum_{n=1}^{N} v_n a_n \log a_n - \sum_{m=1}^{M} u_m A_m \log A_m = F(1).$$

Hence, F is continuous on $\mathbb{R}$. To prove that it is exponentially convex, it suffices to check condition (4.3). Fix $k \in \mathbb{N}$ and $\alpha_i \in \mathbb{R}$, $s_i \in \mathbb{R}^+$, for $i \in \mathbb{N}_k$. Denote $\Phi = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \Phi_{\frac{s_i+s_j}{2}}$. Lemma 4.1 implies

$$\begin{aligned} \Phi''(x) &= \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \Phi''_{\frac{s_i+s_j}{2}}(x) = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j x^{\frac{s_i+s_j}{2}-2} \\ &= \left(\sum_{i=1}^{k} \alpha_i x^{\frac{s_i}{2}-1}\right)^2 \geq 0, \ \ x \in \mathbb{R}^+, \end{aligned}$$

so Φ is a convex function on $\mathbb{R}^+$. Thus, applying Corollary 2.6 to Φ and $t = 1$, we get

$$\sum_{n=1}^{N} v_n \Phi(a_n) - \sum_{m=1}^{M} u_m \Phi(A_m) \geq 0$$

and finally

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i\alpha_j F\left(\frac{s_i+s_j}{2}\right)\\
&\quad=\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i\alpha_j\left(\sum_{n=1}^{N}v_n\Phi_{\frac{s_i+s_j}{2}}(a_n)-\sum_{m=1}^{M}u_m\Phi_{\frac{s_i+s_j}{2}}(A_m)\right)\\
&\quad=\sum_{n=1}^{N}v_n\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i\alpha_j\Phi_{\frac{s_i+s_j}{2}}(a_n)-\sum_{m=1}^{M}u_m\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i\alpha_j\Phi_{\frac{s_i+s_j}{2}}(A_m)\\
&\quad=\sum_{n=1}^{N}v_n\Phi(a_n)-\sum_{m=1}^{M}u_m\Phi(A_m)\geq 0.
\end{aligned}$$

Therefore, (4.3) holds and F is exponentially convex. Since every exponentially convex function is log-convex, (4.8) follows directly from (4.1).

By using similar arguments, we prove exponential convexity of the function G.

Theorem 4.3. *Suppose $M,N\in\mathbb{N}$. For $m\in\mathbb{N}_M$ and $n\in\mathbb{N}_N$, suppose that $a_n\in\mathbb{R}$, u_m, k_{mn}, K_m and A_m are as in Theorem 2.1, and that v_n is as in (2.14). Then the function $G:\mathbb{R}\to\mathbb{R}$, given by (4.7), is exponentially convex and*

$$G(s_2)^{s_3-s_1}\leq G(s_1)^{s_3-s_2}G(s_3)^{s_2-s_1} \tag{4.11}$$

holds for all s_1, s_2, $s_3\in\mathbb{R}$ such that $s_1<s_2<s_3$.

Proof. Combining (4.9), (4.10), L'Hospital's rule and the definition of the function G, we obtain

$$\begin{aligned}
\lim_{s\to 0}G(s) &= \lim_{s\to 0}\frac{1}{s^2}\left(\sum_{n=1}^{N}v_n e^{sa_n}-\sum_{m=1}^{M}u_m e^{sA_m}\right)\\
&= \lim_{s\to 0}\frac{1}{2s}\left(\sum_{n=1}^{N}v_n a_n e^{sa_n}-\sum_{m=1}^{M}u_m A_m e^{sA_m}\right)\\
&= \lim_{s\to 0}\frac{1}{2}\left(\sum_{n=1}^{N}v_n a_n^2 e^{sa_n}-\sum_{m=1}^{M}u_m A_m^2 e^{sA_m}\right)\\
&= \frac{1}{2}\left(\sum_{n=1}^{N}v_n a_n^2-\sum_{m=1}^{M}u_m A_m^2\right)=G(0).
\end{aligned}$$

Since the mapping $s\mapsto\frac{e^{as}}{s^2}$ is continuous on $\mathbb{R}\setminus\{0\}$, we conclude that G is continuous on $\mathbb{R}$. To prove that G is an exponentially convex function, fix $k\in\mathbb{N}$ and $\alpha_i, s_i\in\mathbb{R}$, for $i\in\mathbb{N}_k$. Applying Lemma 4.1 to the function $\Psi=\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i\alpha_j\Psi_{\frac{s_i+s_j}{2}}$, for all $x\in\mathbb{R}$ we get

$$\Psi''(x)=\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i\alpha_j\Psi''_{\frac{s_i+s_j}{2}}(x)=\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i\alpha_j e^{\frac{s_i+s_j}{2}x}=\left(\sum_{i=1}^{k}\alpha_i e^{\frac{s_i}{2}x}\right)^2\geq 0,$$

so Ψ is convex on $\mathbb{R}$ and

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i\alpha_j G\left(\frac{s_i+s_j}{2}\right)\geq 0$$

holds as in the proof of Theorem 4.2. Thus, G is exponentially convex and then also log-convex. Relation (4.11) follows directly from (4.1).

Remark 4.4. Observe that each of inequalities (4.8) and (4.11) implies three further relations suitable for establishing lower and upper bounds for values of F and G. Namely, from (4.8) we obtain that inequalities

$$F(s_2)\leq F(s_1)^{\frac{s_3-s_2}{s_3-s_1}}F(s_3)^{\frac{s_2-s_1}{s_3-s_1}}, \tag{4.12}$$

$$F(s_1)\geq F(s_2)^{\frac{s_3-s_1}{s_3-s_2}}F(s_3)^{\frac{s_1-s_2}{s_3-s_2}} \quad \text{and} \quad F(s_3)\geq F(s_1)^{\frac{s_2-s_3}{s_2-s_1}}F(s_2)^{\frac{s_3-s_1}{s_2-s_1}} \tag{4.13}$$

hold for all s_1, s_2, $s_3\in\mathbb{R}$ such that $s_1<s_2<s_3$, while the same inequalities for G follow from (4.11).

Remark 4.5. In (4.6) and (4.7), the functions F and G were defined as finite sums of functions, so there were no further conditions on the sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(a_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$ needed to apply methods used in the proofs of Theorem 4.2 and Theorem 4.3. Of course, we can also consider the case when $M=N=\infty$, that is, to define F and G respectively by

$$F(s)=\sum_{n=1}^{\infty}v_n\Phi_s(a_n)-\sum_{m=1}^{\infty}u_m\Phi_s(A_m)$$

and

$$G(s)=\sum_{n=1}^{\infty}v_n\Psi_s(a_n)-\sum_{m=1}^{\infty}u_m\Psi_s(A_m).$$

Obviously, then we have to deal with function series and, in order to apply L'Hospital's rule, be able to take limits and differentiate them term by term. Therefore, the sequences of real numbers mentioned above should be such that the function series $\sum_{n=1}^{\infty}v_na_n^s$ and $\sum_{m=1}^{\infty}u_mA_m^s$ are uniformly convergent in neighbourhoods of $s=0$ and $s=1$ and that the function series $\sum_{n=1}^{\infty}v_ne^{a_ns}$ and $\sum_{m=1}^{\infty}u_me^{A_ms}$ are uniformly convergent in some neighbourhood of $s=0$. Some such sufficient conditions follow, for example, from the usual Weierstrass's test for uniform convergence.

Theorem 4.2 and Theorem 4.3, along with Remark 4.4 and Remark 4.5, can be applied to all particular cases of Theorem 2.1 and Corollary 2.4 explored in detail in Section 3. However, owing to the lack of space, here we mention just the cases related to our improvements of Godunova's and Carleman's inequality.

The following result provides a new lower and upper bound for the left-hand side of the refined weighted Godunova's inequality (3.11).

Corollary 4.6. *Let $N \in \mathbb{N}$ and $p \in \mathbb{R} \setminus \{0, 1\}$. If $(w_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers, such that $w_1 > 0$ and the sequence $(W_n)_{n\in\mathbb{N}}$ is defined by (3.6), then the inequalities*

$$\begin{aligned} & p(p-1) \inf_{(s,t)\in\mathcal{S}_p} F(s)^{\frac{t-p}{t-s}} F(t)^{\frac{p-s}{t-s}} \\ & \geq \sum_{n=1}^{N} \left(1 - \frac{W_n}{W_{N+1}}\right) \frac{w_n}{W_n} a_n^p - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} \left(\frac{1}{W_m} \sum_{n=1}^{m} w_n a_n\right)^p \\ & \geq p(p-1) \sup_{(s,t)\in\mathcal{T}_p} F(s)^{\frac{t-p}{t-s}} F(t)^{\frac{p-s}{t-s}} \end{aligned} \tag{4.14}$$

hold for all sequences $(a_n)_{n\in\mathbb{N}}$ of positive real numbers and $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(s) = \sum_{n=1}^{N} \left(1 - \frac{W_n}{W_{N+1}}\right) \frac{w_n}{W_n} \Phi_s(a_n) - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} \Phi_s\left(\frac{1}{W_m} \sum_{n=1}^{m} w_n a_n\right),$$

where Φ_s is defined by (4.4) and

$$\mathcal{S}_p = \{(s,t) \in \mathbb{R}^2 : s < p < t\}, \ \mathcal{T}_p = \{(s,t) \in \mathbb{R}^2 : p < s < t \ \text{ or } \ s < t < p\}.$$

Proof. Follows directly from Theorem 4.2, applied with $M = N$, u_m and k_{mn} as in the proof of Theorem 3.5 and with v_n defined by (3.10). The first inequality in (4.14) is obtained from (4.12), rewritten with $s_1 = s$, $s_2 = p$, and $s_3 = t$, where $s < p < t$. On the other hand, the second inequality in (4.14) is a consequence of both relations in (4.13). The first of them is rewritten with $s_1 = p$, $s_2 = s$ and $s_3 = t$, where $p < s < t$ and the second with $s_1 = s$, $s_2 = t$ and $s_3 = p$, where $s < t < p$.

Remark 4.7. In particular, for $w_n = 1$, $n \in \mathbb{N}$, we have

$$F(s) = \sum_{n=1}^{N} \left(1 - \frac{n}{N+1}\right) \frac{1}{n} \Phi_s(a_n) - \sum_{m=1}^{N} \frac{1}{m+1} \Phi_s\left(\frac{1}{m} \sum_{n=1}^{m} a_n\right),$$

so (4.14) becomes

$$\begin{aligned} & p(p-1) \inf_{(s,t)\in\mathcal{S}_p} F(s)^{\frac{t-p}{t-s}} F(t)^{\frac{p-s}{t-s}} \\ & \geq \sum_{n=1}^{N} \left(1 - \frac{n}{N+1}\right) \frac{a_n^p}{n} - \sum_{m=1}^{N} \frac{1}{m+1} \left(\frac{1}{m} \sum_{n=1}^{m} a_n\right)^p \\ & \geq p(p-1) \sup_{(s,t)\in\mathcal{T}_p} F(s)^{\frac{t-p}{t-s}} F(t)^{\frac{p-s}{t-s}}. \end{aligned}$$

Under the conditions of Remark 4.5, Corollary 4.6 holds also for $N = \infty$. In that case, W_{N+1} is replaced with W_∞ defined by (3.12) and covers also the case when $W_\infty = \infty$.

Our final result in this paper is the following refinement of the weighted Carleman's inequality.

Corollary 4.8. *Suppose $N \in \mathbb{N}$, $(w_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers such that $w_1 > 0$ and the sequence $(W_n)_{n\in\mathbb{N}}$ is defined by (3.6). Then the inequalities*

$$\begin{aligned}\inf_{(s,t)\in\mathcal{S}_1} G(s)^{\frac{t-1}{t-s}} G(t)^{\frac{1-s}{t-s}} &\geq \sum_{n=1}^{N}\left(1-\frac{W_n}{W_{N+1}}\right) w_n a_n - \sum_{m=1}^{N}\frac{w_{m+1}}{W_{m+1}} G_m \\ &\geq \sup_{(s,t)\in\mathcal{T}_1} G(s)^{\frac{t-1}{t-s}} G(t)^{\frac{1-s}{t-s}} \end{aligned} \tag{4.15}$$

hold for all sequences $(a_n)_{n\in\mathbb{N}}$ of positive real numbers, $(G_n)_{n\in\mathbb{N}}$ defined by (3.16), and $G : \mathbb{R} \to \mathbb{R}$ given by

$$G(s) = \sum_{n=1}^{N}\left(1-\frac{W_n}{W_{N+1}}\right)\frac{w_n}{W_n}\Psi_s\left(\log(W_n a_n)\right) - \sum_{m=1}^{N}\frac{w_{m+1}}{W_{m+1}}\Psi_s\left(\log G_m\right),$$

where Ψ_s is defined by (4.5) and

$$\mathcal{S}_1 = \{(s,t)\in\mathbb{R}^2 : s < 1 < t\},\ \mathcal{T}_1 = \{(s,t)\in\mathbb{R}^2 : 1 < s < t \quad or \quad s < t < 1\}.$$

Proof. A direct consequence of Theorem 4.3, rewritten with $M = N$, u_m and k_{mn} as in the proof of Theorem 3.5, v_n defined by (3.10) and with the sequence $(\log(W_n a_n))_{n\in\mathbb{N}}$ instead of $(a_n)_{n\in\mathbb{N}}$. The first inequality in (4.15) follows from (4.12), rewritten with G, $s_1 = s$, $s_2 = 1$, and $s_3 = t$, for $(s,t) \in \mathcal{S}_1$. The second inequality in (4.15) is obtained by combining both relations in (4.13), rewritten with G. In the first of them we set $s_1 = 1$, $s_2 = s$ and $s_3 = t$, where $1 < s < t$, while in the second relation we substitute $s_1 = s$, $s_2 = t$ and $s_3 = 1$, where $s < t < 1$.

Remark 4.9. Note that for $w_n = 1$, $n \in \mathbb{N}$, we have

$$G(s) = \sum_{n=1}^{N}\frac{1}{n}\left(1-\frac{n}{N+1}\right)\Psi_s\left(\log(n a_n)\right) - \sum_{m=1}^{N}\frac{1}{m+1}\Psi_s\left(\log H_m\right),$$

where $H_m = \left(m!\prod_{n=1}^{m} a_n\right)^{\frac{1}{m}}$. Hence, in this setting (4.15) becomes

$$\begin{aligned}\inf_{(s,t)\in\mathcal{S}_1} G(s)^{\frac{t-1}{t-s}} G(t)^{\frac{1-s}{t-s}} &\geq \sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right) a_n - \sum_{m=1}^{N}\frac{1}{m+1}\left(m!\prod_{n=1}^{m} a_n\right)^{\frac{1}{m}} \\ &\geq \sup_{(s,t)\in\mathcal{T}_1} G(s)^{\frac{t-1}{t-s}} G(t)^{\frac{1-s}{t-s}}. \end{aligned}$$

If the sequence $(a_n)_{n\in\mathbb{N}}$ fulfills the conditions of Remark 4.5, Corollary 4.8 holds also for $N = \infty$ and W_{N+1} replaced with W_∞ defined by (3.12). The case with $W_\infty = \infty$ is included as well.

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