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BANACH–SAKS PROPERTIES OF C^*-ALGEBRAS AND HILBERT C^*-MODULES

MICHAEL FRANK[1]* AND ALEXANDER A. PAVLOV[2]

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Abstract. The investigation of C^*-algebras and Hilbert C^*-modules with respect to the classical, the weak and the uniform weak Banach–Saks properties is completed giving a full picture, in particular in the non-unital cases. This way some open questions by M. Kusuda and C.-H. Chu are answered. Criteria and structural characterizations are given. In particular, the weak and the uniform weak Banach–Saks property turn out to be invariant under strong Morita equivalence for non-unital C^*-algebras.

1. Introduction

The study of Banach–Saks type properties of Banach spaces was initiated by S. Banach and S. Saks in [6] in 1930. They focused on Banach spaces of type $L^p([0,1])$ with $1 < p < \infty$. In the sequel a number of case studies appeared. Concrete results were discovered e.g. for Banach spaces of type $C(X)$ where X has been assumed to be a compact metric space (N. R. Farnum, [13]), for symmetric sequence spaces and for certain compact operator algebras over them (J. Arazy, [1]), for commutative and non-commutative C^*-algebras deriving criterions (C.-H. Chu, [10], M. Kusuda, [19, 21, 22]), for Banach spaces of vector-valued functions (C. Nuñez, [24]), for Hilbert C^*-modules (M. Kusuda, [19, 22]), and for other types of Banach spaces. Concise expositions of known results for Banach

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* Corresponding author.

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spaces have been published e.g. by J. Diestel ([11, 12]). Related investigations can be found in [18, 25, 27].

In the present paper we are interested in Hilbert C^*-modules over (non-unital, in general) C^*-algebras and in C^*-algebras as particular classes of examples for the study of various Banach–Saks type properties of Banach spaces. We are going to demonstrate the overall invariance of three types of Banach–Saks properties in the context of strong Morita equivalence in this context, in particular in the still incompletely understood case of non-unital C^*-algebras of coefficients of full Hilbert C^*-modules. For the transfer of properties from the non-unital to the unital case , and vice versa, we describe a general new method for Hilbert C^*-modules. We get the full picture in the cases of the classical Banach–Saks property, the weak and the uniformly weak Banach–Saks property. Also, we are able to describe the inner structure of C^*-algebras and of Hilbert C^*-modules with these properties following ideas by C.-H. Chu ([10]).

In the seek for good classes of coefficients of Hilbert C^*-modules the class of C^*-algebras of compact operators (i.e. of dual C^*-algebras) has been emphasized several times, [15, 16, 17]. In the present paper it is emphasized again as one of the invariant with respect to strong Morita equivalence classes with weak and uniformly weak Banach–Saks property. However, there are other invariant with respect to strong Morita equivalence classes of C^*-algebras built from this class by exact sequences. So it would be interesting to reveal more about the properties of Hilbert C^*-modules over these other classes of C^*-algebras in the future.

Another motivation comes from general properties of unbounded regular operators on certain classes of Hilbert C^*-modules, because they seem to be related to special kinds of convergence properties of nets and sequences in classes of Hilbert C^*-modules over C^*-algebras of compact operators (i.e. over dual C^*-algebras), cf. [16, 17]. The convergence conditions which serve as definitions of weak and uniformly weak Banach–Saks properties might give hints for a geometrical understanding of the background of the theory of unbounded modular operators.

2. Preliminaries

C^*-algebras can be faithfully $*$-represented as norm-closed $*$-subalgebras of sets of all bounded linear operators on fixed Hilbert spaces by the Gel'fand-Naimark-Segal theorem. We use italics for their denotation like A. The symbol A^{**} is reserved for the bidual Banach space of A, a W*-algebra. The multiplier algebra $M(A)$ of a C^*-algebra A can be defined by $\mathrm{M}(A) = \{a \in A^{**} : aA \subseteq A, Aa \subseteq A\}$, [26].

A (left) *pre-Hilbert C^*-module* over a (not necessarily unital) C^*-algebra A is a (left) A-module E equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$, which is A-linear in the first variable and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, x \rangle \geq 0 \text{ with equality if and only if } x = 0 .$$

We always suppose that the linear structures of A and E are compatible. A pre-Hilbert A-module E is called a *Hilbert A-module* if E is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. A Hilbert A-module is *full* if the range of the A-valued inner product on E is dense in A. If E, F are two Hilbert A-modules then the set of all ordered pairs of elements $E \oplus F$ from E and F is a Hilbert A-module with respect to the A-valued inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_E + \langle y_1, y_2 \rangle_F$. It is called the direct *orthogonal sum of* E *and* F.

Beside the Hilbert A-modules we shall consider A-linear bounded operators T on them. The set $End_A(E)$ of all bounded module operators on E forms a Banach algebra, whereas the set $End_A^*(E)$ of all bounded module operators which possess an adjoint operator inside $End_A(E)$ has the structure of a unital C^*-algebra. Note that these two sets do not coincide in general. An important subset of $End_A^*(E)$ is the set $K_A(E)$ of *"compact"* operators, which is defined as the norm-closure of the set $K_A^0(E)$ of all finite linear combinations of the specific operators

$$\{\theta_{x,y} \in End_A(E) : x, y \in E\,,\ \theta_{x,y}(z) = \langle z, x \rangle y \quad \text{for every} \quad z \in E\}.$$

It is a C^*-subalgebra and a two-sided ideal of $End_A^*(E)$, and the C^*-algebra $End_A^*(E)$ can be isometrically identified with the multiplier algebra of $K_A(E)$ ([28]). Note, that E is a right (full!) Hilbert $K_A(E)$-module at the same time, with $K_A(E)$-valued inner product $\langle x, y \rangle_{op} = \theta_{x,y}$ for $x, y \in E$. If E is a full Hilbert A-module then the picture is symmetric - the C^*-algebras of coefficients A and $K_A(E)$ have equal rights with respect to E.

Two C^*-algebras A and B are *strongly Morita equivalent* if there exists a left full Hilbert A-module E which is a right full Hilbert B-module at the same time, with the properties (i) $\langle x, y \rangle_A z = x \langle y, z \rangle_B$ for any $x, y, z \in E$, (ii) A and B act as the module operator C^*-algebras of "compact" B-linear and A-linear operators on E respectively. In this situation E is called an *A-B imprimitivity bimodule*. For the A-B imprimitivity bimodule E there exists a related C^*-algebra $L = K_A(A \oplus E)$ over the orthogonal sum $A \oplus E$ of the Hilbert A-modules A and E. The C^*-algebra L is called the *linking algebra*, cf. [7, 9]. The multiplier algebra $M(L)$ contains two orthogonal projections p, q such that $1_{M(L)} = p + q$, $pLp = K_A(A) = A$, $qLq = K_A(E) = B$ and $qLp = K_A(A, E) = E$ and $pLq = K_A(E, A) = \tilde{E}$ - the dual to E B-A imprimitivity bimodule, cf. [9, §2]. So one may write

$$L = \begin{pmatrix} A & E \\ \tilde{E} & B \end{pmatrix} .$$

3. A general construction

In the present section we are going to construct canonical extensions for any full Hilbert C^*-module E over a non-unital C^*-algebra A that are full Hilbert B-modules E_B for a given C^*-algebra $A \subset B \subseteq M(A)$ with a canonical A-linear isometric embedding $\Gamma : E \to E_B$, where $AE_B \equiv \Gamma(E)$, and with $*$-isomorphic C^*-algebras of adjointable bounded module operators $End_A^*(E) \equiv End_B^*(E_B) \equiv End_{M(A)}^*(E_{M(A)})$ which are identified by the respective operator extensions of bounded module operators via Γ, cf. [5, Thm. 2.3]. In particular, we associate to every Hilbert A-module E over a non-unital C^*-algebra A a canonical Hilbert A_1-module $E_{A_1} := E_c$ over its unitization A_1. It will be the right object to solve the open problems with the investigation of various Banach–Saks properties of Hilbert C^*-modules and C^*-algebras in the non-unital case.

We rely on the respective canonical construction of a multiplier module $E_{M(A)} := E_d$ for any full Hilbert C^*-module E introduced by D. Bakić and B. Guljaš in [4, 5]. Let E_d denote the Hilbert C^*-module $End^*_A(A, E)$ over the multiplier algebra $M(A)$ of A consisting of all adjointable bounded A-linear maps from A into E. Its $M(A)$-valued inner product is defined by the formula $\langle x, y \rangle = x^*y$. In [3, Thm. 1.2] E_d is proved to be the largest essential extension of E in full analogue to the respective property of multiplier algebras for C^*-algebras. Moreover, the map $\Gamma : E \to E_d$ defined by $a\Gamma(x) = ax$ with $a \in A$, $x \in E$ is an isometric A-linear embedding, and the image $\Gamma(E)$ coincides with the subset AE_d. The sets of adjointable bounded C^*-linear operators on both E_d and on $\Gamma(E)$ coincide, i.e. $End^*_A(E) \equiv End^*_{M(A)}(E_d)$. The Hilbert $M(A)$-module E_d is a full Hilbert $M(A)$-module in case E has been a full Hilbert A-module. Now, define

$$E_{A_1} := E_c := \{x \in E_d : \langle x, x \rangle \in A_1\} ,$$

$$E_B := \{x \in E_d : \langle x, x \rangle \in B\} .$$

Obviously, the sets $E_B \subseteq E_d$ are invariant under the action of $B \subseteq M(A)$. Furthermore, for any positive functional $f : M(A) \to \mathbb{C}$ vanishing on $B \subseteq M(A)$ the bilinear form $f(\langle ., . \rangle)$ is a semi-inner product on E_d. Therefore, the triangle inequality gives

$$f(\langle ax + by, ax + by \rangle)^{1/2} \leq f(\langle ax, ax \rangle)^{1/2} + f(\langle by, by \rangle)^{1/2}$$

for any $a, b \in B$, $x, y \in E_c$. Since the set of all positive functionals on $M(A)$ that vanish on $B \subseteq M(A)$ characterizes precisely elements of the subset B in $M(A)$ the set E_B turns out to be a B-module. So E_B is a Hilbert B-module which contains the isometric copy $\Gamma(E)$ of the Hilbert A-module E in such a way that $AE_B = \Gamma(E)$. In terms of [3, Def. 1.1] the triple (E_B, B, Γ) is an essential extension of the Hilbert A-module E.

Denote by $\pi : B \to B/A$ and by $q : E_B \to E_B/E$ the quotient maps. The left action of the C^*-algebra B/A on the B/A-module E_B/E is defined by $\pi(a)q(x) = q(ax)$ for any $a \in B$, $x \in E_B$, cf. [5, Def. 1.4]. Moreover, E_B/E can be equipped with a B/A-valued inner product setting $\langle q(x), q(y) \rangle = \pi(\langle x, y \rangle)$ for any $x, y \in E_B$, turning E_B/E into a Hilbert B/A-module, [5, Thm. 1.6].

Now, let us resort to the case of $B = A_1$, the unitization of A. Consider the exact sequence $0 \to E \to E_c \to E_c/E \to 0$ of Hilbert C^*-modules. In this final step we shall prove that the Banach space E_c always splits into a direct sum of the Banach spaces E and an isometric copy of the Hilbert space E_c/E. Consider the short exact sequence

$$0 \to E \to E_c \to E_c/E \to 0$$

and an orthonormal basis $\{e_\alpha : \alpha \in I\}$ of the Hilbert space E_c/E. Let $\rho : A_1 \to A$ be the linear bounded map which is the projection onto the first summand in the direct sum decomposition $A_1 = A + \mathbb{C}1$. Define a candidate for a new norm on E_c setting

$$\|x\|_c := \|\rho(\langle x, x \rangle_{E_c})\|_A^{1/2} + |(\mathrm{id}_{A_1} - \rho)(\langle x, x \rangle_{E_c})|^{1/2} .$$

The only non-trivial part is to demonstrate subadditivity of $\|\cdot\|_c$. Since the second summand is generated by a single state on A_1 it fulfils the triangle inequality. So we focus on the first summand. Let f be a state on A. Then $f \circ \rho$ is a state on A_1, and $f(\rho(\langle \cdot, \cdot \rangle_{E_c}))^{1/2}$ is a semi-norm on E_c fulfilling the triangle inequality. For given elements $x, y \in E_c$ fix a (existing by [8, Lemma 2.3.23]) state f_0 on A such that $f_0(\rho(\langle x+y, x+y \rangle_{E_c}))^{1/2} = \|\rho(\langle x+y, x+y \rangle_{E_c})^{1/2}\|$. Then

$$\begin{aligned}
\|\rho(\langle x+y, x+y \rangle_{E_c})\|^{1/2} &= \|\rho(\langle x+y, x+y \rangle_{E_c})^{1/2}\| \\
&= f_0(\rho(\langle x+y, x+y \rangle_{E_c}))^{1/2} \\
&\leq f_0(\rho(\langle x, x \rangle_{E_c}))^{1/2} + f_0(\rho(\langle y, y \rangle_{E_c}))^{1/2} \\
&\leq \|\rho(\langle x, x \rangle_{E_c})^{1/2}\| + \|\rho(\langle y, y \rangle_{E_c})^{1/2}\| \\
&= \|\rho(\langle x, x \rangle_{E_c})\|^{1/2} + \|\rho(\langle y, y \rangle_{E_c})\|^{1/2}
\end{aligned}$$

So the mapping $\|\cdot\|_c$ is really a norm the equivalence of which to the Hilbert norm on E_c follows from the way of construction of it as a special norm on A_1 equivalent to the C^*-norm on A_1, combined with the A_1-valued inner product on E_c.

For any $\alpha \in I$ the set $\mathcal{M}_\alpha = \{y \in E_c : q(y) = e_\alpha\}$ is not empty and, moreover, the difference of any two elements of $\mathcal{M}_\alpha$ belongs to $E_c A = \Gamma(E) \subseteq E_c$, i.e. these sets are sufficiently rich. Now, for any $\alpha \in I$ select an element $y_\alpha \in \mathcal{M}_\alpha$ such that $q(y_\alpha) = e_\alpha$. By construction

$$\langle e_\alpha, e_\beta \rangle_{E_c/E} = (\mathrm{id}_{A_1} - \rho)(\langle y_\alpha, y_\beta \rangle_{E_c}) = \pi(\langle y_\alpha, y_\beta \rangle_{E_c}) = \langle q(y_\alpha), q(y_\beta) \rangle_{E_c/E}$$

for any $\alpha, \beta \in I$. Therefore, fixing an index $\alpha \in I$, we have

$$\begin{aligned}
1 &= \langle e_\alpha, e_\alpha \rangle_{E_c/E}^{1/2} \\
&= \inf_{y \in \mathcal{M}_\alpha} \|y\|_c \\
&= \inf_{z \in \Gamma(E)} \|y_\alpha - z\|_c \\
&= \inf_{z \in \Gamma(E)} \|\rho(\langle y_\alpha - z, y_\alpha - z \rangle_{E_c})\|^{1/2} + \langle e_\alpha, e_\alpha \rangle_{E_c/E}^{1/2}
\end{aligned}$$

forcing $\inf_{z \in \Gamma(E)} \|\rho(\langle y_\alpha - z, y_\alpha - z \rangle_{E_c})\| = 0$. Thus, there has to exist a sequence $\{z_{\alpha,k} : k \in \mathbb{N}\} \in \Gamma(E) \subseteq E_c$ with the property $\lim_{k \to \infty} \|\rho(\langle y_\alpha - z_{\alpha,k}, y_\alpha - z_{\alpha,k} \rangle_{E_c})\| = 0$. In other words, the sequence $\{z_{\alpha,k} : k \in \mathbb{N}\} \in \Gamma(E)$ is a Cauchy sequence, and since $\Gamma(E)$ is complete in E_c there exists a norm-limit z_α of $\{z_{\alpha,k}\}$ inside $\Gamma(E) \subseteq E_c$. However, $\rho(\langle y_\alpha - z_\alpha, y_\alpha - z_\alpha \rangle) = 0$ in A. So for every element e_α of the selected orthonormal basis of E_c/E we obtain an element $y'_\alpha \in \mathcal{M}_\alpha$ such that $\rho(\langle y'_\alpha, y'_\alpha \rangle_{E_c}) = 0$ and $(\mathrm{id}_{A_1} - \rho)(\langle y'_\alpha, y'_\alpha \rangle_{E_c}) = \langle e_\alpha, e_\alpha \rangle_{E_c/E}$. Now, consider the linear subspace H_c of E_c which arises as the norm-closed linear hull of the selected elements $\{y'_\alpha : \alpha \in I\} \subset E_c$. By construction, $E_c = \Gamma(E) + H_c$ and H_c is isometrically isomorphic to the Hilbert space E_c/E, since $\pi(\langle y, y \rangle_{E_c}) = \langle q(y), q(y) \rangle_{E_c/E}$ for any $y \in H_c$ by construction.

It is not clear to us, whether the splitting works always for general C^*-algebras $A \subset B \subseteq M(A)$ and general Hilbert A-modules E, or not, since the Hilbert

B/A-module E_B/E might not contain neither orthonormal basises nor module frames, cf. [23]. Here some further investigations have to be made in the future.

We would like to shed some light on the construction of the Hilbert A_1-module E_c from the Hilbert A-module E in the case of non-unital C^*-algebras A and full Hilbert C^*-modules. If $A = E$ it is just the construction of an unitization of the C^*-algebra A, i.e. $A_1 = E_c$ as a Hilbert A_1-module. However, in more general situations the construction of E_c from E has the character of a maximally possible extension of E keeping the coefficients of the extended C^*-valued inner product still in A_1. Full Hilbert A-modules E give rise to full Hilbert A_1-modules E_c by this construction.

Example 3.1. Let A be a non-unital C^*-algebra and E be a full Hilbert A-module. The key idea of the previous proof was to complete E to the extend to force the extension to be a full Hilbert A_1-module, where $A_1 = A + \mathbb{C}1$. The first idea which comes to mind is an operation like "adding an identity to a Hilbert C^*-module" analogous to the existing minimal C^*-extension of C^*-algebras which is unique up to isometric $*$-isomorphisms. However, the construction of E_c from E celebrated in the previous proof has more of a minimax principle as the following example shows. For the C^*-algebras $K(l_2)$ and $B(l_2)$ of all compact and bounded linear operators on the separable Hilbert space l_2, respectively, set

$$\begin{aligned} A &= \begin{pmatrix} K(l_2) & 0 & 0 \\ 0 & B(l_2) & 0 \\ 0 & 0 & B(l_2) \end{pmatrix}, \\ E &= \begin{pmatrix} K(l_2) & 0 & 0 \\ 0 & B(l_2) & 0 \\ 0 & 0 & K(l_2) \end{pmatrix} \oplus \begin{pmatrix} K(l_2) & 0 & 0 \\ 0 & K(l_2) & 0 \\ 0 & 0 & B(l_2) \end{pmatrix}. \end{aligned}$$

Two corresponding minimal extensions of the Hilbert A-module E to Hilbert A_1-modules are

$$E_1 = \begin{pmatrix} K(l_2)+\mathbb{C}1 & 0 & 0 \\ 0 & B(l_2) & 0 \\ 0 & 0 & K(l_2) \end{pmatrix} \oplus \begin{pmatrix} K(l_2) & 0 & 0 \\ 0 & K(l_2) & 0 \\ 0 & 0 & B(l_2) \end{pmatrix},$$

$$E_2 = \begin{pmatrix} K(l_2) & 0 & 0 \\ 0 & B(l_2) & 0 \\ 0 & 0 & K(l_2) \end{pmatrix} \oplus \begin{pmatrix} K(l_2)+\mathbb{C}1 & 0 & 0 \\ 0 & K(l_2) & 0 \\ 0 & 0 & B(l_2) \end{pmatrix}.$$

Obviously, E_1 and E_2 are minimal extensions of the sought kind, however they are non-isomorphic as Hilbert A_1-modules. So the construction of

$$E_c = \begin{pmatrix} K(l_2)+\mathbb{C}1 & 0 & 0 \\ 0 & B(l_2) & 0 \\ 0 & 0 & K(l_2) \end{pmatrix} \oplus \begin{pmatrix} K(l_2)+\mathbb{C}1 & 0 & 0 \\ 0 & K(l_2) & 0 \\ 0 & 0 & B(l_2) \end{pmatrix}$$

gives a more correct result, the maximum of all possible minimal essential extensions of the Hilbert A-module E to a Hilbert A_1-module. The Hilbert space

$H_c \subseteq E_c$ constructed in the proof above equals to

$$H_c = \begin{pmatrix} \mathbb{C}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} \mathbb{C}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the present example.

4. The classical Banach–Saks property

The aim of the reminder of the present paper is to complete the classification of C^*-algebras and Hilbert C^*-modules which are Banach spaces with (classical, weak, uniform weak) Banach–Saks properties. Special emphasis is put on the invariance of Banach–Saks properties with respect to strong Morita equivalence. We focus on the still open non-unital case since the appropriate results for the unital case have been discovered by Cho-Ho Chu and M. Kusuda in [10, 19, 20, 21, 22].

A Banach space E has the *Banach–Saks property* if every bounded sequence $\{x_n\}_n \subset E$ has a subsequence $\{x_{n(k)}\}_k$ such that the derived from it sequence of partial arithmetic means converges in norm, i.e.

$$\lim_{k\to\infty} \left\| \frac{(x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)})}{k} - y \right\| = 0$$

with some $y \in E$. It is known that Banach spaces E with the Banach–Saks property have to be reflexive as normed spaces, [11, p. 85]. Therefore, C^*-algebras with the Banach–Saks property have to be finite-dimensional linear spaces, i.e. a finite direct sum of unital matrix algebras ([22, Lemma 3.1]). The following proposition has been proved by M. Kusuda for the unital case, cf. [22, Thm. 3.6]:

Proposition 4.1. *Let A be a (non-unital, in general) C^*-algebra and E be a full Hilbert A-module. Suppose, that E has the Banach–Saks property. Then A has to be finite-dimensional as a linear space, i.e. A is a finite direct sum of unital matrix algebras. In particular, any full Hilbert A-module over a non-trivial non-unital C^*-algebra A does not possess the Banach–Saks property, neither such C^*-algebras A themselves.*

Proof. Let A be a non-unital C^*-algebra and E be a full Hilbert A-module. Suppose, E has the Banach–Saks property. Construct the Hilbert A_1-module E_c from A and from E. By the results of the previous section the Hilbert A_1-module E_c can be written as a direct sum of the Banach subspace $\Gamma(E)$ of E_c and of a Banach space H_c which is isometrically isomorphic to the Hilbert space E_c/E. Both the isometric copy $\Gamma(E) \subset E_c$ and the Hilbert space H_c have the Banach–Saks property, by supposition or as a matter of fact, respectively. So the (full) Hilbert A_1-module E_c has the Banach–Saks property, too, since it is the direct sum of two Banach spaces with that property. By [22, Thm. 3.6] the C^*-algebra A_1 has to be finite-dimensional as a Banach space, i.e. it has to be a matrix algebra with finite center. However, for non-trivial non-unital C^*-algebras A their unitization A_1 is never a finite-dimensional C^*-algebra, a contradiction. □

Corollary 4.2. *Let A be a (non-unital, in general) C^*-algebra. Then A has the Banach–Saks property if and only if A is reflexive as a Banach space.*

The fact follows immediately from the general representation theory of C^*-algebras because a C^*-algebra is reflexive as a Banach space if and only if it is finite-dimensional as a linear space. It is non-trivial since for Banach spaces reflexivity does not imply the Banach–Saks property, in general.

Corollary 4.3. *Let A be a C^*-algebra with Banach–Saks property and E be a Hilbert A-module. Then E has the Banach–Saks property.*

Proof. We know that A has to be finite-dimensional and unital. Since any two-sided ideal of A is finite-dimensional, too, and has the Banach–Saks property, we can assume E to be full, without loss of generality. Moreover, the C^*-algebra admits a faithful trace functional $tr(\cdot)$, and the norms $\|\langle .,. \rangle\|^{1/2}$ and $tr(\langle .,. \rangle)^{1/2}$ are equivalent on E. So E admits a $\mathbb{C}$-valued inner product $tr(\langle .,. \rangle)$ and, hence, the structure of a Hilbert space. However, Hilbert spaces are known to possess the Banach–Saks property, so does E. □

5. The weak Banach–Saks property

More interesting is the weak Banach–Saks property which is defined in the following way: if for any given weakly null sequence $\{x_n\}_n$ of a Banach space E, one can extract a subsequence $\{x_{n(k)}\}_k$ such that the derived from it sequence of partial arithmetic means converges in norm to zero, i.e.

$$\lim_{k\to\infty} \left\| \frac{(x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)})}{k} \right\| = 0 \, ,$$

then E is said to admit the *weak Banach–Saks property*. Note, that the weak Banach–Saks property inherits to any (closed) subspace of a Banach space with weak Banach–Saks property by definition. Beside this, if A is a non-unital C^*-algebra and $A_1 = A + \mathbb{C}1$ is its unitization, then A has the weak Banach–Saks property if and only if A_1 has the weak Banach–Saks property, [10]. For C^*-algebras and Hilbert C^*-modules as classes of Banach spaces we prove the following fact relying on a key result by M. Kusuda [22, Thm. 2.2] and on a new technique for a certain standard extension of full Hilbert C^*-modules over non-unital C^*-algebras.:

Theorem 5.1. *Let A and B be two strongly Morita equivalent C^*-algebras and E be an A-B imprimitivity bimodule. The following four conditions are equivalent:*

(i) A has the weak Banach–Saks property.
(ii) B has the weak Banach–Saks property.
(iii) E has the weak Banach–Saks property.
(iv) L has the weak Banach–Saks property.

Proof. By [22, Thm. 2.2] the first three conditions are equivalent if either A or B are unital. If neither A nor B are unital then the conditions (i) and (ii) are equivalent and imply condition (iii). So we are going to show that for non-unital A and B condition (iii) implies both conditions (i) and (ii).

Let A be a C^*-algebra and E be a full Hilbert A-module with the weak Banach–Saks property. Consider the derived Hilbert A_1-module E_c of E, cf. third section. As demonstrated E_c can be decomposed into the direct sum of two Banach spaces, $\Gamma(E) \subset E_c$ isometrically isomorphic to the Hilbert A-module E and H_c isometrically isomorphic to the Hilbert space E_c/E. Both these summands admit the weak Banach–Saks property by supposition or by the equivalence of the conditions (i) and (iii) above for the C^*-algebra of all complex numbers $\mathbb{C}$, so does their direct sum E_c. Because E_c is a full Hilbert A_1-module, the already proven equivalence (i)$\equiv$(iii) for unital C^*-algebras forces A_1 to admit the weak Banach–Saks property, and so its Banach subspace A has the weak Banach–Saks property, too, what is to demonstrate.

The equivalent conditions (i)-(iii) imply L to admit the weak Banach–Saks property since L has a block structure consisting of these building blocks and of their (anti-)isomorphic copies. Conversely, if L has the weak Banach–Saks property then each of its linear subspaces admits the same property. □

6. The uniform weak Banach–Saks property and structure theorems

Relying on the classical results by N. R. Farnum [13] and by C.-H. Chu [10] we can derive a number of concrete results from Theorem 5.1. The key role is played by dual C^*-algebras, i.e. by C^*-algebras that admit a faithful $*$-representation in some C^*-algebra $K_{\mathbb{C}}(H)$ of all linear compact operators on some Hilbert space H, cf. [2]. C.-H. Chu proved in [10] that a C^*-algebra A has the weak Banach–Saks property if and only if A admits a finite chain $\{I_i\}$ of two-sided norm-closed ideals such that $I_1 = \{0\}$, $I_n = A$ and any I_{i+1}/I_i, $i = 0, ..., n-1$, is a dual C^*-algebra. So the class of dual C^*-algebras is one large class of C^*-algebras with weak Banach–Saks property, cf. [16, 17]. Note, that the class of dual C^*-algebras is invariant under strong Morita equivalence. Another class of such C^*-algebras can be constructed by taking finite block-diagonal direct sums of C^*-algebras with weak Banach–Saks property. However, there are far more C^*-algebras with weak Banach–Saks property, even unital ones, which can be easily constructed:

Example 6.1. Let A be any non-unital dual C^*-algebra, for example $A = c_0$ or $A = K_{\mathbb{C}}(l_2)$. Then the unitization A_1 of A always serves as an example of a non-dual, unital C^*-algebra with weak Banach–Saks property. Indeed, $\{0\} \triangleleft A \triangleleft A_1 = A + \mathbb{C} \cdot 1$. The example is non-trivial since the short exact sequence $0 \to A \to A_1 \to A_1/A = \mathbb{C} \to 0$ does not split as an exact sequence of C^*-algebras for any non-unital dual C^*-algebra A. (It always splits as an exact sequence of Banach spaces.)

A third Banach–Saks type property of Banach spaces has been introduced by C. Nuñez in [24]. A Banach space E has the *uniform weak Banach–Saks property* if there is a null sequence $\{\delta_n\}_n$ of positive real numbers such that, for any weakly null sequence $\{x_n\}_n$ in E with uniform bound $\|x_n\| \leq 1$ and for any natural number k, there exist natural numbers $n(1) < n(2) < \cdots < n(k)$ such that

$$\frac{\|x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)}\|}{k} < \delta_k \,.$$

C.-H. Chu has shown in [10, Thm. 2] that C^*-algebras are Banach spaces for which the uniform weak and the weak Banach–Saks properties are equivalent. In [22, Thm. 2.2, Cor. 2.3] M. Kusuda found that for full Hilbert C^*-modules over unital C^*-algebras both these properties are equivalent. Moreover, for full Hilbert C^*-modules over C^*-algebras with weak Banach–Saks property again both these properties hold at the same time, cf. [19, Thm. 2.3] and [22, Thm. 2.2]. So we can formulate an analog to Theorem 5.1:

Theorem 6.2. *Let A and B be two strongly Morita equivalent C^*-algebras and E be an A-B imprimitivity bimodule. The following four conditions are equivalent:*

- *(i) A has the uniform weak Banach–Saks property.*
- *(ii) B has the uniform weak Banach–Saks property.*
- *(iii) E has the uniform weak Banach–Saks property.*
- *(iv) L has the uniform weak Banach–Saks property.*

In particular, conditions (i)-(iv) hold in case either A or B or E or L have the weak Banach–Saks property. Conversely, either of conditions (i)-(iv) implies A, B, E and L to have the weak Banach–Saks property.

Theorem 6.2 gives the opportunity to describe the inner structure of Hilbert C^*-modules with the weak or uniform weak Banach–Saks property.

Proposition 6.3. *Let A be a C^*-algebra and E be a full Hilbert A-module with the weak or uniform weak Banach–Saks property. Then there exist a finite sequence $\{E_i : i = 0, ..., l\}$ of norm-closed A-submodules of E and a sequence $\{I_i : i = 0, ..., l\}$ of two-sided norm-closed ideals of A such that*

- *(i) $I_l = A$, $I_{i-1} \subset I_i$ and I_{i-1} is a two-sided ideal of I_i for any $i = 1, ..., l$.*
- *(ii) The C^*-algebra I_0 and the factor C^*-algebras $\{I_i/I_{i-1} : i = 1, ..., l\}$ are dual C^*-algebras.*
- *(iii) $E_l = E$, $E_{i-1} \subset E_i$ and the Hilbert A-modules E_i are full Hilbert I_i-modules for any $i = 0, ..., l$. In particular, the values $\langle x, y\rangle$ belong to I_i for any $x \in E_i$ and any $y \in E_j$ with $j \geq i$, $i, j = 0, ..., l$. The factor modules E_i/E_{i-1} are Hilbert C^*-modules over the dual C^*-algebras I_i/I_{i-1}.*

Proof. Since the full Hilbert A-module E has the weak or the uniform weak Banach–Saks property, the C^*-algebra of coefficients has the same property by Theorem 6.2, as well as the linking algebra L. By [10] there exists a finite sequence $\{J_i : i = 0, ..., l\}$ of two-sided norm-closed ideals of L such that the C^*-algebra J_0 and the factor C^*-algebras J_i/J_{i-1}, $i = 1, ..., l$, are dual C^*-algebras, $J_{i-1} \subset J_i$, and J_{i-1} is a two-sided ideal of J_i for any $i = 1, ..., l$, and $J_l = L$. For the pair of orthogonal projections p, q associated to the linking algebra L of E set $E_i = pJ_iq$ and $I_i = pJ_ip$ for $i = 0, ..., l$. The sets I_i are C^*-algebras and the sets E_i are full Hilbert I_i-modules, $i = 0, ..., l$. The demonstration of the particular properties of these sets listed in the proposition above is an easy exercise. □

As a conclusion we have characterized two strongly Morita equivalent classes of C^*-algebras which admit the weak and uniform weak Banach–Saks property: the C^*-algebras of compact operators (i.e. the dual C^*-algebras) and the C^*-algebras constructed by finite chains of growing ideals the pairwise quotients of which are again C^*-algebras of compact operators. The latter class may be divided into a countable set of strongly Morita invariant subclasses by the length of the decomposing chains of ideals.

In a forthcoming paper we will study modular analogues of the Schur and of various types of Banach–Saks properties for Hilbert C^*-modules ([14]).

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[1] HOCHSCHULE FÜR TECHNIK, WIRTSCHAFT UND KULTUR (HTWK) LEIPZIG, FACHBEREICH IMN, PF 301166, D-04251 LEIPZIG, F.R. GERMANY
E-mail address: `mfrank@imn.htwk-leipzig.de`

[2] MOSCOW STATE UNIVERSITY, 119 922 MOSCOW, RUSSIA, AND UNIVERSITÀ DEGLI STUDI DI TRIESTE, I-34127 TRIESTE, ITALY
E-mail address: `axpavlov@mail.ru`