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# NOTE ON EXTREME POINTS IN MARCINKIEWICZ FUNCTION SPACES

ANNA KAMIŃSKA[1*] AND ANCA M. PARRISH[1]

*Dedicated to Professor Lars-Erik Persson on his 65th birthday*

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Abstract. We show that the unit ball of the subspace $M_W^0$ of ordered continuous elements of $M_W$ has no extreme points, where $M_W$ is the Marcinkiewicz function space generated by a decreasing weight function $w$ over the interval $(0,\infty)$ and $W(t) = \int_0^t w$, $t \in (0,\infty)$. We also present here a proof of the fact that a function $f$ in the unit ball of $M_W$ is an extreme point if and only if $f^* = w$.

## 1. Introduction and preliminaries

In [9, 10], Ryff considered extreme points of the convex set $\Omega(w)$ of functions on $[0,1]$ that is an orbit of a given function $w$. An orbit of a decreasing weight function $w$ is in fact a unit ball of the Marcinkiewicz space $M_W$ corresponding to the weight $w$. Thus the Ryff's description can be applied directly to the characterization of extreme points of the unit ball of the Marcinkiewicz function space $M_W$ on the interval $[0,1]$. Further in [3], the analogous description has been given in the spaces of functions on the interval $(0,\infty)$. Here we consider the Marcinkiewicz spaces $M_W$ over $(0,\infty)$. We first show that the unit ball in the subspace $M_W^0$ of all ordered continuous elements of $M_W$ has no extreme points. Moreover we provide a detailed proof, different than that given in [3, 9, 10], of the fact that $f$ is an extreme point of the unit ball in $M_W$ if and only if $f^* = w$.

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* Corresponding author.

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Extreme and smooth points of the unit ball in Lorentz and Marcinkiewicz spaces have been a subject of investigation already for some time. The characterization of the extreme points in (Orlicz)-Lorentz function spaces has been done in [4], while the extreme and smooth points in both Lorentz and Marcinkiewicz sequence spaces corresponding to decreasing weight have been described in [6]. Smooth points in Marcinkiewicz function spaces over $(0,\infty)$ have been determined in [7].

We will start by agreeing on some notations. Let $L^0$ be the set of all real-valued $|\cdot|$-measurable functions defined on $(0,\infty)$, where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}$. By $\operatorname{supp} f$ we denote the support of $f$, i.e. $\{t : f(t) \neq 0\}$. Given $A \subset (0,\infty)$, denote by $A^c = (0,\infty) \setminus A$.

The *distribution function* $d_f$ of a function $f \in L^0$ is given by $d_f(\lambda) = |\{t > 0 : |f(t)| > \lambda\}|$, for all $\lambda \geq 0$. For $f \in L^0$ we define its *decreasing rearrangement* as $f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}$, $t > 0$. The functions $d_f$ and $f^*$ are right-continuous on $(0,\infty)$ (see [2, 8]).

Let $w : (0,\infty) \to (0,\infty)$ be the *weight function*, with $\lim_{t\to 0^+} w(t) = \infty$ and $\lim_{t\to\infty} w(t) = 0$. Denoting by $W(t) := \int_0^t w$, $t > 0$, we assume that $W(t) < \infty$ for all $t > 0$ and $W(\infty) = \infty$. We assume also here that $w$ is decreasing.

The *Marcinkiewicz space* $M_W$ [8, 5] is the space of all functions $f \in L^0$ satisfying

$$\|f\|_W = \sup_{t>0} \frac{\int_0^t f^*(s)ds}{W(t)} < \infty.$$

We also define the subspace

$$M_W^0 = \Big\{ f \in M_W : \lim_{t\to 0^+,\infty} \frac{\int_0^t f^*}{W(t)} = 0 \Big\}.$$

The space $M_W$ equipped with the norm $\|\cdot\|_W$ is a Banach function space. The set $M_W^0$ is a closed subspace of $M_W$ and it consists of all order continuous elements of $M_W$ which also coincides with the closure of all bounded functions of finite measure supports [8, 5].

Given a Banach space $(X, \|\cdot\|)$, we will denote by $S_X$ and $B_X$ respectively, the unit sphere and the unit ball of the space. An element $x \in S_X$ is an extreme point of the ball $B_X$ if $x = (x_1 + x_2)/2$ with $x_1, x_2 \in S_X$ implies that $x = x_1 = x_2$.

## 2. Main results

**Lemma 2.1.** *Let $f \in M_W^0$ with $\|f\|_W \leq 1$ or $f \in S_{M_W}$ with $|\operatorname{supp} f| < \infty$. Then there exists a measurable set $G$ with $|G^c| < \infty$ and there exists $\varepsilon > 0$ such that for all functions $g$ with $\|g\|_W \leq 1$, $\|g\|_\infty \leq 1$ and $\operatorname{supp} g \subset G$, we have that*

$$\|f + \lambda g\|_W \leq 1, \qquad \textit{for all } |\lambda| < \varepsilon.$$

*Proof.* The case when $f \in M_W^0$ with $\|f\|_W \leq 1$ is proved in [1, Proposition 2]. So we only discuss here the case when $f \in S_{M_W}$ with $|\operatorname{supp} f| < \infty$. Since $\|f\|_W = 1$, denote by

$$T := \sup\{t > 0 : \frac{\int_0^t f^*}{W(t)} = 1\}$$

and let $S := |\operatorname{supp} f|$. So by assumption $\operatorname{supp} f^* = [0, S)$ and $S < \infty$. Since $|\operatorname{supp} f| < \infty$ ,

$$\lim_{t\to\infty} \frac{\int_0^t f^*}{W(t)} = \lim_{t\to\infty} \frac{\int_0^S f^*}{W(t)} = 0,$$

so $0 \le T < \infty$.

If $T = 0$, then $f^*(T) = f^*(0) = \lim_{t\to 0^+} f^*(t) > 0$.

If $T > 0$, consider $0 < c < T$. Then

$$1 = \frac{\int_0^T f^*}{\int_0^T w} = \frac{\int_0^{T-c} f^*}{\int_0^{T-c} w} \frac{\int_0^{T-c} w}{\int_0^T w} + \frac{\int_{T-c}^T f^*}{\int_0^T w} \le \frac{\int_0^{T-c} w}{\int_0^T w} + \frac{\int_{T-c}^T f^*}{\int_0^T w}.$$

Therefore

$$\int_{T-c}^T w \le \int_{T-c}^T f^*,$$

and it follows that

$$1 \le \frac{\int_{T-c}^T f^*}{\int_{T-c}^T w} \le \frac{f^*(T-c)c}{w(T)c},$$

So we have that $0 < w(T) \le f^*(T-c)$, for all $0 < c < T$. Then

$$f^*_-(T) := \lim_{t\to T^-} f^*(t) \ge w(T) > 0.$$

**Case 1:** Let $T = S$. Then $f^*(T) = \lim_{t\to T^+} f^*(t) = 0$ and $T > 0$.
Let $t_1 > T$ and let

$$0 < \varepsilon < \min\left\{\frac{W(t_1) - W(T)}{t_1 - T}, f^*_-(T), 1 - a\right\},$$

where $0 < a < 1$ such that $\frac{\int_0^t f^*}{W(t)} \le a$, for all $t \ge t_1$. Let $G = (\operatorname{supp} f)^c$ and choose $g$ such that $\operatorname{supp} g \subset G$, $\|g\|_\infty \le 1$ and $\|g\|_W \le 1$.
If $0 < t \le T$, then by $\varepsilon < f^*_-(T)$ we have

$$\frac{\int_0^t (|f| + \varepsilon|g|)^*}{W(t)} = \frac{\int_0^t f^*}{W(t)} \le 1.$$

If $T < t \le t_1$, then from the choice of $\varepsilon$ we have that

$$\begin{aligned}\frac{\int_0^t (|f| + \varepsilon|g|)^*}{W(t)} &\le \frac{\int_0^t (|f| + \varepsilon\chi_{(0,\infty)\setminus \operatorname{supp} f^*})^*}{W(t)} = \frac{\int_0^t (f^* + \varepsilon\chi_{[T,\infty)})}{W(t)} \\ &\le \frac{\int_0^t f^*}{W(t)} + \varepsilon\frac{t-T}{W(t)} = \frac{\int_0^T f^*}{W(T)} \frac{W(T)}{W(t)} + \varepsilon\frac{t-T}{W(t)} \\ &\le \frac{W(T)}{W(t)} + \frac{W(t) - W(T)}{t - T} \frac{t-T}{W(t)} = 1.\end{aligned}$$

If $t > t_1$, then by the subadditivity of the functional $f \mapsto \int_0^t f^*$ and the choice of $\varepsilon$,

$$\frac{\int_0^t (|f| + \varepsilon|g|)^*}{W(t)} \le \frac{\int_0^t f^*}{W(t)} + \varepsilon\frac{\int_0^t g^*}{W(t)} \le 1.$$

So we have that $\|f + \varepsilon g\|_W \le 1$.

**Case 2:** Assume now that $0 \le T < S$ .
If $T = 0$, $\sup_{t>0} \frac{\int_0^t f^*}{W(t)} = \overline{\lim}_{t\to 0^+} \frac{\int_0^t f^*}{W(t)} = 1$ and for all $t \in (0, S)$, $\frac{\int_0^t f^*}{W(t)} < 1$. Since $T < S$, we have that $f^*(T) > 0$, so there exists $t_1 > T$ such that $f^*(t_1) > 0$. Let

$$0 < \varepsilon < \min\left\{\frac{f^*(t_1)}{2}, 1 - a\right\},$$

where $0 < a < 1$ is such that $\frac{\int_0^t f^*}{W(t)} \le a$, for all $t > t_1$. Let $G = \{t > 0 : |f(t)| < \frac{f^*(t_1)}{2} - \varepsilon\}$. So

$$|G^c| = |\{t > 0 : |f(t)| \ge \frac{f^*(t_1)}{2} - \varepsilon\}| < \infty.$$

Let $g$ be such that $\operatorname{supp} g \subset G$, $\|g\|_\infty \le 1$, $\|g\|_W \le 1$. If $0 < t \le t_1$, then

$$\frac{\int_0^t (|f| + \varepsilon|g|)^*}{W(t)} = \frac{\int_0^t f^*}{W(t)} \le 1.$$

If $t > t_1$, then

$$\frac{\int_0^t (|f| + \varepsilon|g|)^*}{W(t)} \le \frac{\int_0^t f^*}{W(t)} + \varepsilon\frac{\int_0^t g^*}{W(t)} \le 1,$$

so again we have that $\|f + \varepsilon g\|_W \le 1$. $\square$

Let $f \in S_{M_W^0}$ or $f \in S_{M_W}$ with $|\operatorname{supp} f| < \infty$. Choose $g$ satisfying the assumptions of Lemma 2.1 and let $h_1 = f + \varepsilon g$ and $h_2 = f - \varepsilon g$. Then $h_1 \ne h_2$, $\|h_1\|_W = \|h_2\|_W \le 1$ and $f = \frac{h_1+h_2}{2}$, so $f$ is not an extreme point. So we have proved the following corollaries.

**Corollary 2.2.** *The unit sphere in $M_W^0$ has no extreme points.*

**Corollary 2.3.** *If $f \in S_{M_W}$ and $|\operatorname{supp} f| < \infty$, then $f$ is not an extreme point.*

The next lemma is well know [3, 9], but for the sake of completeness we provide its proof here.

**Lemma 2.4.** *Let $\|f\|_W = 1$ and $\operatorname{supp} f^* = (0, \infty)$. If $f$ is an extreme point of $M_W$, then $f^*$ is also an extreme point.*

*Proof.* By [2, Corollary 7.6], there exists $\tau : \operatorname{supp} f \to \operatorname{supp} f^*$, a measure preserving transformation from $\operatorname{supp} f$ onto $\operatorname{supp} f^*$, such that for all $t > 0$, $|f(t)| = f^*(\tau(t))$.
Assume that $f^*(t) = \frac{h(t)+g(t)}{2}$, for all $t > 0$, $g \ne h$ and $\|g\|_W = \|h\|_W = \|f\|_W = 1$. Then

$$f(t) = \frac{\operatorname{sign} f(t) g(\tau(t)) + \operatorname{sign} f(t) h(\tau(t))}{2}.$$

Define

$$\bar{g}(t) = \begin{cases} \operatorname{sign} f(t) g(\tau(t)), & \text{if } t \in \operatorname{supp} f; \\ 0, & \text{if } t \notin \operatorname{supp} f, \end{cases}$$

and

$$\bar{h}(t) = \begin{cases} \operatorname{sign} f(t) h(\tau(t)), & \text{if } t \in \operatorname{supp} f; \\ 0, & \text{if } t \notin \operatorname{supp} f. \end{cases}$$

Since $\tau$ is a measure preserving transformation, $\bar{g}$ and $\bar{h}$ are equimeasurable to $g$ and $h$ respectively, so $\|\bar{g}\|_W = \|\bar{h}\|_W = \|g\|_W = \|h\|_W = 1$. Since the range of $\tau$ is $(0,\infty)$ and $g \neq h$, there exists $A \subset (0,\infty)$ such that $|A| > 0$ and

$$\bar{g}(t) = g(\tau(t)) \neq h(\tau(t)) = \bar{h}(t), \qquad \text{for all } t \in A,$$

therefore $\bar{g} \neq \bar{h}$ and $f(t) = \frac{\bar{g}(t)+\bar{h}(t)}{2}$, for all $t > 0$, which is a contradiction. □

**Theorem 2.5.** *If $f \in M_W$ is such that $f^* = w$, then $f$ is an extreme point of $B_{M_W}$.*

*Proof.* Let $f = \frac{g+h}{2}$, where $\|g\|_W = \|h\|_W = 1$, and $f^* = w$. Then $f^* = w = \left(\frac{g+h}{2}\right)^*$ and for all $s > 0$,

$$1 = \frac{\int_0^s w}{W(s)} = \frac{\int_0^s f^*}{W(s)} = \frac{\int_0^s \left(\frac{g+h}{2}\right)^*}{W(s)} \leq \frac{1}{2}\frac{\int_0^s g^*}{W(s)} + \frac{1}{2}\frac{\int_0^s h^*}{W(s)} \leq 1.$$

So for all $s > 0$,

$$\int_0^s w = \int_0^s \left(\frac{g}{2}\right)^* + \int_0^s \left(\frac{h}{2}\right)^*,$$

and it follows that $w = \frac{g^*+h^*}{2}$ a.e. Since $w$ is decreasing, we can assume it is right-continuous. The functions $g^*$ and $h^*$ are right-continuous.

**Claim 1:** We wish to show that $w = g^* = h^*$.

Assume that $w(t) \neq h^*(t)$. Then there exists an interval $(a,b) \subset (0,\infty)$ such that either $w(t) > h^*(t)$ on $(a,b)$ or $w(t) < h^*(t)$ on $(a,b)$. Let's assume first that $a = 0$. If $w(t) > h^*(t)$, for all $t \in (0,b)$, then $g^*(t) = 2w(t) - h^*(t) > 2w(t) - w(t) = w(t)$ and in this case,

$$\frac{\int_0^t g^*}{W(t)} > \frac{\int_0^t w}{W(t)} = 1,$$

which is a contradiction. If $w(t) < h^*(t)$, for all $t \in (0,b)$, then again a contradiction since

$$\frac{\int_0^t h^*}{W(t)} > \frac{\int_0^t w}{W(t)} = 1.$$

Assume now that there exist $b > a > 0$ such that $w(t) = h^*(t) = g^*(t)$, for all $t \in (0,a)$ and for all $t \in (a,b)$, $w(t) > h^*(t)$ or $w(t) < h^*(t)$. If $w(t) > h^*(t)$ for all $t \in (a,b)$, then $g^*(t) > w(t)$, for all $t \in (a,b)$ and so

$$1 \geq \frac{\int_0^a g^*}{W(t)} + \frac{\int_a^t g^*}{W(t)} = \frac{\int_0^a w}{W(a)}\frac{W(a)}{W(t)} + \frac{\int_a^t g^*}{W(t)} > \frac{W(a)}{W(t)} + \frac{\int_a^t w}{W(t)} = 1,$$

a contradiction. Similarly we get a contradiction if $w(t) < h^*(t)$ for all $t \in (a,b)$, since then

$$1 \geq \frac{\int_0^t h^*}{W(t)} = \frac{\int_0^a h^*}{W(t)} + \frac{\int_a^t h^*}{W(t)} > \frac{W(a)}{W(t)} + \frac{\int_a^t w}{W(t)} = 1.$$

So we have shown that $w = g^* = h^*$ a.e.

**Claim 2:** It holds that $g = h$ a.e.

By the assumption and claim 1,

$$g^* = h^* = w = f^* = \left(\frac{g+h}{2}\right)^* = \frac{g^* + h^*}{2}.$$

By [8, 7,9 page 64], $\left(\frac{g+h}{2}\right)^* = \frac{g^*+h^*}{2}$ if and only if $|h(t) + g(t)| = |h(t)| + |g(t)|$ a.e. and furthermore, for all $s > 0$, there exists a measurable set $e(s)$ such that $|e(s)| = s$ and

$$\int_{e(s)} \left|\frac{g+h}{2}\right| = \int_{e(s)} |g| = \int_{e(s)} |h| = \int_0^s g^* = \int_0^s h^* = \int_0^s w.$$

So in particular, for all $s > 0$,

$$\int_{e(s)} (|g| - |h|) = 0$$

and so $|g| = |h|$ a.e. Since $|h(t) + g(t)| = |h(t)| + |g(t)|$ a.e., we have that $\operatorname{sign} g = \operatorname{sign} h$ a.e., so $h = g$ a.e. and $f$ is an extreme point.

□

The following theorem is the converse of Theorem 2.5.

**Theorem 2.6.** *If $w$ is strictly decreasing and $f \in S_{M_W}$ is an extreme point of $B_{M_W}$, then $f^* = w$.*

*Proof.* Let $\|f\|_W = 1$ and $f^* \neq w$. In view of Lemma 2.4, it is enough to show that $f^*$ is not an extreme point. We can assume without loss of generality that $w$ is right-continuous. Also, we can assume that $|\operatorname{supp} f^*| = \infty$, since otherwise by Corollary 2.3, $f$ is not an extreme point.

We claim first that there exists an interval $[a, b]$ such that

$$K := \inf_{t\in[a,b]} (w(t) - f^*(t)) > 0 \qquad \text{and} \qquad f^*(b) > 0. \tag{2.1}$$

Suppose that $f^*(s) > w(s)$ for every $s > 0$. Then

$$W(t) \geq \int_0^t f^* \geq \int_0^t w = W(t), \qquad \text{for all } t > 0,$$

so $f^* = w$, which is a contradiction, since $f^* \neq w$. Thus there exists $a_0 > 0$ such that $f^*(a_0) < w(a_0)$. By right-continuity of $w$ there exists $\delta > 0$ such that for all $t \in [a_0, a_0 + \delta]$

$$w(a_0) - w(t) < \frac{w(a_0) - f^*(a_0)}{2}.$$

Hence for all $t \in [a_0, a_0 + \delta]$

$$w(t) - f^*(t) > -\frac{w(a_0) - f^*(a_0)}{2} + w(a_0) - f^*(a_0) + f^*(a_0) - f^*(t) \geq \frac{w(a_0) - f^*(a_0)}{2}.$$

Setting $a = a_0$ and $b = a_0 + \delta$, we have (2.1).

**Case 1:** Assume now that there exists $c \in (a, b)$ such that

$$f^*_+(a) > f^*(c) > f^*_-(b). \tag{2.2}$$

Let $\varepsilon > 0$ be such that

$$0 < \varepsilon < \frac{1}{2}\min\{f^*_+(a) - f^*(c), f^*(c) - f^*_-(b)\}, \qquad \varepsilon < \frac{K}{2}. \tag{2.3}$$

Let

$$a_1 = \inf\{t : f^*(t) \le f^*(c) + \varepsilon\}, \qquad a_2 = \inf\{t : f^*(t) \le f^*(c) + 2\varepsilon\},$$

$$b_1 = \sup\{t : f^*(t) \ge f^*(c) - \varepsilon\}, \qquad b_2 = \sup\{t : f^*(t) \ge f^*(c) - 2\varepsilon\}.$$

By conditions (2.2) and (2.3) and right-continuity of $f^*$ we have $a < a_2 \le a_1 \le c < b_1 \le b_2 < b$. Let $\alpha = \frac{a_1+b_1}{2}$, and define

$$g = f^*\chi_{(0,a_1)} + (f^* + \varepsilon)\chi_{(a_1,\alpha)} + (f^* - \varepsilon)\chi_{(\alpha,b_1)} + f^*\chi_{(b_1,\infty)},$$

and

$$h = f^*\chi_{(0,a_1)} + (f^* - \varepsilon)\chi_{(a_1,\alpha)} + (f^* + \varepsilon)\chi_{(\alpha,b_1)} + f^*\chi_{(b_1,\infty)}.$$

Then $f^* = \frac{g+h}{2}$. Now we have

$$g^* = f^*\chi_{(0,a_2)} + g^*\chi_{(a_2,\alpha)} + g^*\chi_{(\alpha,b_2)} + f^*\chi_{(b_2,\infty)},$$

where $g^*\chi_{(a_2,\alpha)}$ is equimeasurable to $f^*\chi_{(a_2,a_1)} + (f^* + \varepsilon)\chi_{(a_1,\alpha)}$ and $g^*\chi_{(\alpha,b_2)}$ is equimeasurable to $(f^* - \varepsilon)\chi_{(\alpha,b_1)} + f^*\chi_{(b_1,b_2)}$. It follows that

$$\int_{a_2}^{\alpha} g^* = \int_{a_2}^{\alpha} f^* + \varepsilon(\alpha - a_1) \quad \text{and} \quad \int_{\alpha}^{b_2} g^* = \int_{\alpha}^{b_2} f^* - \varepsilon(b_1 - \alpha).$$

Hence

$$\int_a^b g^* = \int_a^{a_2} f^* + \int_{a_2}^{\alpha} f^* + \varepsilon(\alpha - a_1) + \int_{\alpha}^{b_2} f^* - \varepsilon(b_1 - \alpha) + \int_{b_2}^{b} f^* = \int_a^b f^*. \tag{2.4}$$

Moreover, by (2.1) and (2.3) we have that

$$g^*(t) \le w(t), \qquad \text{for all } t \in (a,b). \tag{2.5}$$

If $t \in (0,a)$, then $\int_0^t g^* = \int_0^t f^* \le W(t)$. If $t \in (a,b)$ or $t \in (b,\infty)$, then by (2.4) and (2.5), $\int_0^t g^* \le W(t)$. So $\|g\|_W \le 1$, and similarly, $\|h\|_W \le 1$.

**Case 2:** Suppose now that (2.2) is not satisfied. Then

$$f^*_+(a) = f^*(c) \qquad \text{or} \qquad f^*_-(b) = f^*(c), \qquad \text{for all } c \in (a,b).$$

We shall consider several cases.

**(a)** Let $f^*_+(a) > f^*_-(b)$ and there exists $c \in (a,b)$ such that

$$f^*\chi_{(a,c)} = f^*_+(a) \qquad \text{and} \qquad f^*\chi_{(c,b)} = f^*_-(b). \tag{2.6}$$

Let

$$d = \inf\{t : f^*(t) = f^*(a)\}.$$

Since $w$ is decreasing, we will also have similarly to (2.1) that

$$\inf_{t\in[d,b]} (w(t) - f^*(t)) \ge K > 0.$$

**(a-i)** If $f^*$ is not continuous at $d > 0$ or $d = 0$, then pick up $0 < \varepsilon < \min\{f^*_-(d) - f^*_+(d), f^*_+(d) - f^*_-(b), K\}$, for $d > 0$, and $0 < \varepsilon < \min\{f^*_+(d) - f^*_-(b), K\}$, for $d = 0$. Let

$$g = f^*\chi_{(0,d)\cup(c,\infty)} + (f^* + \varepsilon)\chi_{(d,\frac{d+c}{2})} + (f^* - \varepsilon)\chi_{(\frac{d+c}{2},d)},$$

and

$$h = f^*\chi_{(0,d)\cup(c,\infty)} + (f^* - \varepsilon)\chi_{(d,\frac{d+c}{2})} + (f^* + \varepsilon)\chi_{(\frac{d+c}{2},c)}.$$

Then $f^* = \frac{g+h}{2}$ and $g^* = g = h^*$. Since $g^*(t) \le w(t)$ for all $t \in (d,c)$, $g^*(t) = f^*(t)$, for all $t \notin (d,c)$ and $\int_d^c g^* = \int_d^c f^*$, we can prove that $\int_0^t g^* \le W(t)$, for all $t \ge 0$, so $\|g\|_W \le 1$ and $\|h\|_W \le 1$.

**(a-ii)** If $d > 0$ and $f^*$ is continuous at $d$, then for all $0 < t < d$, $f^*(t) > f^*_+(d) = f^*_-(d)$ and by continuity of $f^*$ at $d$, we shall find $0 < d_2 < d_1 < d$ such that

$$\inf_{t\in[d_2,d]}(w(t) - f^*(t)) > 0$$

and

$$f^*_+(d_2) > f^*(d_1) > f^*_-(d) = f^*(d).$$

However, this is the situation (2.2), so we are done.

**(b)** Now let the alternative to (2.6) be satisfied, that is

$$f^*(t) = f^*(a) = f^*_+(a), \qquad \text{for all } t \in [a,b)$$

**(b-i)** If $d > 0$ and $f^*$ is continuous at $d$, then we have the situation (a-ii).

**(b-ii)** Assume that either $d = 0$ or $f^*$ is discontinuous at $d > 0$. Let

$$b_0 = \sup\{t : f^*(t) = f^*(a)\}.$$

Let $d < t < b_0$. Then, for $d > 0$,

$$F(t) = \frac{\int_0^t f^*}{W(t)} = \frac{\left(\frac{\int_0^d f^*}{W(d)}\right)W(d) + f^*(d)(t-d)}{W(t)} = \frac{\lambda W(d) + f^*(d)(t-d)}{W(t)},$$

where $0 < \lambda = \frac{\int_0^d f^*}{W(d)} \le 1$, and for $d = 0$,

$$F(t) = \frac{f^*(d)(t-d)}{W(t)},$$

that is the same formula as for $d > 0$ if we agree that $\lambda > 0$ and $W(0) = 0$. We have for all $d < t < b_0$ that $F(t) \le 1$, so

$$\frac{t-d}{W(t) - \lambda W(d)} f^*(d) \le 1.$$

Let, for $t \in (d, b_0)$,

$$H(t) = \frac{t-d}{W(t) - \lambda W(d)}.$$

For $t \in (d, b_0)$, since $w$ is strictly decreasing, we have

$$W(t) - \lambda W(d) \ge \int_d^t w > (t-d)w(t).$$

Hence for $t \in (d, b_0)$,

$$H'(t) = \frac{W(t) - \lambda W(d) - (t-d)w(t)}{(W(t) - \lambda W(d))^2} > 0,$$

thus H is strictly increasing on $(d, b_0)$. $H(t)$ is strictly increasing on $(d, b_0)$ and $H(b_0)f^*(d) \leq 1$, so $H(t)f^*(d) < 1$ for all $t \in (d, b_0)$, which is equivalent to $F(t) < 1$ on $(d, b_0)$.

- Assume first that $f^*$ is discontinuous at $b_0$. We first observe that $K = \max_{t \in [d+\delta, b_0-\delta]} F(t) < 1$ for any $\delta > 0$. We first choose $\delta > 0$ and $\varepsilon_1 > 0$ such that $H(d+\delta)(f^*(d) + \varepsilon_1) < 1$. With the same $\delta$ we then choose $0 < \varepsilon \leq \varepsilon_1$ such that

$$\varepsilon < f^*_-(d) - f^*_+(d) \quad \text{(only in the case } d = 0), \quad \varepsilon < f^*_+(b_0) - f^*_-(b_0), \tag{2.7}$$

$$H(d+\delta)(f^*(d) + \varepsilon) \leq 1, \tag{2.8}$$

$$K + \frac{\varepsilon\delta}{W(d+\delta)} \leq 1, \tag{2.9}$$

$$K + \frac{\varepsilon\delta}{W(b_0-\delta)} \leq 1. \tag{2.10}$$

Define

$$g = f^*\chi_{(0,d)} + (f^* + \varepsilon)\chi_{(d,d+\delta)} + (f^* - \varepsilon)\chi_{(d+\delta,d+2\delta)} + f^*\chi_{(d+2\delta,\infty)}, \tag{2.11}$$

and

$$h = f^*\chi_{(0,d)} + (f^* - \varepsilon)\chi_{(d,d+\delta)} + (f^* + \varepsilon)\chi_{(d+\delta,d+2\delta)} + f^*\chi_{(d+2\delta,\infty)}, \tag{2.12}$$

Then by (2.7)

$$g^* = f^*\chi_{(0,d)} + (f^* + \varepsilon)\chi_{(d,d+\delta)} + f^*\chi_{(d+\delta,b_0-\delta)} + (f^* - \varepsilon)\chi_{(b_0-\delta,b_0)} + f^*\chi_{(b_0,\infty)}.$$

For $t \in (0, d)$, $\int_0^t g^* = \int_0^t f^* \leq W(t)$.
For $t \in (d, d+\delta)$,

$$\frac{\int_0^t g^*}{W(t)} = \frac{\int_0^d f^* + (f^*(d) + \varepsilon)(t-d)}{W(t)} = \frac{\lambda W(d) + (f^*(d) + \varepsilon)(t-d)}{W(t)} \leq 1$$

if and only if

$$\frac{t-d}{W(t) - \lambda W(d)}(f^*(d) + \varepsilon) = H(t)(f^*(d) + \varepsilon) \leq 1.$$

Since by (2.8), for all $t \in (d, d+\delta)$,

$$H(t)(f^*(d) + \varepsilon) < H(d+\delta)(f^*(d) + \varepsilon) < 1,$$

we have that $\int_0^t g^* \leq W(t)$.
For $t \in (d+\delta, b_0-\delta)$, by (2.9) we have

$$\frac{\int_0^t g^*}{W(t)} = \frac{\int_0^t f^*}{W(t)} + \frac{\varepsilon\delta}{W(t)} \leq K + \frac{\varepsilon\delta}{W(d+\delta)} \leq 1.$$

Now let $t \in (b_0 - \delta, b_0)$. Then

$$\frac{\int_0^t g^*}{W(t)} = \frac{\int_0^{b_0} f^* - \int_t^{b_0} f^* + \varepsilon(b_0 - t)}{W(t)} = \frac{\beta W(b_0) - (f^*(d) - \varepsilon)(b_0 - t)}{W(t)},$$

where $\beta = \frac{\int_0^{b_0} f^*}{W(b_0)} \leq 1$. We have that for $t \in (b_0 - \delta, b_0)$,

$$\frac{\int_0^t g^*}{W(t)} \leq 1 \quad \text{if and only if} \quad \frac{S(t)}{f^*(d) - \varepsilon} \leq 1, \tag{2.13}$$

where

$$S(t) = \frac{\beta W(b_0) - W(t)}{b_0 - t}.$$

We have that $S$ is strictly decreasing for $t < b_0$. Indeed, since $w$ is strictly decreasing,

$$\beta W(b_0) - W(t) \leq W(b_0) - W(t) < w(t)(b_0 - t).$$

Hence $S'(t) < 0$ for $t < b_0$, and so $S(t)$ is strictly decreasing. Hence, in order to show that $\int_0^t g^* \leq W(t)$ for $t \in (b_0 - \delta, b_0)$, it is enough to prove that

$$S(b_0 - \delta)\frac{1}{f^*(d) - \varepsilon} \leq 1.$$

This is equivalent to

$$\int_0^d f^* + f^*(d)(b_0 - d) - \delta(f^*(d) - \varepsilon) \leq W(b_0 - \delta),$$

that is

$$\frac{\int_0^{b_0-\delta} f^*}{W(b_0 - \delta)} + \frac{\varepsilon\delta}{W(b_0 - \delta)} \leq 1.$$

But by (2.10),

$$F(b_0 - \delta) + \frac{\varepsilon\delta}{W(b_0 - \delta)} \leq K + \frac{\varepsilon\delta}{W(b_0 - \delta)} \leq 1,$$

which implies that for all $t \in (b_0 - \delta, b_0)$,

$$\frac{S(t)}{f^*(d) - \varepsilon} \leq \frac{S(b_0 - \delta)}{f^*(d) - \varepsilon} \leq 1,$$

which in turn yields that $\int_0^t g^* \leq W(t)$ by (2.13).
If $t \in (b_0, \infty)$, then

$$\int_0^t g^* = \int_0^t f^* \leq W(t).$$

So $\|g\|_W = \|h\|_W \leq 1$, $f^* = \frac{g+h}{2}$, and $f^*$ cannot be extreme point.

- Let $f^*$ be continuous at $b_0$ and let $F(b_0) = \frac{\int_0^{b_0} f^*}{W(b_0)} = 1$. Hence we have that for all $\varepsilon > 0$, there exists $b_0 < t < b_0 + \varepsilon$ such that $f^*(t) \leq w(t)$. Indeed, if not, then there exists $\varepsilon > 0$ such that $f^*(t) > w(t)$, for all $t \in (b_0, b_0+\varepsilon)$. Let $t \in (b_0, b_0+\varepsilon)$, then

$$\frac{\int_0^t f^*}{W(t)} = \frac{W(b_0) + \int_{b_0}^t f^*}{W(t)} > \frac{W(b_0) + \int_{b_0}^t w}{W(t)} = 1,$$

which is a contradiction. Hence $f^*(a) = f^*_+(b_0) = \lim_{t\to b_0^+} f^*(t) \le w(b_0) \le w(t)$ for all $t \le b_0$.
Let $0 < \delta < \frac{b_0-d}{4}$. Since $w$ is strictly decreasing, $w(d+\delta) - f^*(d) > 0$. Let

$$0 < \varepsilon < \min\{f^*_-(d) - f^*_+(d), f^*(d), w(d+\delta) - f^*(d)\},$$

where the first term inside the minimum does not exist if $d = 0$.
Let $\varepsilon$ be also chosen such that there exists $\delta_1 < \delta$ such that $\delta_1 = \min\{\delta_2, \delta_3\} > 0$, where

$$\delta_2 = \max (f^*)^{-1}[(f^*(d) - \varepsilon, f^*(d))] \quad \text{and} \quad \delta_3 = \max w^{-1}[(f^*(d) - \varepsilon, f^*(d))].$$

Let $g$ and $h$ be given by (2.11) and (2.12). Then $\frac{g+h}{2} = f^*$ and $g^* = h^*$, where in this case

$$\begin{aligned} g^*(t) &= f^*\chi_{(0,d)}(t) + (f^*+\varepsilon)\chi_{(d,d+\delta)}(t) + f^*\chi_{(d+\delta,b_0-\delta)}(t) \\ &\quad + f^*\chi_{(b_0,b_0+\delta_1)}(t+\delta) + (f^*(d)-\varepsilon)\chi_{(b_0-\delta+\delta_1,b_0+\delta_1)}(t) + f^*\chi_{(b_0+\delta_1,\infty)}(t). \end{aligned} \tag{2.14}$$

If $t \in (0,d)$, then $\int_0^t g^* = \int_0^t f^* \le W(t)$.
If $t \in (d, b_0+\delta_1)$, then $g^*(t) \le w(t)$, so $\int_0^t g^* \le \int_0^t w = W(t)$.
If $t \in (b_0+\delta_1, \infty)$, first compute

$$\begin{aligned} \int_d^{b_0+\delta_1} g^* &= \int_d^{d+\delta} f^* + \varepsilon\delta + \int_{d+\delta}^{b_0-\delta} f^* + \int_{b_0}^{b_0+\delta_1} f^* + \int_{b_0-\delta+\delta_1}^{b_0+\delta_1} f^*(d) - \varepsilon\delta \\ &= \int_d^{d+\delta} f^* + \int_{d+\delta}^{b_0-\delta} f^* + \int_{b_0}^{b_0+\delta_1} f^* + \int_{b_0-\delta}^{b_0} f^* = \int_d^{b_0+\delta_1} f^*. \end{aligned} \tag{2.15}$$

Then $\int_0^t g^* = \int_0^t f^* \le W(t)$, so $\|g\|_W \le 1$, and similarly $\|h\|_W \le 1$.

• Let $f^*$ be continuous at $b_0$ and let $F(b_0) < 1$. Since $F(b_0) < 1$, so $H(b_0)f^*(d) < 1$ and by the fact that $H$ is strictly increasing on $(d, b_0)$ we get that $H(t)f^*(d) < 1$ on $(d, b_0)$.
Hence $F(t) < 1$ on $(d, b_0)$, and by $F(b_0) < 1$ and continuity of $F$ we find $0 < \delta < \frac{b_0-d}{4}$, such that $L := \max_{t\in(d+\delta,b_0+\delta)} F(t) < 1$. Pick up then $\varepsilon > 0$ satisfying the following conditions

$$\varepsilon < f^*_-(d) - f^*_+(d) \quad (\text{only if } d = 0)$$

$$H(b_0)(f^*(d) + \varepsilon) \le 1, \tag{2.16}$$

$$L + \frac{\varepsilon\delta}{W(d+\delta)} \le 1. \tag{2.17}$$

Finally let $0 < \delta_1 < \delta$ be such that

$$(b_0, b_0+\delta_1) \subset (f^*)^{-1}[(f^*(a) - \varepsilon, f^*(a))].$$

Then define $g$ and $h$ as in (2.11) and (2.12), and so $g^* = h^*$ have the form (2.14). Clearly if $t \in (0, d)$, then $\int_0^t g^* = \int_0^t f^* \le W(t)$.
If $t \in (d, d+\delta)$, then by (2.16), $H(t)(f^*(d) + \varepsilon) \le 1$ for all $t \in (d, d+\delta)$. But it implies that

$$\frac{\int_0^t g^*}{W(t)} = \frac{\int_0^t f^* + \varepsilon(t-d)}{W(t)} \le 1.$$

For $t \in (d+\delta, b_0+\delta_1)$, $g^*(t) \le f^*(t)$ and by (2.17)

$$\frac{\int_0^t g^*}{W(t)} \le \frac{\int_0^{d+\delta} f^* + \int_{d+\delta}^t f^* + \varepsilon\delta}{W(t)} \le L + \frac{\varepsilon\delta}{W(d+\delta)} < 1.$$

Now let $t \in (b_0+\delta_1, \infty)$. Since $g^*$ has exactly the same form as in the previous case, we have by (2.15) that $\int_0^t g^* = \int_0^t f^* \le W(t)$. This completes the proof. $\square$

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[1] Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA.

*E-mail address*: kaminska@memphis.edu

*E-mail address*: abuican1@memphis.edu