Banach J. Math. Anal. 3 (2009), no. 2, 103–108

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
http://www.math-analysis.org

A NOTE ON THE VON NEUMANN ALGEBRA UNDERLYING SOME UNIVERSAL COMPACT QUANTUM GROUPS

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Communicated by M. Skeide

ABSTRACT. We show that for $F \in GL(2, \mathbb{C})$, the von Neumann algebra associated to the universal compact quantum group $A_u(F)$ is a free Araki-Woods factor.

INTRODUCTION

It is a classical theorem that any compact Lie group is a closed subgroup of some $U(n)$. In [4], a class of quantum groups was introduced which plays the same rôle with respect to the compact matrix quantum groups (introduced in [6], but there called compact matrix *pseudo*groups). These universal quantum groups were denoted $A_u(F)$, where the parameter F takes values in invertible matrices over $\mathbb{C}$. In [1], the representation theory of the $A_u(F)$ was investigated, and it was shown that the irreducible representations are naturally labeled by the free monoid with two generators. Also on the level of the 'function algebra' of $A_u(F)$, freeness manifests itself: it was shown in [1] that the (normalized) trace of the fundamental representation is a circular element w.r.t. the Haar state (in the sense of Voiculescu, see [5]). Furthermore, the von Neumann algebra associated to $A_u(I_2)$, where I_2 is the unit matrix in $GL(2, \mathbb{C})$, is actually isomorphic to the free group factor $\mathscr{L}(\mathbb{F}_2)$.

In this note, we generalize this last result by showing that for $0 < q \leq 1$, the von Neumann algebra underlying the universal quantum group $A_u(F)$ with $F =$

Date: Received: 24 July 2009; Revised: 8 September 2009; Accepted: 18 September 2009.
2000 *Mathematics Subject Classification.* Primary 46L10; Secondary 46L54, 16W35.
Key words and phrases. von Neumann algebra, free quantum group, free Araki-Woods factor, free probability.

$\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$ is a free Araki-Woods factor ([3]), namely the one associated to the orthogonal representation

$$t \to \begin{pmatrix} \cos(t \ln q^2) & -\sin(t \ln q^2) \\ \sin(t \ln q^2) & \cos(t \ln q^2) \end{pmatrix}$$

of $\mathbb{R}$ on $\mathbb{R}^2$. The proof of this fact uses a technique similar to the one of Banica for the case $F = I_2$, combined with results from [2] (which are based on the matrix model techniques from [3]). Since

$$A_u(F) = A_u(\lambda U|F|U^*)$$

for any $\lambda \in \mathbb{R}_0^+$ and any unitary U (see [1]), we obtain that all $A_u(F)$ with $F \in GL(2, \mathbb{C})$ have free Araki-Woods factors as their associated von Neumann algebras. We remark that for higher-dimensional F, even $F = I_3$, much less is known about the concrete form of the associated von Neumann algebra, and probably different techniques than the ones used in this paper will be necessary to probe their structure.

Remarks on notation:

- If M is a von Neumann algebra and $x_1, x_2, \ldots$ are elements in M, we denote by $W^*(x_1, x_2, \ldots)$ the von Neumann subalgebra of M which is the σ-weak closure of the unital *-algebra generated by the x_i.
- Matrix units of $B(l^2(\mathbb{N}))$ w.r.t. the canonical basis of $l^2(\mathbb{N})$ are written e_{ij}.
- For $0 < q < 1$, we write ω_q for the normal state $\omega_q(e_{ij}) = \delta_{i,j}(1 - q^2)q^{2i}$ on $B(l^2(\mathbb{N}))$.
- We denote by $S \in \mathscr{L}(\mathbb{Z})$ the shift operator $\xi_k \to \xi_{k+1}$ on $l^2(\mathbb{Z})$. Since we will at times need different copies of $\mathscr{L}(\mathbb{Z})$, we will sometimes use an index for emphasis, using the same index symbol for the generator S (for example, $\mathscr{L}(\mathbb{Z})_I$ and S_I, or $\mathscr{L}(\mathbb{Z})_H$ and S_H).
- We denote by τ the state on $\mathscr{L}(\mathbb{Z})$ which makes S into a Haar unitary with respect to it (i.e. $\tau(S^n) = 0$ for $n \in \mathbb{Z}_0$). We then use the same index convention as in the previous point.

1. Preliminaries

In this preliminary section, we will give, for the sake of economy, ad hoc definitions of the von Neumann algebras associated to the $A_u(F)$ and $A_o(F)$ quantum groups ([4]), and of the free Araki-Woods factors ([3]), for special values of their parameters.

Throughout this section, we fix a number $0 < q < 1$.

Definition 1.1. We define the C*-algebra $C_u(H)$ as the universal enveloping C*-algebra of the unital *-algebra generated by elements a and b, with defining relations

$$\begin{cases} a^*a + b^*b = 1 & ab = qba \\ aa^* + q^2bb^* = 1 & a^*b = q^{-1}ba^* \\ & bb^* = b^*b. \end{cases}$$

Remark 1.2. $C_u(H)$ is the (universal) C*-algebra associated with the quantum group $H = SU_q(2)$. In [1], Proposition 5, it is shown that this equals the quantum group $A_o(\begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix})$.

The following fact is found in [7].

Lemma 1.3. *Let $\mathscr{H}$ be the Hilbert space $l^2(\mathbb{N}) \otimes l^2(\mathbb{Z})$, whose canonical basis elements we denote as $\xi_{n,k}$ (and with the convention $\xi_{n,k} = 0$ when $n < 0$). Then there exists a faithful unital *-representation of $C_u(H)$ on $\mathscr{H}$, determined by*

$$\begin{cases} \pi(a)\,\xi_{n,k} = \sqrt{1-q^{2n}}\,\xi_{n-1,k}, \\ \pi(b)\xi_{n,k} = q^n\,\xi_{n,k+1}. \end{cases}$$

Note that $\pi(a)$ is an amplification of a weighted unilateral shift with all weights different, and that $\pi(b)$ is a normal operator, being the tensor product of a diagonal operator (with all eigenvalues of multiplicity one) and a bilateral shift. As such, one can obtain, for $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, the elements $e_{nn} \otimes S^m \in \pi(C_u(H))''$ by applying Borel calculus to $\pi(b)$, while we have $e_{mn} \otimes 1 \in \pi(C_u(H))''$ for all $m, n \in \mathbb{N}$ by multiplying $\pi(a)^{|m-n|}$ or $\pi(a^*)^{|m-n|}$ to the right with (a scalar multiple of) $e_{nn} \otimes 1$.

Definition 1.4. In the notation of the previous lemma, denote by ψ the state

$$\psi(x) = (1-q^2)\sum_{n\in\mathbb{N}} q^{2n}\langle \pi(x)\xi_{n,0}, \xi_{n,0}\rangle$$

on $C_u(H)$. Then ψ is called the *Haar state* on $C_u(H)$.

Of course, this name is motivated by the further compact quantum group structure on $C_u(H)$, which we will however not need in the following.

Definition 1.5. The von Neumann algebra $\mathscr{L}^\infty(H)$ is defined to be the σ-weak closure of $C_u(H)$ in its GNS-representation with respect to the Haar state ψ.

We then continue to write ψ for the extension of ψ to a normal state on $\mathscr{L}^\infty(H)$.

We will use the terminology 'W*-probability space' when talking about a von Neumann algebra with some fixed normal state on it. An isomorphism between two W*-probability spaces is then a *-isomorphism between the underlying von Neumann algebras, preserving the associated fixed states.

Lemma 1.6. *There is a natural isomorphism*

$$(\mathscr{L}^\infty(H), \psi) \to (B(l^2(\mathbb{N}))\bar{\otimes}\mathscr{L}(\mathbb{Z})_H, \omega_q \otimes \tau_H)$$

of W-probability spaces.*

Proof. We first prove that $\pi(C_u(H))''$ equals $B(l^2(\mathbb{N}))\bar{\otimes}\mathscr{L}(\mathbb{Z})_H$. Clearly, we have that $\pi(C_u(H))'' \subseteq B(l^2(\mathbb{N}))\bar{\otimes}\mathscr{L}(\mathbb{Z})_H$ by the explicit forms of the generators $\pi(a)$ and $\pi(b)$ of $\pi(C_u(H))$. By functional calculus on $\pi(a)$ and $\pi(b)$, we have $e_{ij}\otimes S^n \in \pi(C_u(H))''$ for all $i,j\in\mathbb{N}$ and $n\in\mathbb{Z}$ (see the remark after Lemma 1.3), so in fact equality holds.

Now ψ equals the composition of π with the state $\omega_q\otimes\omega_{\delta_0}$ on $B(l^2(\mathbb{N})\bar{\otimes}l^2(\mathbb{Z}))$, where ω_{δ_0} is the pure state w.r.t. $\delta_0\in l^2(\mathbb{Z})$. So we have a natural normal unital *-homomorphism $\pi(C_u(H))''\rightarrow\mathscr{L}^\infty(H)$, which restricts to π^{-1} on $\pi(C_u(H))$. Since the restriction $\omega_q\otimes\tau_H$ of $\omega_q\otimes\omega_{\delta_0}$ to $B(l^2(\mathbb{N}))\bar{\otimes}\mathscr{L}(\mathbb{Z})_H=\pi(C_u(H))''$ is faithful, this homomorphism is an isomorphism. □

In the following, we then simply identify the W*-probability spaces appearing in the previous lemma (by the isomorphism appearing in its proof).

Definition 1.7. The W*-probability space $(\mathscr{L}^\infty(G),\varphi)$ is defined as

$$(W^*(S_Ia,S_Ib,S_Ia^*,S_Ib^*),(\tau_I*\psi)_{|\mathscr{L}^\infty(G)})\subseteq(\mathscr{L}(\mathbb{Z})_I,\tau_I)*(\mathscr{L}^\infty(H),\psi).$$

Remark 1.8. By [1], Théorème 1.(iv), the von Neumann algebra $\mathscr{L}^\infty(G)$ will coincide with the von Neumann algebra associated with the universal quantum group $A_u(\begin{pmatrix}1&0\\0&q\end{pmatrix})$, and φ with its Haar state.

Recall that the state ω_q was introduced at the end of the introduction.

Definition 1.9. ([3], Corollary 4.9) By a *free Araki-Woods factor (at parameter q^2)*, we mean a W*-probability space (M,ϕ) isomorphic to the free product W*-probability space $(\mathscr{L}(\mathbb{Z}),\tau)*(B(l^2(\mathbb{N})),\omega_q)$.

We end this section with a small alteration of Lemme 8 of [1].

Lemma 1.10. *Let (A,ϕ) be a unital *-algebra together with a functional ϕ on it. Let $B\subseteq A$ be a unital sub-*-algebra, and $d\in B$ a unitary in the center of B such that $\phi(d)=\phi(d^*)=0$. Let $u\in A$ be a Haar unitary which is *-free from B w.r.t. ϕ. Then ud is a Haar unitary which is *-free from B w.r.t. ϕ.*

Proof. This is precisely Lemme 8 of [1], with the condition 'ϕ is a trace' replaced by 'd is in the center of B'. However, the proof of that lemma still applies verbatim. □

2. $\mathscr{L}^\infty(G)$ is a free Araki-Woods factor

Throughout this section, we again fix a number $0<q<1$. We also continue to use the notations introduced in the previous section.

We proceed to prove the following theorem.

Theorem 2.1. *The W*-probability space $(\mathscr{L}^\infty(G),\varphi)$ is a free Araki-Woods factor at parameter q^2.*

By the remark after Definition 1.7 and the remarks in the introduction, this will imply that if $F \in GL(2,\mathbb{C})$, then the von Neumann algebra associated to $A_u(F)$ is the free Araki-Woods factor at parameter $\frac{\lambda_1}{\lambda_2}$, where $\lambda_1 \leq \lambda_2$ are the eigenvalues of F^*F (where we take $\mathscr{L}(\mathbb{F}_2)$ to be the free Araki-Woods factor at parameter 1).

The proof of Theorem 2.1 will be preceded by three lemmas. Consider the following von Neumann subalgebras of $(\mathscr{L}(\mathbb{Z})_I, \tau_I) * (\mathscr{L}^\infty(H), \psi)$:

$$(M_1, \varphi_1) = (W^*(S_I(1 \otimes S_H)), (\tau_I * \psi)_{|M_1})$$

and

$$(M_2, \varphi_2) = (W^*((1 \otimes S_H^*)a, (1 \otimes S_H^*)b, (1 \otimes S_H^*)a^*, (1 \otimes S_H^*)b^*), (\tau_I * \psi)_{|M_2}).$$

Lemma 2.2. *The von Neumann algebras M_1 and M_2 are free with respect to each other, and $\mathscr{L}^\infty(G)$ is the smallest von Neumann subalgebra of $\mathscr{L}(\mathbb{Z})_I * \mathscr{L}^\infty(H)$ which contains them.*

Proof. The proof is similar to the one of Théorème 6 in [1]. First of all, remark that $S_I(1 \otimes S_H)$ is the unitary part in the polar decomposition of $S_I b$, so that $S_I(1 \otimes S_H)$ is in $\mathscr{L}^\infty(G)$. Then of course

$$(1 \otimes S_H^*)a = (1 \otimes S_H^*)S_I^* \cdot S_I a$$

is in $\mathscr{L}^\infty(G)$, and similarly for the other generators of M_2. Hence M_1 and M_2 indeed generate $\mathscr{L}^\infty(G)$.

We now apply Lemma 1.10 to get that $S_I(1 \otimes S_H)$ is *-free w.r.t. $\mathscr{L}^\infty(H)$, by taking $(A, \phi) = (\mathscr{L}(\mathbb{Z})_I, \tau_I) * (\mathscr{L}^\infty(H), \psi)$, $B = \mathscr{L}^\infty(H)$, $d = 1 \otimes S_H$ and $u = S_I$. *A fortiori*, we will then have M_1 free w.r.t. M_2. □

Lemma 2.3. *We have*

$$(M_1, \varphi_1) \cong (\mathscr{L}(\mathbb{Z}), \tau)$$

and

$$(M_2, \varphi_2) \cong (B(l^2(\mathbb{N}))\bar{\otimes}\mathscr{L}(\mathbb{Z}), \omega_q \otimes \tau).$$

Proof. The fact that $(M_1, \varphi_1) \cong (\mathscr{L}(\mathbb{Z}), \tau)$ is of course trivial. We want to show that $(M_2, \varphi_2) \cong (B(l^2(\mathbb{N}))\bar{\otimes}\mathscr{L}(\mathbb{Z}), \omega_q \otimes \tau)$.

We have that $1 \otimes S_H^2$ is in M_2, since this is the adjoint of the unitary part of the polar decomposition of $(1 \otimes S_H^*)b^*$ (recall that $b = D \otimes S_H$ with D some diagonal positive operator). Also all $e_{ii} \otimes 1$ are in M_2, by functional calculus on the positive part of this polar decomposition. Hence, by multiplying $(1 \otimes S_H^*)a$ or $(1 \otimes S_H^*)a^*$ to the left with the $e_{ii} \otimes 1$, and possibly multiplying with $1 \otimes S_H^2$, we conclude that the $e_{ij} \otimes S_H^{i-j}$ with $|i-j| = 1$ are in M_2. But then also the matrix units $f_{ij} = e_{ij} \otimes S_H^{i-j}$ with $i, j \in \mathbb{N}$ are in M_2, and it is not hard to see that in fact $M_2 = W^*(f_{ij}, (1 \otimes S_H^2))$. An easy calculation further shows that $\psi(f_{ij}(1 \otimes S_H^2)^n) = (\omega_q \otimes \tau)(e_{ij} \otimes S^n)$. Hence the assignment

$$f_{ij}(1 \otimes S_H^2)^n \to e_{ij} \otimes S^n$$

extends (uniquely) to an isomorphism between the W*-probability spaces (M_2, φ_2) and $(B(l^2(\mathbb{N}))\bar{\otimes}\mathscr{L}(\mathbb{Z}), \omega_q \otimes \tau)$. □

Lemma 2.4. *We have that* $(M, \phi) := (\mathscr{L}(\mathbb{Z}), \tau) * (\mathscr{L}(\mathbb{Z})\bar{\otimes}B(l^2(\mathbb{N})), \tau \otimes \omega_q)$ *is a free Araki-Woods factor at parameter* q^2.

Proof. The proof is similar to the one of Theorem 3.1 in [2]. Denote $(N, \theta) = (\mathscr{L}(\mathbb{Z}), \tau) * (B(l^2(\mathbb{N})), \omega_q)$, and denote $\phi_0 = \frac{1}{1-q^2}\phi$ and $\theta_0 = \frac{1}{1-q^2}\theta$. Then by Proposition 3.10 of [2], we will have that

$$(e_{00}Me_{00}, \phi_0) \cong (\mathscr{L}(\mathbb{Z}), \tau) * (e_{00}Ne_{00}, \theta_0).$$

By Proposition 2.7 in [2] (which is based on the proof of Theorem 5.4 and Proposition 6.3 in [3]) and the remark before it, we know that $(e_{00}Ne_{00}, \theta_0)$ as well as $(N, \theta) \cong (e_{00}Ne_{00}, \theta_0)\bar{\otimes}(B(l^2(\mathbb{N})), \omega_q)$ are free Araki-Woods factors at parameter q^2. By the free absorption property ([3], Corollary 5.5), $(e_{00}Me_{00}, \phi_0)$ is a free Araki-Woods factor at parameter q^2, and hence also

$$(M, \phi) \cong (e_{00}Me_{00}, \phi_0)\bar{\otimes}(B(l^2(\mathbb{N})), \omega_q)$$

is a free Araki-Woods factor at parameter q^2.

□

Proof (of Theorem 2.1). By Lemmas 2.2 and 2.3, $(\mathscr{L}^\infty(G), \varphi)$ is isomorphic to the free product of $(\mathscr{L}(\mathbb{Z}), \tau)$ with $(B(l^2(\mathbb{N}))\bar{\otimes}\mathscr{L}(\mathbb{Z}), \omega_q \otimes \tau)$, which by Lemma 2.4 is a free Araki-Woods factor at parameter q^2. □

Acknowledgements: The motivation for this paper comes from a question posed by Stefaan Vaes concerning the validity of Theorem 2.1.

References

1. T. Banica, *Le groupe quantique compact libre* $U(n)$, Comm. Math. Phys. **190** (1997), 143–172.
2. C. Houdayer, *On some free products of von Neumann algebras which are free Araki-Woods factors*, Int. Math. Res. Notices **2007** (2007), article ID rnm098, 21 pages.
3. D. Shlyakhtenko, *Free quasi-free states*, Pacific J. Math. **177** (2) (1997), 329–368.
4. A. Van Daele and S.Z. Wang, *Universal quantum groups*, Int. J. Math. **7** (2) (1996), 255–264.
5. D.V. Voiculescu, *Circular and semicircular systems and free product factors, Operator Algebras, Unitary Representations*, Enveloping Algebras and Invariant Theory, Progress in Mathematics **92**, Birkhäuser, Boston (1990).
6. S.L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), 613–665.
7. S.L. Woronowicz, *Twisted* $SU(2)$ *group. An example of a non-commutative differential calculus*, Publ. RIMS, Kyoto University **23** (1987), 117–181.

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