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EXPONENTIAL ANALYSIS OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED TERMS

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Abstract. Non-negative definite Lyapunov functionals are employed to obtain sufficient conditions that guarantee boundedness of solutions of system of functional differential equations with unbounded terms. The theory is illustrated with several examples regarding Volterra integro-differential equations.

1. Introduction

In this paper, we make use of non-negative definite Lyapunov functionals and obtain sufficient conditions that guarantee the boundedness of all solutions of the system of functional differential equations with unbounded terms, of the form

$$x'(t) = G(t, x(s);\ 0 \le s \le t) \stackrel{def}{=} G(t, x(\cdot)) \tag{1.1}$$

where $x \in \mathbb{R}^n$, $G : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear continuous function in t and x. For a vector $x \in \mathbb{R}^n$ we take $|x|$ to be the Euclidean norm of x. Let $t_0 \ge 0$, then for each continuous function $\phi : [0, t_0] \to \mathbb{R}^n$, there is at least one continuous function $x(t) = x(t, t_0, \phi)$ on an interval $[t_0, I]$ satisfying (1.1) for $t_0 \le t \le I$ and such that $x(t, t_0, \phi) = \phi(t)$ for $0 \le t_0 \le I$. It is assumed that the right hand derivative, $x'(t)$ of $x(t)$ exist at $t = t_0$. For conditions ensuring existence, uniqueness and continuability of solutions of (1.1) we refer the reader to [3] .

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A stereotype of equation (1.1) is the Volterra integro-differential equation

$$x'(t) = h(x(t)) + \int_0^t B(t,s)f(x(s))ds + g(t). \tag{1.2}$$

We are mainly interested in applying our results to Volterra integro-differential equations of the forms (1.2) with $f(x) = x^n$ where n is positive and rational. Most importantly, we are interested in proving boundedness of solutions of equations of the form (1.2) when $g(t)$ is unbounded.

For application, we will apply our obtained results to nonlinear Volterra integro-differential equations. At the end of the paper we will compare our theorems to those obtained in [8] and [9] and show that our results are different when it comes to applications. For more on the boundedness and stability of solutions of (1.1), we refer the interested reader to [1, 2, 4, 5, 6, 7, 10].

Let D be unbounded subset of $\mathbb{R}^n$. For motivational purpose, suppose there exists a continuously differentiable Lyapunov functional $V : \mathbb{R}^+ \times D \to \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of non-negative real numbers, that satisfies, along the solutions of (1.2)

$$W_1(|x|) \le V(t, x(\cdot)) \le W_2(|x|) + \int_0^t \varphi_1(t,s)W_3(|x(s)|)ds \tag{1.3}$$

and

$$V'(t, x(\cdot)) \le -\eta(t)V(t, x(\cdot)) + F(t). \tag{1.4}$$

Here the function $F : [0,t] \to \mathbb{R}$ is continuous and $W_i : [0,\infty) \to [0,\infty)$ are continuous in x with $W_i(0) = 0$, $W_i(s) > 0$ if $s > 0$ and W_i is strictly increasing. Such a function W_i is called a wedge. (In this paper wedges are always denoted by W or W_i, where i is a positive integer). The function η is continuous and non-negative. Let $t_0 \ge 0$, then for each continuous function $\phi : [0, t_0] \to \mathbb{R}^n$, there is at least one continuous function $x(t) = x(t, t_0, \phi)$ on an interval $[t_0, I]$ satisfying (1.2) for $t_0 \le t \le I$ and such that $x(t, t_0, \phi) = \phi(t)$ for $0 \le t \le t_0$. From (1.4) one obtains the variational of parameters formula

$$V(t, x(\cdot)) \le \Big[V(t_0, \phi)e^{-\int_{t_0}^t \eta(u)du} + \int_{t_0}^t |F(s)|e^{-\int_s^t \eta(u)du}ds\Big]. \tag{1.5}$$

Let $||\cdot||$ denote the supremum norm. To relate V back to the solution x we use the left hand side of (1.3) to obtain

$$||x|| \le W^{-1}\Big[V(t_0, |\phi(t)|)e^{-\int_{t_0}^t \eta(u)du} + \int_{t_0}^t |F(s)|e^{-\int_s^t \eta(u)du}ds\Big].$$

Thus, if

$$V(t_0, |\phi(t)|)e^{-\int_{t_0}^t \eta(u)du} + \int_{t_0}^t |F(s)|e^{-\int_s^t \eta(u)du}ds\Big] \le K,$$

for some positive constant K, then (1.5) yields that all solutions of (1.2) are bounded.

The variational of parameters formula (1.5) was easily obtained due to the nature of (1.4). However, finding a Lyapunov functional V such that (1.4) is satisfied is extremely difficult, if not impossible, in some cases.

This research is mainly concerned with the following two issues.
1) We will prove a theorem in which condition (1.4) is replaced with

$$V'(t,x(\cdot)) \leq -\eta(t)V^q(t,x(\cdot)) + F(t),$$

where $F(t)$ may be unbounded and $q > 1$.
2) We will prove a theorem in which we offer a systematic approach to the construction of such a Lyapunov functional that satisfies (1.4).

2. Boundedness of Solutions

In this section we use non-negative definite Lyapunov type functionals and establish sufficient conditions to obtain boundedness results on all solutions $x(t)$ of (1.1). For $t_0 \geq 0$, we let $\phi : [0,t_0] \to \mathbb{R}^n$ be continuous, we define $\|\phi\| = \sup\{|\phi(s)| : 0 \leq s \leq t_0\}$.

Definition 2.1. We say that solutions of system (1.1) are bounded, if any solution $x(t,t_0,\phi)$ of (1.1) satisfies

$$||x(t,t_0,\phi)|| \leq C\Big(|\phi|, t_0\Big), \quad \text{for all } t \geq t_0,$$

where $C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a constant that depends on t_0 and ϕ is a given continuous and bounded initial function. We say that solutions of system (1.1) are uniformly bounded if C is independent of t_0.

If $x(t)$ is any solution of system (1.1), then for a continuously differentiable function

$$V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+,$$

we define the derivative V' of V by

$$V'(t,x) = \frac{\partial V(t,x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t,x)}{\partial x_i} f_i(t,x).$$

A continuous function $W : [0,\infty) \to [0,\infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$ and W strictly increasing is called a wedge. (In this paper wedges are always defined by W or W_i where i is a positive integer).

Theorem 2.2. *Let $q \geq 1$ and suppose there exists a continuously differentiable Lyapunov functional $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ that satisfies*

$$W_1(||x||) \leq V(t,x),\ V(t,x) \neq 0 \textit{ if } x \neq 0, \tag{2.1}$$

$$V'(t,x) \leq -\alpha(t)V^q(t,x) + F(t) \tag{2.2}$$

and

$$V(t,x) - V^q(t,x) \leq \gamma$$

for some nonnegative constant γ where $\alpha(t)$ and $F(t)$ are positive continuous functions. Then all solutions $x(t)$ of (1.1) satisfy

$$||x|| \leq W^{-1}\Big[V(t_0,\|\phi\|)e^{-\int_{t_0}^{t}\alpha(u)du} + \int_{t_0}^{t}\big[\gamma\alpha(u) + F(u)\big]e^{-\int_{u}^{t}\alpha(s)ds}du\Big]. \tag{2.3}$$

Proof. For any initial time $t_0 \geq 0$, let $x(t)$ be any solution of (1.1) with $x(t) = \phi(t)$, for $0 \leq t \leq t_0$. Rewrite (2.2) as

$$V'(t,x) + \alpha(t)V(t,x) \leq \alpha(t)V(t,x) - \alpha(t)V^q(t,x) + F(t). \tag{2.4}$$

Multiply (2.4) with the integrating factor $\int_{t_0}^{t} \alpha(s)ds$ and then integrate from t_0 to t to get

$$\begin{aligned} V(t,x) &\leq W^{-1}\Big[V(t_0, ||\phi||)e^{-\int_{t_0}^t \alpha(u)du} \\ &+ \int_{t_0}^t \big[\alpha(u)(V(u,x) - V^q(u,x)) + F(u)\big]e^{-\int_u^t \alpha(s)ds}du\Big] \\ &\leq W^{-1}\Big[V(t_0, ||\phi||)e^{-\int_{t_0}^t \alpha(u)du} \\ &+ \int_{t_0}^t \big[\gamma\alpha(u) + F(u)\big]e^{-\int_u^t \alpha(s)ds}du\Big]. \end{aligned}$$

Now use (2.1) to obtain (2.3). This completes the proof.
In [8] the author proved a theorem parallel to Theorem 2.2 where $q = 1$ and $F(t) = 0$. ☐

Example 2.3. To illustrate the application of Theorem 2.2, we consider the following two dimensional system of nonlinear Volterra integro-differential equations

$$\begin{aligned} y_1' &= y_2 - y_1\,|y_1| - y_1\;y_2^2 \int_0^t |B(t,s)|\, f(y_1(s), y_2(s))ds + g_1(t) \\ y_2' &= -y_1 - y_2\,|y_2| + y_1^2\;y_2 \int_0^t C(t,s) g(y_1(s), y_2(s))ds + g_2(t) \\ &(y_1(t), y_2(t)) = (\varphi_1(t), \varphi_2(t)), \end{aligned}$$

for some given initial continuous and bounded functions $\varphi_1(t), \varphi_2(t),\ 0 \leq t \leq t_0$. The scalar functions $|B(t,s)|, C(t,s)$ are continuous in t and s and $|B(t,s)| \geq |C(t,s)|$. Also, the scalar $f(y_1(s), y_2(s))$ and $g(y_1(s), y_2(s))$ are continuous in y_1 and y_2. We assume that

$$f(y_1(s), y_2(s)) \geq 0, |g(y_1(s), y_2(s))| \leq f(y_1(s), y_2(s)), \text{ for all } \ y_1, y_2 \in \mathbb{R}.$$

Let us take $V(y_1, y_2) = \frac{1}{2}(y_1^2 + y_2^2)$. Then

$$\begin{aligned} V'(y_1, y_2) &= -y_1^2\,|y_1| - y_2^2\,|y_2| + y_1 g_1(t) + y_2 g_2(t) \\ &\quad - y_1^2 y_2^2\Big(\int_0^t |B(t,s)|\, f(y_1(s), y_2(s))ds - \int_0^t C(t,s) g(y_1(s), y_2(s))ds\Big) \\ &\leq -\left(|y_1|^3 + |y_2|^3\right) + y_1^2 y_2^2 \int_0^t \Big(|C(t,s)| - |B(t,s)|\Big) f(y_1(s), y_2(s))ds \\ &\leq -\Big[\,|y_1|^3 + |y_2|^3\Big] + |y_1||g_1(t)| + |y_1||g_1(t)|. \end{aligned}$$

To further simplify the above inequality we make use of Young's inequality, which says for any two nonnegative real numbers w and z, we have

$$wz \leq \frac{w^e}{e} + \frac{z^f}{f}, \quad \text{with } 1/e + 1/f = 1.$$

Thus, for $e=3$ and $f=3/2$, we get

$$|y_1||g_1(t)| \le \frac{|y_1|^3}{3} + 2/3(|g_1(t)|^{3/2}).$$

Similarly,

$$|y_2||g_2(t)| \le \frac{|y_2|^3}{3} + 2/3(|g_2(t)|^{3/2}).$$

Thus,

$$\begin{aligned} V'(y_1,y_2) &\le -\frac{4}{3}\Big[\frac{|y_1|^3}{2} + \frac{|y_2|^3}{2}\Big] + 2/3(|g_1(t)|^{3/2}) + 2/3(|g_2(t)|^{3/2}) \\ &= -\frac{4}{3}\Big[\frac{\big(|y_1|^2\big)^{3/2}}{2} + \frac{\big(|y_2|^2\big)^{3/2}}{2}\Big] + 2/3(|g_1(t)|^{3/2}) + 2/3(|g_2(t)|^{3/2}) \\ &\le -\frac{4}{3}\,\big(|y_1|^2 + |y_2|^2\big)^{3/2}\, 2^{-3/2} + F(t) \\ &= -\frac{4}{3}V^{3/2}(y_1,y_2) + F(t), \end{aligned}$$

where we have used the inequality $\big(\frac{a+b}{2}\big)^l \le \frac{a^l}{2} + \frac{b^l}{2}$, $a,b>0$, $l>1$ and $F(t) = 2/3(|g_1(t)|^{3/2}) + 2/3(|g_2(t)|^{3/2})$. Next,

$$\begin{aligned} V(t,y) - V^q(t,y) &= V(x,t) - V^{\frac{3}{2}}(x,t) \\ &= y_1^2 + y_2^2 - (y_1^2+y_2^2)^{\frac{3}{2}} 2^{-3/2} \le \frac{4}{27}. \end{aligned}$$

Hence, we have $\alpha(t) = \frac{4}{3}$ and $\gamma = \frac{4}{27}$. By Theorem 2.2 all solutions of the above two dimensional system satisfy

$$\begin{aligned} \frac{1}{2}(y_1^2+y_2^2) \le \frac{1}{2}(\varphi_1^2(t) + \varphi_2^2(t)) e^{-\int_{t_0}^t \frac{4}{3} ds} \\ + \int_{t_0}^t \big[\frac{4}{27}\frac{4}{3} + F(u)\big] e^{-\int_u^t \frac{4}{3} ds} du\Big]. \end{aligned}$$

Next, we turn our attention to issue 2).

Theorem 2.4. *Let D be a set in $\mathbb{R}^n$. Suppose there exists a continuously differentiable Lyapunov functional $V : \mathbb{R}^+ \times D \to \mathbb{R}^+$ that satisfies*

$$W_1(||x||) \le V(t,x) \le W_2(||x||) + \int_0^t \varphi_1(t,s) W_3(|x(s)|) ds \tag{2.5}$$

and

$$V'(t,x) \le -\alpha_1(t) W_4(||x||) - \alpha_2(t) \int_0^t \varphi_2(t,s) W_5(|x(s)|) ds + F(t) \tag{2.6}$$

for positive continuous functions constants $\alpha_1(t), \alpha_2(t)$ and $F(t)$ where $\varphi_i(t,s) \ge 0$ is a scalar function continuous for $0 \le s \le t < \infty, i = 1,2,$ such that for some constant $\gamma \ge 0$ the inequality

$$W_2(||x||) - W_4(||x||) + \int_0^t \Big(\varphi_1(t,s) W_3(|x(s)|) - \varphi_2(t,s) W_5(|x(s)|)\Big) ds \le \gamma \tag{2.7}$$

holds. Then all solutions of (1.1) that starts in D satisfy the variational of parameters inequality

$$\|x\| \leq W_1^{-1}\Big\{V(t_0,\|\phi\|)e^{-\int_{t_0}^t \alpha(s)ds} + \int_{t_0}^t \Big(\gamma\alpha(u)+F(u)\Big)e^{-\int_u^t \alpha(s)ds}du\Big\},$$

for all $t \geq t_0$, *where* $\alpha(t) = \max\{\alpha_1(t), \alpha_2(t)\}$.

Proof. Let $\alpha(t) = \max\{\alpha_1(t), \alpha_2(t)\}$. For any initial time $t_0 \geq 0$, let $x(t)$ be any solution of (1.1) with $x(t) = \phi(t)$, for $0 \leq t \leq t_0$. Then,

$$\frac{d}{dt}\Big(V(t,x(t))e^{\int_{t_0}^t \alpha(s)ds}\Big) = \Big[V'(t,x(t)) + \alpha(t)V(t,x(t))\Big]e^{\int_{t_0}^t \alpha(s)ds}.$$

For $x(t) \in \mathbb{R}^n$, using (2.6) and (2.7)we get

$$\begin{aligned}
\frac{d}{dt}\Big(V(t,x(t))e^{\int_{t_0}^t \alpha(s)ds}\Big) &\leq \Big[-\alpha_1(t)W_4(||x||) - \alpha_2(t)\int_0^t \varphi_2(t,s)W_5(|x(s)|)ds \\
&+ \alpha(t)W_2(||x||) + \alpha(t)\int_0^t \varphi_1(t,s)W_3(|x(s)|)ds + F(t)\Big]e^{\int_{t_0}^t \alpha(s)ds} \\
&\leq \Big\{\alpha(t)\Big[W_2(||x||) - W_4(||x||) \\
&+ \int_0^t \Big(\varphi_1(t,s)W_3(|x(s)|) - \varphi_2(t,s)W_5(|x(s)|)\Big)ds\Big] + F(t)\Big\}e^{\int_{t_0}^t \alpha(s)ds} \\
&\leq \Big(\gamma\alpha(t) + F(t)\Big)e^{\int_{t_0}^t \alpha(s)ds}. \qquad (2.8)
\end{aligned}$$

Integrating (2.8) from t_0 to t we obtain,

$$V(t,x(t))e^{\int_{t_0}^t \alpha(s)ds} \leq V(t_0,\phi) + \int_{t_0}^t \Big(\gamma\alpha(u) + F(u)\Big)e^{\int_{t_0}^u \alpha(s)ds}du.$$

Consequently,

$$V(t,x(t)) \leq V(t_0,\phi)e^{-\int_{t_0}^t \alpha(s)ds} + \int_{t_0}^t \Big(\gamma\alpha(u) + F(u)\Big)e^{-\int_u^t \alpha(s)ds}du.$$

From condition (2.5) we have $W_1(|x|) \leq V(t,x(t))$, which implies that

$$||x|| \leq W_1^{-1}\Big\{V(t_0,\|\phi\|)e^{-\int_{t_0}^t \alpha(s)ds} + \int_{t_0}^t \Big(\gamma\alpha(u)+F(u)\Big)e^{-\int_u^t \alpha(s)ds}du\Big\}, \qquad (2.9)$$

for all $t \geq t_0$. □

We remark that it is clear from (2.9) that if

$$V(t_0,\|\phi\|)e^{-\int_{t_0}^t \alpha(s)ds} + \int_{t_0}^t \Big(\gamma\alpha(u)+F(u)\Big)e^{-\int_u^t \alpha(s)ds}du \leq K,$$

for some positive constant K, then all solution of (1.1) are uniformly bounded.

Example 2.5. Let $\phi(t)$ be a given bounded continuous initial function and consider the scalar Volterra integro-differential equation

$$x' = \sigma(t)x(t) + \int_0^t B(t,s)x(s)ds + g(t),\ t \geq 0, \tag{2.10}$$
$$x(t) = \phi(t) \ \text{ for } \ 0 \leq t \leq t_0.$$

Assume for $k > 1$ that

$$-2\sigma(t) - \int_0^t |B(t,s)|ds - k\int_t^\infty |B(u,t)|du - 1 > 0,$$

and

$$(k-1)|B(t,s)| \geq k\int_t^\infty |B(u,s)|du. \tag{2.11}$$

Then all solutions of (2.10) satisfy

$$|x| \leq \Big\{V(t_0, \|\phi\|)e^{-\int_{t_0}^t \alpha(s)ds} + \int_{t_0}^t \Big(\gamma\alpha(u) + F(u)\Big)e^{-\int_u^t \alpha(s)ds}du\Big\}^{1/2}, \tag{2.12}$$

with

$$\alpha(t) = \max\{-2\sigma(t) - \int_0^t |B(t,s)|ds - k\int_t^\infty |B(u,t)|du - 1,\ 1\},$$

$$\gamma = 0,\ \text{and}\ \ F(t) = g^2(t),$$

and

$$V(t_0, \phi) = \varphi^2 + k\int_0^{t_0}\int_{t_0}^\infty |B(u,s)|du\phi^2(s)ds.$$

To see this we let

$$V(t,x) = x^2 + k\int_0^t\int_t^\infty |B(u,s)|dux^2(s)ds.$$

Then along solutions of (2.10) we have

$$\begin{aligned} V'(t,x) &= 2xx' + k\int_t^\infty |B(u,t)|x^2(t)du - k\int_0^t |B(t,s)|x^2(s)ds \\ &\leq 2\sigma(t)x^2 + 2\int_0^t |B(t,s)|\ |x(t)|x(s)ds + 2|x(t)||g(t)| \\ &\quad +k\int_t^\infty |B(u,t)|x^2(t)du - k\int_0^t |B(t,s)|x^2(s)ds. \end{aligned}$$

Using the fact that $ab \leq a^2/2 + b^2/2$, the above inequality simplifies to

$$\begin{aligned} V'(t,x) &\leq 2\sigma(t)x^2 + \int_0^t |B(t,s)|(x^2(t) + x^2(s))ds \\ &\quad +k\int_t^\infty |B(u,t)|x^2(t)du - k\int_0^t |B(t,s)|x^2(s)ds \\ &+ x^2(t) + g^2(t) \\ &\leq -\Big(-2\sigma(t) - \int_0^t |B(t,s)|ds - k\int_t^\infty |B(u,t)|du - 1\Big)x^2(t) \\ &\quad -(k-1)\int_0^t |B(t,s)|x^2(s)ds + g^2(t). \end{aligned}$$

Let $\alpha_1(t) = -2\sigma(t) - \int_0^t |B(t,s)|ds - k\int_t^\infty |B(u,t)|du - 1 > 0$ and $\alpha_2(t) = 1$. By taking $W_1 = W_2 = W_4 = x^2(t)$, $W_3 = W_5 = x^2(s)$, and $\varphi_1(t,s) = k\int_t^\infty |B(u,s)|du$, $\varphi_2(t,s) = (k-1)|B(t,s)|$, we see that all the conditions of Theorem 2.4 are satisfied. Hence all solutions of (2.10) satisfy (2.12).

In the following remark we specify the functions $B(t,s), \sigma(t)$ and $g(t)$.

Remark 2.6. Note that if $B(t,s) = e^{-3(t-s)}$, then condition (2.11) is satisfied for $k \geq 3/2$. Also, condition (2.7) is satisfied with $\gamma = 0$. Let $\phi(t)$ be a given bounded continuous initial function for $0 \leq t \leq 1$. Then for $t \geq t_0 \geq 1$ and for $\sigma(t) = -t/2 - \frac{1}{6}(1 - e^{-3t}) - k/6 - 1/2$ we have $\alpha(t) = \alpha_1(t) = t$. Hence from inequality (2.9) we will have for $g(t) = t^{1/2}$ that

$$\begin{aligned} ||x|| &\leq \Big\{V(t_0,\phi)e^{-\int_{t_0}^t \alpha(s)ds} + \int_{t_0}^t \Big(\gamma\alpha(u) + F(u)\Big)e^{-\int_u^t \alpha(s)ds}du\Big\}^{1/2} \\ &\leq \Big\{V(t_0,\phi)e^{-\int_{t_0}^t sds} + \int_{t_0}^t ue^{-\int_u^t sds}du\Big\}^{1/2} \\ &\leq \{(1 + k/9)\|\phi\|^2 + 1\}^{1/2}. \end{aligned}$$

Thus we have shown that every solution of the Volterra integro-differential equation

$$\begin{aligned} &x' = [-t/2 - \frac{1}{6}(1 - e^{-3t}) - k/6 - 1/2]x(t) + \int_0^t e^{-3(t-s)}x(s)ds + t^{1/2}, \\ &t \geq t_0 \geq 1, \text{ with } x(t) = \phi(t) \text{ for } 0 \leq t \leq t_0 \leq 1 \end{aligned}$$

satisfies the inequality

$$||x|| \leq \{(1 + k/9)\|\phi\|^2 + 1\}^{1/2}, \text{ for } k \geq 3/2.$$

In the next example, we take $f(x)$ to be nonlinear.

Example 2.7. Consider the scalar nonlinear Volterra integro-differential equation

$$x' = \sigma(t)x(t) + \int_0^t B(t,s)x^{2/3}(s)ds + g(t),\ t \geq 0,\ x(t) = \phi(t) \text{ for } 0 \leq t \leq t_0. \tag{2.13}$$

If

$$-2\sigma(t) - \int_0^t |B(t,s)|ds - \int_t^\infty |B(u,t)|du - 1 > 0,$$

and

$$\frac{|B(t,s)|}{3} \geq \int_t^\infty |B(u,s)|du$$

then all solutions of (2.13) satisfy inequality (2.12) with $k = 1$
where

$$\alpha(t) = \max\{-2\sigma(t) - \int_0^t |B(t,s)|ds - \int_t^\infty |B(u,t)|du - 1 > 0,\ 1\},$$

$$\gamma = 0, \text{ and } F(t) = g^2(t),$$

and

$$V(t_0,\phi) = \varphi^2 + \int_0^{t_0}\int_{t_0}^\infty |B(u,s)|du\phi^2(s)ds.$$

To see this we let

$$V(t,x) = x^2 + \int_0^t\int_t^\infty |B(u,s)|dux^2(s)ds.$$

Then along solutions of (2.13) we have

$$\begin{aligned} V'(t,x) &= 2xx' + \int_t^\infty |B(u,t)|x^2(t)du - \int_0^t |B(t,s)|x^2(s)ds \\ &\leq 2\sigma(t)x^2 + 2\int_0^t |B(t,s)|\,|x(t)|x^{2/3}(s)ds \\ &\quad + \int_t^\infty |B(u,t)|x^2(t)du - \int_0^t |B(t,s)|x^2(s)ds \\ &+ 2|x||g(t)|. \end{aligned}$$

Using the fact that $ab \leq a^2/2 + b^2/2$, the above inequality simplifies to

$$\begin{aligned} V'(t,x) &\leq 2\sigma(t)x^2 + \int_0^t |B(t,s)|(x^2(t) + x^{4/3}(s))ds \\ &\quad + \int_t^\infty |B(u,t)|x^2(t)du - \int_0^t |B(t,s)|x^2(s)ds \\ &+ x^2 + g^2(t). \end{aligned} \tag{2.14}$$

To further simplify (2.14) we make use of Young's inequality. Thus, for $e = 3/2$ and $f = 3$, we get

$$\begin{aligned}\int_0^t |B(t,s)|x^{4/3}(s)ds &= \int_0^t |B(t,s)|^{1/3}|B(t,s)|^{2/3}x^{4/3}(s)ds \\ &\leq \int_0^t \Big(\frac{|B(t,s)|}{3} + \frac{2}{3}|B(t,s)|x^2(s)\Big)ds.\end{aligned}$$

By substituting the above inequality into (2.14), we arrive at

$$\begin{aligned}V'(t,x) &\leq \Big(2\sigma(t) + \int_0^t |B(t,s)|ds + \int_t^\infty |B(u,t)|du + 1\Big)x^2(t) \\ &\quad - \int_0^t \Big(|B(t,s)| - \frac{2}{3}|B(t,s)|\Big)x^2(s)ds + L \\ &\leq -\alpha(t)\big(x^2(t) + \int_0^t \frac{|B(t,s)|}{3}x^2(s)ds\big) + F(t)\end{aligned}$$

where $F(t) = \frac{1}{3}\int_0^t |B(t,s)|ds + g^2(t)$. Let $\alpha_1(t) = -2\sigma(t) - \int_0^t |B(t,s)|ds - \int_t^\infty |B(u,t)|du - 1 > 0$ and $\alpha_2(t) = 1$. By taking $W_1 = W_2 = W_4 = x^2(t),\ W_3 = W_5 = x^2(s)$, $\varphi_1(t,s) = \int_t^\infty |B(u,s)|du$, and $\varphi_2(t,s) = \frac{|B(t,s)|}{3}$, we see that all the conditions of Theorem 2.4 are satisfied. Hence all solutions of (2.13) satisfy inequality (2.12).

We give our last example, in which the displayed Volterra integro-differential equation is totally nonlinear.

Example 2.8. Let $D = \{x \in \mathbb{R} : ||x|| \geq 1\}$. Let $\phi(t)$ be a given bounded continuous initial function such that $\sup\{|\phi(t)| = 1 : 0 \leq t \leq t_0\}$. Consider the scalar nonlinear Volterra integro-differential equation

$$\begin{aligned}x' &= \sigma(t)x^3(t) + \int_0^t B(t,s)x^{1/3}(s)ds + g(t),\ t \geq 0, \\ &\quad x(t) = \phi(t)\ \text{ for }\ 0 \leq t \leq t_0. \qquad (2.15)\end{aligned}$$

If

$$-2\sigma(t) - \frac{1}{2}\int_0^t |B(t,s)|^{\frac{1}{2}}ds - \int_t^\infty |B(u,t)|du - 1 > 0,$$

and

$$\frac{5|B(t,s)|}{6} \geq \int_t^\infty |B(u,s)|du,$$

then all solutions of (2.15) initiating in the set D satisfy inequality (2.3). where

$$\alpha(t) = \max\{-2\sigma(t) - \int_0^t |B(t,s)|ds - \int_t^\infty |B(u,t)|du - \frac{1}{2} > 0,\ 1\},$$

$$\gamma = 0,\ \text{and}\ \ F(t) = g^2(t),$$

and

$$V(t_0,\phi)=\phi^2+\phi^4\int_0^{t_0}\int_{t_0}^{\infty}|B(u,s)|duds.$$

To see this, we consider the Lyapunov functional $V(t,x):\mathbb{R}^+\times D\to\mathbb{R}^+$,

$$V(t,x)=x^2+\int_0^t\int_t^{\infty}|B(u,s)|dux^4(s)ds.$$

Then for $x\in D$, we have along solutions of (2.15)

$$\begin{aligned}
V'(t,x) &= 2xx'+\int_t^{\infty}|B(u,t)|x^4(t)du-\int_0^t|B(t,s)|x^4(s)ds\\
&\le 2\sigma(t)x^4+2\int_0^t|B(t,s)|\;|x(t)||x(s)|^{1/3}ds\\
&\quad +\int_t^{\infty}|B(u,t)|x^4(t)du-\int_0^t|B(t,s)|x^4(s)ds\\
&+ x^2+g^2(t).
\end{aligned}$$

By noting that $2|x(t)||x(s)|^{1/3}\le x^2(t)+x^{2/3}(s)$ we have from the above inequality that

$$\begin{aligned}
V'(t,x) &\le 2\sigma(t)x^4+\int_0^t|B(t,s)|\;(x^2(t)+|x(s)|^{2/3})ds\\
&\quad +\int_t^{\infty}|B(u,t)|x^4(t)du-\int_0^t|B(t,s)|x^4(s)ds.
\end{aligned}$$

Next we note that

$$\begin{aligned}
\int_0^t|B(t,s)|\;x^2(t)dt &= \int_0^t|B(t,s)|^{1/2}|B(t,s)|^{1/2}\;x^2(t)ds\\
&\le \int_0^t|B(t,s)|^{1/2}[\frac{|B(t,s)|}{2}+\frac{x^4(t)}{2}]ds.
\end{aligned}$$

Also

$$x^2\le\frac{x^4}{2}+\frac{1}{2}.$$

using Young's inequality with $e=6$ and $f=6/5$, we get

$$\begin{aligned}
x(s)^{2/3}|B(t,s)| &= x(s)^{2/3}|B(t,s)|^{1/6}|B(t,s)|^{5/6}\\
&\le \frac{x^4(s)|B(t,s)|}{6}+\frac{5}{6}|B(t,s)|.
\end{aligned}$$

$$\begin{aligned}
V'(t,x) &\le \Big(2\sigma(t)+\frac{1}{2}\int_0^t|B(t,s)|ds+\int_t^{\infty}|B(u,t)|du+\frac{1}{2}\Big)x^4(t)\\
&\quad -\int_0^t\Big(|B(t,s)|-\frac{|B(t,s)|}{6}\Big)x^4(s)ds+L\\
&\le -\alpha(t)\big(x^4(t)+\int_0^t\frac{5|B(t,s)|}{6}x^4(s)ds\big)+F(t),
\end{aligned}$$

where
$$F(t) = \frac{5}{6}\int_0^t |B(t,s)|ds + \frac{1}{2}\int_0^t |B(t,s)|^{3/2}ds + \frac{1}{2}.$$
By taking $W_1 = W_2 = x^2(t), W_3 = W_4 = W_5 = x^4(s)$, and
$$\varphi_1(t,s) = \int_t^\infty |B(u,s)|du,\ \varphi_2(t,s) = \frac{5|B(t,s)|}{6}$$
we see that conditions (2.5) and (2.6) of Theorem 2.4 are satisfied. Left to show that condition (2.7) hold. Since $\frac{5|B(t,s)|}{6} \geq \int_t^\infty |B(u,s)|du$ we have, for $x \in D$ that
$$\begin{aligned} & W_2(|x|) - W_4(|x|) + \int_0^t \Big(\varphi_1(t,s)W_3(|x(s)|) - \varphi_2(t,s)W_5(|x(s)|)\Big)ds \\ = \ & x^2(t) - x^4(t) + \int_0^t \Big(\int_t^\infty |B(u,s)|du - \frac{5|B(t,s)|}{6}\Big)x^4(s)ds \\ \leq \ & x^2(1-x^2) \leq 0. \end{aligned}$$
Thus, condition (2.7) is satisfied for $\gamma = 0$. An application of Theorem 2.4 yields
$$\begin{aligned} |x(t)| \ \leq \ & \Big[\Big(1 + \int_0^{t_0}\int_{t_0}^\infty |B(u,s)|du\ ds\Big)e^{-\int_{t_0}^t \alpha(s)ds} \\ & + \int_{t_0}^t F(u)e^{-\int_u^t \alpha(s)ds}du\Big]^{1/2}, \end{aligned}$$
for all $t \geq t_0$. Hence, every solution x with $x(t) \in D$ satisfies
$$\begin{aligned} 1 \ \leq \ & |x(t)| \leq \Big[\Big(1 + \int_0^{t_0}\int_{t_0}^\infty |B(u,s)|du\ ds\Big)e^{-\int_{t_0}^t \alpha(s)ds} \\ + & \int_{t_0}^t F(u)e^{-\int_u^t \alpha(s)ds}du\Big]^{1/2}. \end{aligned}$$
Remark 2.9. Note that if $B(t,s) = e^{-k(t-s)}$, then the second condition of Example 2.8 is satisfied for $k = 6/5$. Also, condition (2.7) is satisfied with $\gamma = 0$. Let $\phi(t)$ be a given bounded continuous initial function for $0 \leq t \leq 1$. Then for $t \geq t_0 \geq 1$ and for $\sigma(t) = -t/2 - \frac{1}{k} + \frac{e^{-kt}}{2k} - 1/2$ we have $\alpha(t) = \alpha_1(t) = t$. Hence from inequality (2.9) we will have for $g(t) = t^{1/2}$ that
$$\begin{aligned} ||x|| \ \leq \ & \Big\{V(t_0,\phi)e^{-\int_{t_0}^t \alpha(s)ds} + \int_{t_0}^t \Big(\gamma\alpha(u) + F(u)\Big)e^{-\int_u^t \alpha(s)ds}du\Big\}^{1/2} \\ \leq \ & \Big\{V(t_0,\phi)e^{-\int_{t_0}^t sds} + \int_{t_0}^t ue^{-\int_u^t sds}du\Big\}^{1/2} \\ \leq \ & \{(1 + \|\phi\|^2/k^2)\|\phi\|^2 + 1\}^{1/2}. \end{aligned}$$
Thus we have shown that every solution of the Volterra integro-differential equation
$$x' = [-t/2 - \frac{1}{k} + \frac{e^{-kt}}{2k} - 1/2]x(t) + \int_0^t e^{-k(t-s)}x(s)ds + t^{1/2},$$
$$t \geq t_0 \geq 1,\ \text{with}\ \ x(t) = \phi(t)\ \ \text{for}\ \ 0 \leq t \leq t_0 \leq 1$$

satisfies the inequality

$$1 \leq ||x|| \leq \{(1 + \|\phi\|^2/k^2)\|\phi\|^2 + 1\}^{1/2}, \text{ for } k = 6/5.$$

3. Comparison

In [9] the author considered the scalar Volterra integro-differential equation

$$x'(t) = Af(x(t)) + \int_0^t B(t,s)g(x(s))ds + h(t), \tag{3.1}$$

where f, g and h are continuous in their respective arguments and proved the following theorem.

Theorem 3.1. [9] *Assume $xf(x) > 0$ for all $x \neq 0$. Suppose there is a constant $m > 0$ such that*

$$g^2(x) \leq m^2 f^2(x) \text{ for all } x \in \mathbb{R}. \tag{3.2}$$

If

$$A(t) + k\int_0^t |B(t,s)ds + \frac{1}{2}\int_t^\infty |B(u,t)|du \leq -\rho, \ t \geq 0$$

for some positive constant ρ and k such that $m^2 < 2k$,

$$\int_0^x f(x)dx \to \infty \text{ as } |x| \to \infty,$$

and

$$h(\cdot) \in L^2[0,\infty),$$

then all solutions of (3.1) are bounded.

In Example 2.7 we considered the scalar nonlinear Volterra integro-differential equation

$$x' = \sigma(t)x(t) + \int_0^t B(t,s)x^{2/3}(s)ds + h(t) \ \ t \geq 0, \ x(t) = \phi(t) \tag{3.3}$$

for $0 \leq t \leq t_0$, where $h(t)$ is continuous in t. Theorem 3.1 of [9] can not be applied to (3.3) since condition (3.2) can not hold for a positive constant m and for all $x \in \mathbb{R}$. Moreover, we have only required that $h(t)$ satisfies

$$\int_{t_0}^t h^2(u)e^{-\int_u^t \alpha(s)ds}du \leq Q, \tag{3.4}$$

for some positive constant Q. On the other hand, in [9], it was required that $h(t)$ be an $L^2[0,\infty)$ function. It is worth mentioning that in [9], the author had to require that $h(t)$ be bounded. We conclude that condition (3.4) is improvement over [8] and [9].

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