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ESSENTIALLY SLANT TOEPLITZ OPERATORS

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ABSTRACT. The notion of an essentially slant Toeplitz operator on the space L^2 is introduced and some of the properties of the set $\mathrm{ESTO}(L^2)$, the set of all essentially slant Toeplitz operators on L^2, are investigated. In particular the conditions under which the product of two operators in $\mathrm{ESTO}(L^2)$ is in $\mathrm{ESTO}(L^2)$ are discussed. The notion is generalized to kth-order essentially slant Toeplitz operators.

The notion of Toeplitz operators was introduced by O. Toeplitz [8] in the year 1911. Subsequently many researchers like Devinatz [4], Abrahamse [1], Barria and Halmos [3] came up with various generalizations of the notion of Toeplitz operators. The essential commutant of the unilateral forward shift has been the object of study for several years for its far reaching applications to various branches like probability, statistics, oscillation signal processing etc. Barria and Halmos [3] brought much attention to this set and mooted an idea of deriving ways to characterize completely this set. The essential commutant of the forward shift has sometimes been referred to as the set of essentially Toeplitz operators.

Ho [7], in the year 1995, began a systematic study of yet another class of operators having the property that the matrices of such operators with respect to the standard orthonormal basis could be obtained from those of Toeplitz operators just by eliminating every other row. Such operators were termed as slant Toeplitz operators [7]. Villemoes [9] associated the Besov regularity of solutions of the refinement equation with the spectral radius of an associated slant Toeplitz

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operator and Goodman, Micchelli and Ward [6] showed the connection between the spectral radii and conditions for the solutions of certain differential equations being in Lipschitz classes.

Ever since the introduction of the class of slant Toeplitz operators, the study has gained voluminous importance due to its multidirectional applications and hence it is desirable to consider those operators which behave essentially in the same manner as slant Toeplitz operators do.

Motivated by the work of Barria, Halmos and Ho, in this paper we introduce a new class of operators on the space L^2 called essentially slant Toeplitz operators and study some algebraic properties of this class of operators. The study is also carried to the counterpart of these operators on the space H^2. For the spaces L^2, H^2 and L^∞ one can see [5]. We begin with the following definitions:

Definition 1. A bounded linear operator A on the space H^2 is said to be an essentially Toeplitz operator if $T_z^*AT_z - A$ is a compact operator on H^2, where T_z denotes the Toeplitz operator on H^2 induced by z.

Definition 2. A slant Toeplitz operator on the space L^2 is an operator of the form WM_ϕ, where M_ϕ denotes the multiplication operator on L^2 induced by ϕ in L^∞ and W is defined on L^2 as

$$\left.\begin{array}{l} W(z^{2n}) = z^n \\ W(z^{2n-1}) = 0 \end{array}\right\} \forall\ n \in \mathbb{Z},$$

where $\{e_n : n \in \mathbb{Z}, e_n(z) = z^n\}$ denotes the standard orthonormal basis of L^2.

It is known that [7] an operator A on the space L^2 is a slant Toeplitz operator if and only if $M_zA = AM_{z^2}$, where M_z is the multiplication operator on L^2 induced by z.

1. Essentially slant Toeplitz operators on L^2

We introduce the following:

Definition 1.1. A bounded linear operator A on the space L^2 is said to be an essentially slant Toeplitz operator if $M_zA - AM_{z^2} = K$, for some compact operator K on L^2.

We denote the set of all essentially slant Toeplitz operators on L^2 by $\mathrm{ESTO}(L^2)$. Since the zero operator on L^2 is a compact operator, every slant Toeplitz operator on L^2 is trivially in $\mathrm{ESTO}(L^2)$. In fact, if T is any compact perturbation of a slant Toeplitz operator on L^2 then $T \in \mathrm{ESTO}(L^2)$. It is known that the only compact slant Toeplitz operator is the zero operator. Also, from the definition itself, every compact operator on L^2 is in $\mathrm{ESTO}(L^2)$. So if $\mathrm{STO}(L^2)$ denotes the set of all slant Toeplitz operators on L^2 and $\mathcal{K}$ denotes the ideal of all compact operators on L^2 then

$$\mathrm{STO}(L^2) \cap \mathcal{K} = \{0\}$$

and

$$\mathrm{ESTO}(L^2) \cap \mathcal{K} = \mathcal{K}\,.$$

Thus any non-zero compact operator on L^2 is an essentially slant Toeplitz operator but not a slant Toeplitz operator.

We now present an example of a non-compact essentially slant Toeplitz operator on L^2 which is not a slant Toeplitz operator:

Example 1.2. Let A on L^2 be defined as

$$Ae_n = \begin{cases} e_1 & \text{if } \ n = 0 \\ 0 & \text{if } \ n \neq 0, \ n \text{ is even} \\ e_m, \ \text{where } \ m = \left(\dfrac{n+1}{2}\right), & \text{if } \ n \text{ is odd} \end{cases}$$

where $e_n(z) = z^n \ \forall \ n \in \mathbb{Z}$. The matrix representation of A with respect to $\{e_n\}_{n\in\mathbb{Z}}$ is given by

$$\left[\begin{array}{ccccccccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}\right]$$

If W is defined on L^2 as $\left.\begin{array}{l} W(z^{2n}) = z^n \\ W(z^{2n-1}) = 0 \end{array}\right\} \forall \ n \in \mathbb{Z}$

and K is defined on L^2 as $Ke_n = \begin{cases} e_1 & \text{if } \ n = 0 \\ 0 & \text{otherwise} \end{cases}$

for all $n \in \mathbb{Z}$, then we can write

$$A = WM_z + K$$

It is clear that

$$M_zA - AM_{z^2} = M_zWM_z - WM_{z^3} + K'$$

where $K' \in \mathcal{K}$. Therefore $M_zA - AM_{z^2} = 0 + K' \in \mathcal{K}$. Hence $A \in \mathrm{ESTO}(L^2)$ but A is not a slant Toeplitz operator on L^2. Some basic properties of the set $\mathrm{ESTO}(L^2)$ are as follows

(i) $\mathrm{ESTO}(L^2)$ *is a norm-closed vector subspace of* $\mathcal{B}(L^2)$*, the set of all bounded linear operators on the space* L^2*.*

Proof. For $T_1, T_2 \in \mathrm{ESTO}(L^2)$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} & M_z(\alpha T_1 + \beta T_2) - (\alpha T_1 + \beta T_2)M_{z^2} \\ = \ & \alpha(M_zT_1 - T_1M_{z^2}) + \beta(M_zT_2 - T_2M_{z^2}) \in \mathcal{K}\,. \end{aligned}$$

Also, if for each n, T_n is in $\mathrm{ESTO}(L^2)$ and $T_n \to T$ uniformly in $\mathcal{B}(L^2)$ then $M_zT_n - T_nM_{z^2} \to M_zT - TM_{z^2}$ uniformly in $\mathcal{B}(L^2)$. Since $\mathcal{K}$ is uniformly closed it follows that $T \in \mathrm{ESTO}(L^2)$. Thus, $\mathrm{ESTO}(L^2)$ is a norm-closed vector subspace of $\mathcal{B}(L^2)$.

(ii) $\mathrm{ESTO}(L^2)$ *is not an algebra of operators on* L^2 since the product of two essentially slant Toeplitz operators on L^2 is not necessarily an essentially slant Toeplitz operator as is shown in the following:

Example 1.3. Let $A = B = WM_z + K$, where W and K are as defined in Example 1.2. Then $A, B \in \mathrm{ESTO}(L^2)$ but $C = AB \notin \mathrm{ESTO}(L^2)$ because

$$\begin{aligned} M_zC - CM_{z^2} &= M_zAB - ABM_{z^2} \\ &= M_z(WM_z)^2 - (WM_z)^2M_{z^2} \ (\mathrm{mod}\, \mathcal{K}). \end{aligned}$$

Therefore $M_zC - CM_{z^2} \in \mathcal{K}$ if and only if $M_z(WM_z)^2 - (WM_z)^2M_{z^2} \in \mathcal{K}$.
But

$$(M_z(WM_z)^2 - (WM_z)^2M_{z^2})e_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ e_2 & \text{if } n = 1 \\ -e_2 & \text{if } n = 3 \\ e_3 & \text{if } n = 5 \\ -e_3 & \text{if } n = 7 \\ e_4 & \text{if } n = 9 \\ \quad \vdots \end{cases}$$

Therefore $M_z(WM_z)^2 - (WM_z)^2M_{z^2} \notin \mathcal{K}$. Hence $C \notin \mathrm{ESTO}(L^2)$.

(iii) $\mathrm{ESTO}(L^2)$ *is not a self-adjoint set.*
For the operator $A = WM_z + K$ (as above) belongs to $\mathrm{ESTO}(L^2)$ and $A^* \notin \mathrm{ESTO}(L^2)$.

(iv) If $T_1, T_2 \in \mathrm{ESTO}(L^2)$ then $T_1T_2 \in \mathrm{ESTO}(L^2)$ if and only if $T_1M_zT_2 = T_1M_{z^2}T_2 (\mathrm{mod}\, \mathcal{K})$.
For if $T_1, T_2 \in \mathrm{ESTO}(L^2)$ then

$$\begin{aligned} M_zT_1T_2 - T_1T_2M_{z^2} &= T_1M_{z^2}T_2 - T_1T_2M_{z^2} \ (\mathrm{mod}\, \mathcal{K}) \\ &= T_1M_{z^2}T_2 - T_1M_zT_2 \ (\mathrm{mod}\, \mathcal{K}). \end{aligned}$$

Therefore, $T_1T_2 \in \mathrm{ESTO}(L^2)$ if and only if $T_1M_zT_2 = T_1M_{z^2}T_2 \ (\mathrm{mod}\, \mathcal{K})$.

(v) Let $A \in \mathrm{ESTO}(L^2)$ and $p \in \mathbb{N}$, $p > 1$. If $n(p)$ denotes the number of partitions of p as sum of two natural numbers, $p = m_i + n_i$ ($m_i, n_i \in \mathbb{N}; i = 1, 2, \ldots, n(p)$), $A^{m_i}, A^{n_i} \in \mathrm{ESTO}(L^2)$ then the following are equivalent:
(a) $A^p \in \mathrm{ESTO}(L^2)$
(b) $A^{m_i}M_zA^{n_i} = A^{m_i}M_{z^2}A^{n_i}$ $(\mathrm{mod}\, \mathcal{K})$, $i = 1, 2, \ldots, n(p)$
(c) $A^{n_i}M_zA^{m_i} = A^{n_i}M_{z^2}A^{m_i}$ $(\mathrm{mod}\, \mathcal{K})$, $i = 1, 2, \ldots, n(p)$
In addition to these properties, we have the following

Theorem 1.4. *If $T_1, T_2 \in \mathrm{ESTO}(L^2)$ such that either T_1 commutes essentially with M_z or T_2 commutes essentially with M_{z^2} then $T_1T_2 \in \mathrm{ESTO}(L^2)$.*

Proof. Let $T_1, T_2 \in \mathrm{ESTO}(L^2)$

Case (i): $T_1 M_z = M_z T_1 \pmod{\mathcal{K}}$

Then

$$\begin{aligned} M_z T_1 T_2 - T_1 T_2 M_{z^2} &= M_z T_1 T_2 - T_1 M_z T_2 \pmod{\mathcal{K}} \\ &= T_1 M_z T_2 - T_1 M_z T_2 \pmod{\mathcal{K}} \\ &= 0 \pmod{\mathcal{K}}. \end{aligned}$$

Therefore $T_1 T_2 \in \mathrm{ESTO}(L^2)$.

Case (ii): $T_2 M_{z^2} = M_{z^2} T_2 \pmod{\mathcal{K}}$

Then

$$\begin{aligned} M_z T_1 T_2 - T_1 T_2 M_{z^2} &= T_1 M_{z^2} T_2 - T_1 T_2 M_{z^2} \pmod{\mathcal{K}} \\ &= T_1 T_2 M_{z^2} - T_1 T_2 M_{z^2} \pmod{\mathcal{K}} \\ &= 0 \pmod{\mathcal{K}}. \end{aligned}$$

Therefore $T_1 T_2 \in \mathrm{ESTO}(L^2)$.

Remark 1.5. From the proof of the above theorem we obtain the following:

(i) If T_1 commutes essentially with M_z and $T_2 \in \mathrm{ESTO}(L^2)$ then $T_1 T_2 \in \mathrm{ESTO}(L^2)$.

(ii) If $T_1 \in \mathrm{ESTO}(L^2)$ and T_2 commutes essentially with M_{z^2} then $T_1 T_2 \in \mathrm{ESTO}(L^2)$.

As a consequence we have the following:

If M_ϕ is a multiplication operator on L^2 induced by ϕ in L^∞ and $A \in \mathrm{ESTO}(L^2)$ then AM_ϕ and $M_\phi A$ both are in $\mathrm{ESTO}(L^2)$.

Theorem 1.6. *If $A, A^* \in \mathrm{ESTO}(L^2)$ then $TA^* = A^*T^* \pmod{\mathcal{K}}$ where $T = M_z + M_{\bar{z}^2}$.*

Proof. Let $A, A^* \in \mathrm{ESTO}(L^2)$. Then

$$M_z A - AM_{z^2} = K_1 \tag{1.1}$$

$$M_z A^* - A^* M_{z^2} = K_2 \tag{1.2}$$

where $K_1, K_2 \in \mathcal{K}$. Taking adjoint on both the sides of (1.1) and subtracting (1.2) we have

$$(M_z + M_{\bar{z}^2})A^* - A^*(M_{z^2} + M_{\bar{z}}) = K\,, \quad \text{for some} \quad K \in \mathcal{K}\,.$$

Therefore $TA^* = A^*T^* \pmod{\mathcal{K}}$ where $T = M_z + M_{\bar{z}^2}$.

Corollary 1.7. *A necessary condition for any operator $A \in \mathrm{ESTO}(L^2)$ to be self adjoint is that TA is essentially self adjoint, where $T = M_z + M_{\bar{z}^2}$.*

2. Compressions of essentially slant Toeplitz operators

In 2001, Arora and Zegeye [10] obtained a characterization of the compression of a slant Toeplitz operator to H^2 as follows:

An operator B on H^2 is the compression of a slant Toeplitz operator to H^2 if and only if $B = T_z^* B T_{z^2}$, where T_z is the Toeplitz operator induced by z. Motivated by this we define the compression of an essentially slant Toeplitz operator to H^2 as follows:

Definition 2.1. An operator B on the space H^2 is termed as the compression of an essentially slant Toeplitz operator to H^2 if $B - T_z^* B T_{z^2} = K$, for some compact operator K on H^2.

As T_z is essentially unitary, we can equivalently give the definition in the following way:

An operator B on the space H^2 is the compression of an essentially slant Toeplitz operator to H^2 if $T_z B - B T_{z^2} = K$, for some compact operator K on H^2.

We denote the set of all compressions of essentially slant Toeplitz operators to H^2 by $\mathrm{ESTO}(H^2)$. Clearly if T is the compression of a slant Toeplitz operator to H^2 then $T \in \mathrm{ESTO}(H^2)$. The set $\mathrm{ESTO}(H^2)$ has the following properties:

(i) $\mathrm{ESTO}(H^2)$ is a norm-closed vector subspace of $\mathcal{B}(H^2)$.
(ii) $\mathrm{ESTO}(H^2)$ is not an algebra of operators on H^2.
(iii) $\mathrm{ESTO}(H^2)$ is not a self-adjoint set.
(iv) If $\mathcal{K}(H^2)$ denotes the space of all compact operators on H^2, then $\mathcal{K}(H^2) \cap \mathrm{ESTO}(H^2) = \mathcal{K}(H^2)$.
(v) If $A, B \in \mathrm{ESTO}(H^2)$ then $AB \in \mathrm{ESTO}(H^2)$ if and only if $AT_z B = AT_{z^2} B \ (\mathrm{mod}\, \mathcal{K}(H^2))$.
(vi) If $A, B \in \mathrm{ESTO}(H^2)$ such that either A commutes essentially with T_z or B commutes essentially with T_{z^2} then $AB \in \mathrm{ESTO}(H^2)$.
(vii) A necessary condition for an operator $A \in \mathrm{ESTO}(H^2)$ to be self adjoint is that TA is essentially self adjoint where $T = T_z + T_{\bar{z}^2}$.

Note. Using the fact that any two multiplication operators on L^2 commute,it has been observed in Remark 1.5 that if $A \in \mathrm{ESTO}(L^2)$ and M_ϕ is any multiplication operator on L^2 then AM_ϕ and $M_\phi A$ both are in $\mathrm{ESTO}(L^2)$. Although any two Toeplitz operators do not commute in general still we have an analogous result here as is shown in the following:

Theorem 2.2. *If T_ϕ is a Toeplitz operator on H^2 induced by symbol ϕ in L^∞ and $A \in \mathrm{ESTO}(H^2)$ then AT_ϕ and $T_\phi A$ both are in* $\mathrm{ESTO}(H^2)$.

Proof. Let T_ϕ be a Toeplitz operator on H^2 induced by ϕ in L^∞. Using the characterization of Toeplitz operators it is easy to see that the commutator of T_ϕ and T_z is compact. In fact for any positive integer n, the commutator of T_ϕ and T_{z^n} is a compact operator on H^2. Now let us suppose that A is in $\mathrm{ESTO}(H^2)$. Consider

$$\begin{aligned} T_z(T_\phi A) - (T_\phi A)T_z^2 &= T_\phi T_z A - T_\phi A T_z^2 (\mathrm{mod}\, \mathcal{K}) \\ &= T_\phi (T_z A - A T_{z^2}) (\mathrm{mod}\, \mathcal{K}) \in \mathcal{K}\,. \end{aligned}$$

Also

$$\begin{aligned} T_z(AT_\phi) - (AT_\phi)T_z^2 &= T_zAT_\phi - AT_z^2T_\phi (\bmod \mathcal{K}) \\ &= (T_zA - AT_z^2)T_\phi (\bmod \mathcal{K}) \in \mathcal{K}\,. \end{aligned}$$

This concludes the proof.

Theorem 2.3. *The set* $\mathrm{ESTO}(H^2)$ *contains no Fredholm operator.*

Proof. Let A in $\mathrm{ESTO}(H^2)$ be a Fredholm operator of index n. Then $T_zA - AT_{z^2} = K$, for some compact operator K on H^2. This implies that $T_zA = AT_{z^2} + K$.Since A is Fredholm of index n, it follows that T_zA is a Fredholm operator of index $n-1$. On the other hand $AT_{z^2} + K$ is a Fredholm operator of index $n-2$. This leads to $n-1 = n-2$,which is absurd. Thus there is no Fredholm operator in the set $\mathrm{ESTO}(H^2)$.

3. Generalization

The notion of kth-order slant Toeplitz operators on the space L^2 and its compression to H^2 was initiated by Arora and Batra [2] in the year 2003. Motivated by their work, we introduce the concept of generalized essentially slant Toeplitz operator on L^2 and its compression to H^2 as follows:

Definition 3.1. A bounded linear operator A on the space L^2 is said to be a kth-order essentially slant Toeplitz operator on L^2 ($k \geq 2,\ k$ *an integer*) if $M_zA - AM_{z^k} = K$ for some $K \in \mathcal{K}$. We denote by k-$\mathrm{ESTO}(L^2)$, the set of all kth-order essentially slant Toeplitz operators on L^2. The set k-$\mathrm{ESTO}(L^2)$, contains all kth-order slant Toeplitz operators [2] on L^2.

Definition 3.2. A bounded linear operator A on the space H^2 is termed as the compression of a kth-order essentially slant Toeplitz operator to H^2 ($k \geq 2,\ k$ *an integer*) if $T_zA - AT_{z^k} = K$ for some $K \in \mathcal{K}(H^2)$. We denote the set of all compressions of kth-order essentially slant Toeplitz operators to H^2 by k-$\mathrm{ESTO}(H^2)$. If T is the compression of a kth-order slant Toeplitz operator to H^2 then $T \in k$-$\mathrm{ESTO}(H^2)$.

In particular for $k = 2$, the sets 2-$\mathrm{ESTO}(L^2)$ and 2-$\mathrm{ESTO}(H^2)$ are the sets $\mathrm{ESTO}(L^2)$ and $\mathrm{ESTO}(H^2)$ respectively. The results for k-$\mathrm{ESTO}(L^2)$ and k-$\mathrm{ESTO}(H^2)$ ($k \geq 2$) have similar proofs as we have for $\mathrm{ESTO}(L^2)$ and $\mathrm{ESTO}(H^2)$. We list the results here:

For any fixed integer $k \geq 2$,

(1) k-$\mathrm{ESTO}(L^2)$ and k-$\mathrm{ESTO}(H^2)$ are norm-closed vector subspaces of $\mathcal{B}(L^2)$ and $\mathcal{B}(H^2)$ respectively.
(2) $\mathcal{K} \cap k$-$\mathrm{ESTO}(L^2) = \mathcal{K}$
$\mathcal{K}(H^2) \cap k$-$\mathrm{ESTO}(H^2) = \mathcal{K}(H^2)$
(3) If $k_1, k_2 \geq 2$; $k_1 \neq k_2$ then
k_1-$\mathrm{ESTO}(L^2) \cap k_2$-$\mathrm{ESTO}(L^2) = \mathcal{K}$.
(4) (i) If $T_1, T_2 \in k$-$\mathrm{ESTO}(L^2)$ then $T_1T_2 \in k$-$\mathrm{ESTO}(L^2)$ if and only if $T_1M_zT_2 = T_1M_{z^k}T_2 \ (\bmod \mathcal{K})$

(ii) If $T_1, T_2 \in k$-ESTO(H^2) then $T_1T_2 \in k$-ESTO(H^2) if and only if $T_1T_zT_2 = T_1T_{z^k}T_2 \pmod{\mathcal{K}(H^2)}$

(5) (i) If $T_1, T_2 \in k$-ESTO(L^2) such that either T_1 commutes essentially with M_z or T_2 commutes essentially with M_{z^k} then $T_1T_2 \in k$-ESTO(L^2).

(ii) If $T_1, T_2 \in k$-ESTO(H^2) such that either T_1 commutes essentially with T_z or T_2 commutes essentially with T_{z^k} then $T_1T_2 \in k$-ESTO(H^2).

(6) (i) If $T \in k$-ESTO(L^2) and M_ϕ is any multiplication operator on L^2 then TM_ϕ and $M_\phi T$ both are in k-ESTO(L^2).

(ii) If $T \in k$-ESTO(H^2) and T_ϕ is any Toeplitz operator on H^2 then TT_ϕ and $T_\phi T$ both are in k-ESTO(H^2).

(7) (i) A necessary condition for an operator A in k-ESTO(L^2) to be self adjoint is that SA is essentially self adjoint where $S = M_z + M_{\bar{z}^k}$.

(ii) A necessary condition for an operator A in k-ESTO(H^2) to be self adjoint is that TA is essentially self adjoint where $T = T_z + T_{\bar{z}^k}$.

(8) There is no Fredholm operator in the set k-ESTO(H^2).

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