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UNIQUENESS OF ROTATION INVARIANT NORMS

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Abstract. If $N \geq 2$, then there exist finitely many rotations of the sphere $\mathbb{S}^N$ such that the set of the corresponding rotation operators on $L^p(\mathbb{S}^N)$ determines the norm topology for $1 < p \leq \infty$. For $N = 1$ the situation is different: the norm topology of $L^2(\mathbb{S}^1)$ cannot be determined by the set of operators corresponding to the rotations by elements of any 'thin' set of rotations of $\mathbb{S}^1$.

1. Introduction

K. Jarosz showed in [8] that the set of operators on $L^p(\mathbb{S}^1)$, with $1 < p < \infty$, corresponding to all rotations on the circle $\mathbb{S}^1$ determines the norm topology of $L^p(\mathbb{S}^1)$. Following [8] we say that a set $\mathcal{T}$ of continuous linear operators on a given Banach space X *determines the norm topology of* X if any complete norm $|\cdot|$ on X such that the operator $T\colon (X, |\cdot|) \to (X, |\cdot|)$ is continuous for each $T \in \mathcal{T}$ is equivalent to the given norm $\|\cdot\|$ on X. In [13] we found out the complete analogue for Jarosz's result for all the N-dimensional Euclidean spheres $\mathbb{S}^N$: for $N \geq 2$ the set of operators on $L^p(\mathbb{S}^N)$, with $1 < p \leq \infty$, corresponding to all the rotations on $\mathbb{S}^N$ determines the norm topology of $L^p(\mathbb{S}^N)$. It seems natural to study the following question: how many rotation operators are required to determine the norm topology of $L^p(\mathbb{S}^N)$ with $1 < p \leq \infty$ and $N \geq 1$? This is the question we address in this paper. It turns out that there exists a strong dichotomy in the

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answer to this question depending on whether $N = 1$ or $N \geq 2$. We prove that while the norm topology of $L^p(\mathbb{S}^N)$ with $N \geq 2$ can be determined by the set of operators corresponding to appropriate finitely many rotations of $\mathbb{S}^N$, the norm topology of $L^2(\mathbb{S}^1)$ cannot be determined by the set of operators corresponding to the rotations by elements of any 'thin' set of rotations of $\mathbb{S}^1$. It is worth pointing out that the special feature of $\mathbb{S}^N$ which is behind this phenomenon is that the sets of operators on $L^p(\mathbb{S}^N)$ $(1 < p \leq \infty)$ corresponding to the so-called *finite Kazhdan's sets* of the group $SO(N+1)$ of all rotations of $\mathbb{S}^N$ determine the norm topology of $L^p(\mathbb{S}^N)$ and that $SO(N+1)$ has such sets for $N \geq 2$ (see Section 2 for the details). On the other hand, if $E \subset \mathbb{R}$ is such that there is a trigonometric series $\sum[a_n \cos(n\theta) + b_n \sin(n\theta)]$ which converges absolutely in E but not everywhere, then the set of those operators on $L^2(\mathbb{S}^1)$ corresponding to the rotations of $\mathbb{S}^1$ with angles from E does not determine the norm topology of $L^2(\mathbb{S}^1)$ (see Section 3 for the details).

It seems appropriate to point out that the results of this paper are related to some results contained in our previous paper [1] and the third's author paper [13]. Nevertheless it is important to know that throughout this paper we are not using at all the techniques developed in the seminal paper [13], which are still on the basis of [1].

2. The space $L^p(\mathbb{S}^N)$ with $N \geq 2$

Throughout this section, $N \geq 2$ and $\mathbb{S}^N$ stands for the N-dimensional Euclidean sphere endowed with the Lebesgue measure. We denote by $L^p(\mathbb{S}^N)$ $(1 \leq p \leq \infty)$ the Banach space of all complex-valued functions f (or rather, equivalence classes thereof) on $\mathbb{S}^N$ with

$$\|f\|_p = \left(\int_{\mathbb{S}^N} |f(x)|^p \, dx \right)^{1/p} < \infty \quad (p < \infty)$$

$$\|f\|_\infty = \inf \left\{ \sup_{x \in \mathbb{S}^N \setminus Z} |f(x)| : \; Z \text{ has zero measure} \right\} < \infty.$$

Also, G stands for the so-called *special orthogonal group* $SO(N+1)$ consisting of all rotations of $\mathbb{S}^N$. Of course, such rotations are nothing but the restriction to $\mathbb{S}^N$ of the linear isometries of $\mathbb{R}^{N+1}$ which preserve the orientation. One may also think of $SO(N+1)$ as the set of those $(N+1) \times (N+1)$ real matrices A with $A^t A = I$ and $\det A = 1$. Thus it is easy to see that it is a compact group (with respect to the relative topology as a subset of $\mathbb{R}^{(N+1)^2}$). Accordingly, there is a non-zero, regular (positive) Borel measure λ on G which is left invariant, *i.e.* $\lambda(tE) = \lambda(E)$ for each $t \in G$ and each Borel subset E of G. This is the so-called *Haar measure* of G and it is unique up to a constant multiple (we refer the reader to [5, Section 2.2] for a full account about such a measure). From now on, we endow G with the Haar measure λ normalized so that $\lambda(G) = 1$ and we simply write $\int_G (\cdot) dt$ for the integral with respect to this measure. We define the rotations of every function $f : \mathbb{S}^N \to \mathbb{C}$ by $\big(\tau(t)f\big)(x) = f\left(t^{-1}(x)\right)$ for all $x \in \mathbb{S}^N$

and $t \in G$. It is easily seen that

$$\|\tau(t)f\|_p = \|f\|_p, \quad (1 \le p \le \infty,\ f \in L^p(\mathbb{S}^N),\ t \in G).$$

So $\tau(t)$ gives an isometry from $L^p(\mathbb{S}^N)$ onto itself for each $1 \le p \le \infty$ and we write $\tau_p(t)$ for this restriction. These operators are the *rotation operators* and it was shown in [13] that the set $\{\tau_p(t)\colon L^p(\mathbb{S}^N) \to L^p(\mathbb{S}^N),\ t \in G\}$ of all rotation operators on $L^p(\mathbb{S}^N)$ determines the norm topology of $L^p(\mathbb{S}^N)$ for $1 < p \le \infty$.

As usual, a *unitary representation* of G is a group homomorphism π from G into the group of all unitary operators $U(H_\pi)$ on some complex Hilbert space H_π, which is continuous with respect to the given topology of G and the strong operator topology on $U(H_\pi)$. The Banach–Ruziewicz problem for the spheres asks whether the Lebesgue measure on $\mathbb{S}^N$ is the unique normalized, finitely additive rotation invariant measure on all Lebesgue measurable subsets of $\mathbb{S}^N$. When solving this problem it was shown that G has the so-called *strong Kazhdan's property* (T) for $N \ge 2$ ([4] in the cases $N = 2, 3$ and [10, 12] in the case $N \ge 4$). This means that there exists a finite set K of G and a positive ϵ with the property that, whenever a unitary representation π of G satisfies $\sup_{t\in K} \|\pi(t)u - u\| < \epsilon$ for some unit vector $u \in H_\pi$, then π has a non-zero invariant vector. In this case, K is called a *Kazhdan's set*, ϵ a *Kazhdan's constant*, and (K, ϵ) a *Kazhdan's pair* for G. We refer the reader to [6] for a thorough discussion of Kazhdan's property (T). It turned out in [1] that the strong Kazhdan's property (T) has applications to automatic continuity.

Let π be a unitary representation of G on a finite-dimensional Hilbert space H_π (whose norm we denote by $|\cdot|_\pi$). We denote by $L^p(\mathbb{S}^N, H_\pi)$ $(1 \le p \le \infty)$ the Banach space of all (equivalence classes of) functions $F\colon \mathbb{S}^N \to H_\pi$ with

$$\|F\|_p = \left(\int_{\mathbb{S}^N} |F(x)|_\pi^p \, dx\right)^{1/p} < \infty \quad (p < \infty),$$

$$\|F\|_\infty = \inf\left\{\sup_{x\in\mathbb{S}^N\setminus Z} |F(x)|_\pi\colon\ Z \text{ has zero measure}\right\} < \infty.$$

For every $t \in G$ and every function $F\colon \mathbb{S}^N \to H_\pi$ we define

$$(\tau\otimes\pi)(t)F\colon \mathbb{S}^N \to H_\pi, \quad \big((\tau\otimes\pi)(t)F\big)(x) = \pi(t)\Big(F\left(t^{-1}(x)\right)\Big), \quad (x \in \mathbb{S}^N).$$

It is straightforward to check that

$$\|(\tau\otimes\pi)(t)F\|_p = \|F\|_p, \quad (1 \le p \le \infty,\ F \in L^p(\mathbb{S}^N, H_\pi),\ t \in G).$$

We write $(\tau_p\otimes\pi)(t)$ for the restriction of $(\tau\otimes\pi)(t)$ to $L^p(\mathbb{S}^N, H_\pi)$, which gives an isometry from $L^p(\mathbb{S}^N, H_\pi)$ onto itself. It should be pointed out that the space $L^p(\mathbb{S}^N, H_\pi)$ can be algebraically identified with the tensor product $L^p(\mathbb{S}^N) \otimes H_\pi$ by means of the natural map $f \otimes u \mapsto f(\cdot)u$ and, with this identification, it is clear that $(\tau_p\otimes\pi)(t) = \tau_p(t)\otimes\pi(t)$ for each $t \in G$. We now define, for every $1 \le p \le \infty$, a continuous linear operator

$$\tau_p \odot \pi\colon L^p(\mathbb{S}^N, H_\pi) \to L^p(\mathbb{S}^N, H_\pi)$$

by

$$\big((\tau_p \odot \pi)F\big)(x) = \int_G \pi(t)\Big(F\left(t^{-1}(x)\right)\Big)\, dt, \quad (F \in L^p(\mathbb{S}^N, H_\pi),\ x \in \mathbb{S}^N).$$

Lemma 2.1. *Let $1 \le p \le \infty$ and let π be a unitary representation of G on a finite-dimensional Hilbert space H_π. Then*

$$(\tau_p \odot \pi) \circ \big((\tau_p \otimes \pi)(t)\big) = \big((\tau_p \otimes \pi)(t)\big) \circ (\tau_p \odot \pi) = \tau_p \odot \pi$$

for each $t \in G$ and the operator $\tau \odot \pi$ is a continuous linear projection of norm one from $L^p(\mathbb{S}^N, H_\pi)$ onto

$$\mathfrak{N}_\pi^p = \left\{F \in L^p(\mathbb{S}^N, H_\pi)\colon\ (\tau_p \otimes \pi)(t)F = F,\ \forall t \in G\right\}.$$

Furthermore, $\dim \mathfrak{N}_\pi^p < \infty$.

Proof. We begin by proving that

$$\|(\tau_p \odot \pi)F\|_p \le \|F\|_p$$

for each $F \in L^p(\mathbb{S}^N, H_\pi)$. If $1 \le p < \infty$, then

$$\left|\big((\tau_p \odot \pi)F\big)(x)\right|_\pi \le \int_G \left|\pi(t)\Big(F\left(t^{-1}(x)\right)\Big)\right|_\pi dt =$$

$$\int_G \left|F\left(t^{-1}(x)\right)\right|_\pi dt \le \left(\int_G \left|F\left(t^{-1}(x)\right)\right|_\pi^p dt\right)^{1/p}$$

and therefore

$$\|(\tau_p \odot \pi)F\|_p^p = \int_{\mathbb{S}^N} \left|\big((\tau_p \odot \pi)F\big)(x)\right|_\pi^p dx \le \int_{\mathbb{S}^N}\int_G \left|F\left(t^{-1}(x)\right)\right|_\pi^p dtdx =$$

$$\int_G\int_{\mathbb{S}^N} \left|F\left(t^{-1}(x)\right)\right|_\pi^p dxdt = \int_G\int_{\mathbb{S}^N} |F(y)|_\pi^p\ dydt = \int_G \|F\|_p^p dt = \|F\|_p^p.$$

In the case where $p = \infty$ the proof is straightforward and is left to the reader.

Since the linearity of $\tau_p \odot \pi$ is obvious, it may be concluded that $\tau_p \odot \pi$ is continuous with $\|\tau_p \odot \pi\| \le 1$.

On the other hand, for every $F \in L^p(\mathbb{S}^N, H_\pi)$, we have

$$\Big((\tau_p \odot \pi)\big((\tau_p \odot \pi)F\big)\Big)(x) = \int_G \pi(t)\Big(\big((\tau_p \odot \pi)F\big)\left(t^{-1}(x)\right)\Big)dt =$$

$$\int_G \pi(t)\left(\int_G \pi(s)\Big(F\left(s^{-1}\left(t^{-1}(x)\right)\right)\Big)ds\right)dt =$$

$$\int_G\int_G \pi(ts)\Big(F\left((ts)^{-1}(x)\right)\Big)dsdt =$$

$$\int_G \underbrace{\int_G \pi(r)\Big(F\left(r^{-1}(x)\right)\Big)dr}_{=\left((\tau_p\odot\pi)F\right)(x)}\, dt = \int_G \big((\tau_p \odot \pi)F\big)(x)dt = \big((\tau_p \odot \pi)F\big)(x).$$

This entails that $(\tau_p \odot \pi)^2 = \tau_p \odot \pi$.

Let $t \in G$, $F \in L^p(\mathbb{S}^N, H_\pi)$, and $x \in \mathbb{S}^N$. Then

$$\Big((\tau_p \odot \pi)\big((\tau_p \otimes \pi)(t)F\big)\Big)(x) = \int_G \pi(s)\Big(\big((\tau_p \otimes \pi)(t)F\big)\left(s^{-1}(x)\right)\Big)ds =$$

$$\int_G \pi(s)\Big(\pi(t)\Big(F\left(t^{-1}\left(\left(s^{-1}(x)\right)\right)\right)\Big)\Big)ds = \int_G \pi(st)\Big(F\left((st)^{-1}(x)\right)\Big)ds =$$

$$\int_G \pi(r)\Big(F\left(r^{-1}(x)\right)\Big)dr = \big((\tau_p \odot \pi)F\big)(x),$$

which shows that $(\tau_p \odot \pi) \circ \big((\tau_p \otimes \pi)(t)\big) = \tau_p \odot \pi$. On the other hand, we have

$$\Big((\tau_p \otimes \pi)(t)\big((\tau_p \odot \pi)F\big)\Big)(x) = \pi(t)\Big(\big((\tau_p \odot \pi)F\big)\left(t^{-1}(x)\right)\Big) =$$

$$\pi(t)\left(\int_G \pi(s)\big(F\left(s^{-1}\left(t^{-1}(x)\right)\right)\big)\right)ds = \int_G \pi(ts)\Big(F\left((ts)^{-1}(x)\right)\Big)ds =$$

$$\int_G \pi(r)\Big(F\left(r^{-1}(x)\right)\Big)dr = \big((\tau_p \odot \pi)F\big)(x)$$

which yields $\big((\tau_p \otimes \pi)(t)\big) \circ (\tau_p \odot \pi) = \tau_p \odot \pi$. Moreover, this identity obviously entails that the range of $\tau_p \odot \pi$ is contained in $\mathfrak{N}_\pi^p$. Now, if $F \in \mathfrak{N}_\pi^p$, then

$$\big((\tau_p \odot \pi)F\big)(x) = \int_G \big((\tau_p \otimes \pi)(t)F\big)(x)dt = \int_G F(x)dt = F(x),$$

and so F is in the range of $\tau_p \odot \pi$.

Finally, we check that $\dim \mathfrak{N}_\pi^p < \infty$. To this end we consider the linear operator

$$\Psi\colon L^p(\mathbb{S}^N, H_\pi) \to L^p(G, H_\pi),$$
$$\Psi(F)(t) = \pi(t^{-1})\big(F(t(\texttt{north pole}))\big), \quad (F \in L^p(\mathbb{S}^N, H_\pi),\ t \in G),$$

which is easily seen to be injective. Furthermore, if $F \in L^p(\mathbb{S}^N, H_\pi)$ and $s, t \in G$, then

$$\Psi\big((\tau_p \otimes \pi)(t)F\big)(s) = \pi(s^{-1})\big(\pi(t)\big(F(t^{-1}(s(\texttt{north pole})))\big)\big) =$$

$$\pi((t^{-1}s)^{-1})\big(F((t^{-1}s)(\texttt{north pole}))\big) = \Psi(F)(t^{-1}s).$$

Hence, if F lies in $\mathfrak{N}_\pi^p$, then

$$\Psi(F)(t^{-1}\ \cdot) = \Psi(F)(\cdot), \quad (t \in G),$$

which clearly forces that $\Psi(F)$ is constant. Consequently,

$$\dim \mathfrak{N}_\pi^p \leq \dim \Psi(\mathfrak{N}_\pi^p) \leq \dim H_\pi.$$

$\square$

Lemma 2.2. *Let $1 \leq p \leq \infty$ and let $f \in L^p(\mathbb{S}^N)$ such that*

$$(\tau_p \odot \pi)(f \otimes u) = 0$$

for each unitary representation π of G on a finite-dimensional Hilbert space H_π and for each $u \in H_\pi$. Then $f = 0$.

Proof. Let $1 \leq q \leq \infty$ with $\frac{1}{p}+\frac{1}{q}=1$ and $g \in L^q(\mathbb{S}^N)$. Take a unitary representation π of G on a finite-dimensional Hilbert space H_π and $u, v \in H_\pi$. For every $x \in \mathbb{S}^N$, we have

$$0=\left\langle \big((\tau_p \odot \pi)(f \otimes u)(x), v\right\rangle_\pi = \int_G f\left(t^{-1}(x)\right)\langle \pi(t)u, v\rangle_\pi dt$$

and hence

$$0=\int_{\mathbb{S}^N}\left(\int_G f\left(t^{-1}(x)\right)\langle \pi(t)u, v\rangle_\pi dt\right) g(x)dx = \int_G\left(\int_{\mathbb{S}^N} f\left(t^{-1}(x)\right) g(x)dx\right)\langle \pi(t)u, v\rangle_\pi dt.$$

This implies that

$$\int_G\left(\int_{\mathbb{S}^N} f\left(t^{-1}(x)\right) g(x)dx\right)\xi(t)dt=0$$

for each ξ in the linear span $\mathcal{E}$ of the functions $t \mapsto \langle \pi(t)u, v\rangle_\pi$ as π ranges over all unitary representations of G on a finite-dimensional Hilbert space H_π and u, v ranges over H_π. The functions in $\mathcal{E}$ are the so-called *trigonometric polynomials* on G and $\mathcal{E}$ is dense in $C(G)$ (the Banach space of all complex-valued continuous functions on G) (see [5, Section 5.2]) . We now claim that the function $t \mapsto \int_{\mathbb{S}^N} f\left(t^{-1}(x)\right) g(x)dx$ is continuous, in which case it follows that vanishes everywhere on G, because of the density of $\mathcal{E}$ on $C(G)$. Indeed, if $p<\infty$, then the claim follows from [2, Chapter VIII, §2, Example 5]. In the case when $p=\infty$ we write

$$\int_{\mathbb{S}^N} f\left(t^{-1}(x)\right) g(x)dx = \int_{\mathbb{S}^N} f(y)g(t(y))dy$$

and then we can apply the preceding case.

Since $\int_{\mathbb{S}^N} f\left(t^{-1}(x)\right) g(x)dx = 0 \ \ \forall t \in G$, by taking $t=\mathbf{1}$ we arrive at

$$\int_{\mathbb{S}^N} f(x)g(x)dx=0.$$

Since the preceding identity holds for each $g \in L^q(\mathbb{S}^N)$, we conclude that $f=0$, as required. □

Now it is important to know that [1, Lemma 4] can be rephrased as follows.

Lemma 2.3. *Let (K,ϵ) be a Kazhdan's pair for G with K finite and let $1 \leq p < \infty$. If π is a unitary representation of G on a finite-dimensional Hilbert space H_π, then*

$$\sup_{t\in K}\|(\tau_p \otimes \pi)(t)F - F\|_p \geq \left(3^3 2^{3p+7}p\right)^{-1}\epsilon^2\|F\|_p$$

for each $F \in L^p(\mathbb{S}^N, H_\pi)$ such that $(\tau_p \odot \pi)F=0$.

Lemma 2.4. *Let $\{t_1,\dots,t_J\}$ be a Kazhdan's set for G and let $1<p\leq\infty$. If π is a unitary representation of G on a finite-dimensional Hilbert space H_π, then the range of the continuous linear operator*

$$\Psi^p_\pi \colon L^p(\mathbb{S}^N, H_\pi)\times \overset{J}{\cdots} \times L^p(\mathbb{S}^N, H_\pi) \to L^p(\mathbb{S}^N, H_\pi)$$

defined by

$$\Psi^p_\pi(F_1,\dots,F_J)=\sum_{j=1}^{J}\Big[F_j-(\tau_p\otimes\pi)(t_j^{-1})F_j\Big],\quad (F_1,\dots,F_J\in L^p(\mathbb{S}^N,H_\pi))$$

is $\mathfrak{M}^p_\pi=\big\{F\in L^p(\mathbb{S}^N,H_\pi)\colon\ (\tau_p\odot\pi)F=0\big\}$.

Proof. We begin the proof by observing that $L^p(\mathbb{S}^N,H_\pi)$ can be identified in a natural way with the dual $L^q(\mathbb{S}^N,H_{\overline{\pi}})^*$ of $L^q(\mathbb{S}^N,H_{\overline{\pi}})$ where, of course, q is such that $\frac{1}{p}+\frac{1}{q}=1$ and $\overline{\pi}$ is the so-called *conjugate representation* of π. Such a representation is defined as follows. The Hilbert space $H_{\overline{\pi}}$ is the conjugate Hilbert space of H_π, which is nothing but H_π with scalar multiplication replaced by $(\lambda,u)\mapsto\overline{\lambda}u$ and inner product replaced by $\langle u,v\rangle_{\overline{\pi}}=\langle v,u\rangle_\pi$. Then $\overline{\pi}(t)$ is the operator on $H_{\overline{\pi}}$ such that coincides with $\pi(t)$ as a set-theoretical transformation on H_π for each $t\in G$. The duality between the spaces $L^p(\mathbb{S}^N,H_\pi)$ and $L^q(\mathbb{S}^N,H_{\overline{\pi}})$ is given by

$$\langle F,G\rangle=\int_{\mathbb{S}^N}\langle F(x),G(x)\rangle_\pi\,dx,\quad (F\in L^p(\mathbb{S}^N,H_\pi),\ G\in L^q(\mathbb{S}^N,H_{\overline{\pi}})).$$

We now consider the spaces

$$\mathfrak{M}^q_{\overline{\pi}}=\big\{F\in L^q(\mathbb{S}^N,H_{\overline{\pi}})\colon\ (\tau_q\odot\overline{\pi})F=0\big\},$$
$$\mathfrak{N}^q_{\overline{\pi}}=\big\{F\in L^q(\mathbb{S}^N,H_{\overline{\pi}})\colon\ (\tau_q\odot\overline{\pi})F=F\big\},$$

and the continuous linear operator

$$\Delta^q_{\overline{\pi}}\colon\mathfrak{M}^q_{\overline{\pi}}\to L^q(\mathbb{S}^N,H_{\overline{\pi}})\times\overset{J}{\cdots}\times L^q(\mathbb{S}^N,H_{\overline{\pi}})$$

defined by

$$\Delta^q_{\overline{\pi}}(F)=\big(F-(\tau_q\otimes\overline{\pi})(t_1)F,\dots,F-(\tau_q\otimes\overline{\pi})(t_J)F\big),\quad (F\in\mathfrak{M}^q_{\overline{\pi}}).$$

On account of Lemma 2.3, there exists $\delta>0$ such that

$$\|\Delta^q_{\overline{\pi}}(F)\|_q\geq\delta\|F\|_q,\quad (F\in\mathfrak{M}^q_{\overline{\pi}}).$$

This implies that $\Delta^q_{\overline{\pi}}$ is injective and that its range is closed, which entails that the adjoint operator $\big(\Delta^q_{\overline{\pi}}\big)^*$ is surjective (see [3, Proposition 1.8 and Theorem 1.10 in Chap. 6, §1]).

If $F\in L^p(\mathbb{S}^N,H_\pi)$, $G\in L^q(\mathbb{S}^N,H_{\overline{\pi}})$, and $t\in G$, then

$$\langle F,(\tau_q\otimes\overline{\pi})(t)G\rangle=\int_{\mathbb{S}^N}\Big\langle F(x),\overline{\pi}(t)\Big(G\left(t^{-1}(x)\right)\Big)\Big\rangle_\pi dx=$$
$$\int_{\mathbb{S}^N}\big\langle\pi(t^{-1})\big(F(x)\big),G\left(t^{-1}(x)\right)\big\rangle_\pi dx=$$
$$\int_{\mathbb{S}^N}\Big\langle\pi(t^{-1})\Big(F\big(t(y)\big)\Big),G(y)\Big\rangle_\pi dy=\langle(\tau_p\otimes\pi)(t^{-1})F,G\rangle$$

and

$$\langle F,(\tau_q\odot\overline{\pi})G\rangle=\int_{\mathbb{S}^N}\Big\langle F(x),\int_G\overline{\pi}(t)\Big(G\left(t^{-1}(x)\right)\Big)dt\Big\rangle_\pi dx=$$

$$\int_{\mathbb{S}^N}\int_G \Big\langle F(x), \overline{\pi}(t)\Big(G\left(t^{-1}(x)\right)\Big)\Big\rangle_\pi dtdx =$$

$$\int_G\int_{\mathbb{S}^N} \Big\langle F(x), \overline{\pi}(t)\Big(G\left(t^{-1}(x)\right)\Big)\Big\rangle_\pi dxdt =$$

$$\int_G\int_{\mathbb{S}^N} \big\langle F\big(t(y)\big), \overline{\pi}(t)\big(G(y)\big)\big\rangle_\pi dydt =$$

$$\int_G\int_{\mathbb{S}^N} \Big\langle \pi(t^{-1})\Big(F\big(t(y)\big)\Big), G(y)\Big\rangle_\pi dydt =$$

$$\int_{\mathbb{S}^N} \left\langle \int_G \pi(t^{-1})\Big(F\big(t(y)\big)\Big)dt, G(y)\right\rangle_\pi dy =$$

$$\int_{\mathbb{S}^N} \left\langle \int_G \pi(s)\Big(F\big(s^{-1}(y)\big)\Big)ds, G(y)\right\rangle_\pi dy = \langle(\tau_p\odot\pi)F, G\rangle,$$

which show that the adjoint operators $(\tau_q\otimes\overline{\pi})(t)^*$ and $(\tau_q\odot\overline{\pi})^*$ are nothing but the operators $(\tau_p\otimes\pi)(t^{-1})$ y $\tau_p\odot\pi$, respectively. The space $\big(\mathfrak{M}^q_{\overline{\pi}}\big)^*$ is identified in a natural way with the quotient space $L^p(\mathbb{S}^N, H_\pi)/\big(\mathfrak{M}^q_{\overline{\pi}}\big)^0$, where $(\cdot)^0$ stands for the polar. On the other hand, since $\tau_q\odot\overline{\pi}$ is a projection from $L^q(\mathbb{S}^N, H_{\overline{\pi}})$ onto $\mathfrak{N}^q_{\overline{\pi}}$, it follows that $\tau_p\odot\pi = (\tau_q\odot\overline{\pi})^*$ is a projection from $L^p(\mathbb{S}^N, H_\pi)$ onto $\big(\ker(\tau_q\odot\overline{\pi})\big)^0 = \big(\mathfrak{M}^q_{\overline{\pi}}\big)^0$. Accordingly, $\mathfrak{N}^p_\pi = \big(\mathfrak{M}^q_{\overline{\pi}}\big)^0$ and $\big(\Delta^q_{\overline{\pi}}\big)^*$ can be thought of as the operator

$$L^p(\mathbb{S}^N, H_\pi)\times \overset{J}{\cdots} \times L^p(\mathbb{S}^N, H_\pi) \to \frac{L^p(\mathbb{S}^N, H_\pi)}{\mathfrak{N}^p_\pi}$$

given by

$$\big(\Delta^q_{\overline{\pi}}\big)^*(F_1,\dots,F_J) = \sum_{j=1}^J \Big[F_j - (\tau_p\otimes\pi)(t_j^{-1})F_j\Big] \ + \ \mathfrak{N}^p_\pi$$

for all $F_1,\dots,F_J \in L^p(\mathbb{S}^N, H_\pi)$. On the other hand, the operator

$$\Lambda^p_\pi\colon \frac{L^p(\mathbb{S}^N, H_\pi)}{\mathfrak{N}^p_\pi} \to \mathfrak{M}^p_\pi, \quad \Lambda^p_\pi(F+\mathfrak{N}^p_\pi) = F - (\tau_p\odot\pi)F, \quad (F\in L^p(\mathbb{S}^N, H_\pi))$$

is an isomorphism and clearly we have

$$\Lambda^p_\pi \circ (\Delta^q_{\overline{\pi}})^* = \Psi^p_\pi,$$

which proves the lemma. □

Theorem 2.5. *Let N be a positive integer with $N\geq 2$. Then there exist finitely many rotations of $\mathbb{S}^N$ such that the set consisting of the corresponding rotation operators on $L^p(\mathbb{S}^N)$ determines the norm topology of that space for $1 < p \leq \infty$. Specifically, the set of operators on $L^p(\mathbb{S}^N)$ corresponding to the rotations $\{t_1,\dots,t_J\}$ of any Kazhdan set for $SO(N+1)$ determines the norm topology of $L^p(\mathbb{S}^N)$.*

Proof. Let $K = \{t_1, \dots, t_J\}$ be a Kazhdan set for G and let $\|\cdot\|$ be a complete norm on $L^p(\mathbb{S}^N)$ with the property that the operator $\tau_p(t_j)$ from $(L^p(\mathbb{S}^N), \|\cdot\|)$ into itself is continuous for each $j = 1, \dots, J$. It is an immediate restatement of the closed graph theorem that the norms $\|\cdot\|$ and $\|\cdot\|_p$ are equivalent if and only if 0 is the only function f in $L^p(\mathbb{S}^N)$ with the property that there exists a sequence of functions (f_n) in $L^p(\mathbb{S}^N)$ converging to 0 with respect to $\|\cdot\|$ and converging to f with respect to $\|\cdot\|_p$. Our procedure consists in proving that for such a function f, we have $\tau_p \odot \pi(f \otimes u) = \{0\}$ for each unitary representation π of G on a finite-dimensional Hilbert space H_π and for each $u \in H_\pi$. Then Lemma 2.2 yields $f = 0$ and so the equivalence of $\|\cdot\|$ and $\|\cdot\|_p$.

It should be noted that the Banach isomorphism theorem entails that the operators $\tau_p(t_j{}^{-1}) : (L^p(\mathbb{S}^N), \|\cdot\|) \to (L^p(\mathbb{S}^N), \|\cdot\|)$ $(j = 1, \dots, J)$ are also continuous.

Let π be a unitary representation of G on a finite-dimensional Hilbert space H_π. We consider the norm $\|\cdot\|_\pi$ on $L^p(\mathbb{S}^N, H_\pi)$ defined by

$$\|F\|_\pi = \sup\big\{\|\langle F(\cdot), u\rangle_\pi\|\colon\ u \in H_\pi,\ |u|_\pi = 1\big\}.$$

Let $\mathfrak{M}^p_\pi$ and $\mathfrak{N}^p_\pi$ the subspaces of $L^p(\mathbb{S}^N, H_\pi)$ introduced in Lemmas 2.1 and 2.4. On account of Lemma 2.1, we have

$$L^p(\mathbb{S}^N, H_\pi) = \mathfrak{M}^p_\pi \oplus \mathfrak{N}^p_\pi$$

and $\dim \mathfrak{N}^p_\pi < \infty$. We consider the operator Ψ^p_π given in Lemma 2.4 acting on $L^p(\mathbb{S}^N, H_\pi)$ equipped with the new norm $\|\cdot\|_\pi$, *i.e.*

$$\Psi^p_\pi\colon \big(L^p(\mathbb{S}^N, H_\pi), \|\cdot\|_\pi\big) \times \overset{J}{\cdots} \times \big(L^p(\mathbb{S}^N, H_\pi), \|\cdot\|_\pi\big) \to \big(L^p(\mathbb{S}^N, H_\pi), \|\cdot\|_\pi\big),$$

$$\Psi^p_\pi(F_1, \dots, F_J) = \sum_{j=1}^{J} \Big[F_j - (\tau_p \otimes \pi)(t_j^{-1})F_j\Big], \quad (F_1, \dots, F_J \in L^p(\mathbb{S}^N, H_\pi)),$$

which, according to our hypothesis, is obviously continuous. On account of Lemma 2.4, the range of that operator is $\mathfrak{M}^p_\pi$, which is finite-codimensional. From [11, Lemma 3.3] it follows that $\mathfrak{M}^p_\pi$ is a closed subspace of $(L^p(\mathbb{S}^N, H_\pi), \|\cdot\|_\pi)$. Since $\mathfrak{N}^p_\pi$ is also closed in $(L^p(\mathbb{S}^N, H_\pi), \|\cdot\|_\pi)$ (because it is finite-dimensional), it may be concluded that the map

$$(\mathfrak{M}^p_\pi, \|\cdot\|_\pi) \times (\mathfrak{N}^p_\pi, \|\cdot\|_\pi) \to \big(L^p(\mathbb{S}^N, H_\pi), \|\cdot\|_\pi\big), \quad (F, G) \mapsto F + G$$

is an isomorphism.

Let (f_n) be a sequence of functions in $L^p(\mathbb{S}^N)$ converging to zero with respect to the norm $\|\cdot\|$ and converging to some function $f \in L^p(\mathbb{S}^N)$ with respect to the norm $\|\cdot\|_p$. Let $u \in H_\pi$. We now write

$$f_n \otimes u = G_n + H_n, \quad G_n \in \mathfrak{M}^p_\pi, \quad H_n \in \mathfrak{N}^p_\pi, \quad (n \in \mathbb{N})$$

with

$$\|\cdot\|_\pi - \lim_{n\to\infty} G_n = 0 \quad \text{and} \quad \|\cdot\|_\pi - \lim_{n\to\infty} H_n = 0.$$

Since $\dim \mathfrak{N}^p_\pi < \infty$, it follows that the norms $\|\cdot\|_\pi$ and $\|\cdot\|_p$ are equivalent on this space. Consequently, (H_n) also converges to zero with respect to the norm $\|\cdot\|_p$

and therefore (G_n) converges to $f \otimes u$ with respect to $\|\cdot\|_p$. Since the operator $\tau_p \odot \pi$ is continuous on $(L^p(\mathbb{S}^N, H_\pi), \|\cdot\|_p)$, it follows that

$$\|\cdot\|_p - \lim_{n\to\infty} \underbrace{(\tau \odot \pi)G_n}_{=0} = (\tau \odot \pi)(f \otimes u)$$

and so $(\tau \odot \pi)(f \otimes u) = 0$, as required. □

3. The space $L^2(\mathbb{S}^1)$ and thin sets of rotations

Throughout this section we restrict our attention to the spaces $L^p(\mathbb{S}^1)$. As in the preceding section, we denote by $\tau_p(t)f$ the rotation of $f \in L^p(\mathbb{S}^1)$ by $t \in SO(2)$. Of course, $\mathbb{S}^1$ can be thought of as the set of complex numbers $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, which turns into a compact abelian group with respect to the usual multiplication of complex numbers. This is the so-called *circle group*. On the other hand, the map $\theta \mapsto \left(\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$ gives an isomorphism from the group $\mathbb{R}/2\pi\mathbb{Z}$ onto $SO(2)$. Moreover, $\mathbb{R}/2\pi\mathbb{Z}$ can be identified with the group $\mathbb{T}$ by mean of the natural map $\theta \mapsto e^{i\theta}$, so that the action τ_p of $\mathbb{T}$ on $L^p(\mathbb{S}^1)$ is given by

$$\big(\tau_p(z)f\big)(w) = f(z^{-1}w), \quad (f \in L^p(\mathbb{S}^1),\ z, w \in \mathbb{T}).$$

A straightforward consequence of [8] is the following result.

Theorem 3.1. *Let E be a subset of $SO(2)$ for which one of the following assertions holds:*

i. *E is a measurable set of positive Lebesgue measure.*
ii. *$SO(2) \setminus E$ is of the first category.*

Then the set of rotation operators on $L^p(\mathbb{S}^1)$ corresponding to the rotations of $\mathbb{S}^1$ by elements of E determines the norm topology of $L^p(\mathbb{S}^1)$ for each $1 < p < \infty$.

Proof. Let $1 < p < \infty$ and let $|\cdot|$ be any complete norm on $L^p(\mathbb{S}^1)$. It is easily seen that the set of those $z \in SO(2)$ such that the rotation operator $\tau_p(z)$ from $(L^p(\mathbb{S}^1), |\cdot|)$ into itself is continuous is a subgroup of $SO(2)$.

If E satisfies one of the properties of the theorem, then it is well known that the subgroup generated by E is $SO(2)$ (see [15, Vol. I, p. 250], for example). Consequently, if the norm $|\cdot|$ makes the operators $\tau_p(z)$ with $z \in E$ continuous, then it makes all operators $\tau_p(z)$ with $z \in SO(2)$ continuous. Hence [8, Theorem 3.5] gives the desired conclusion. □

Our next objective is to prove that the set of operators corresponding to any *thin* set of rotations of $\mathbb{S}^1$ does not determine the norm topology of $L^2(\mathbb{S}^1)$. To this end we involve the following two results.

Lemma 3.2. *Let $\mathcal{T}$ be a set of continuous linear operators on a Banach space X. Suppose that there exist a non-zero vector $x_0 \in X$, a discontinuous linear functional ϕ on X, and a family $(\lambda_T)_{T\in\mathcal{T}}$ of complex numbers such that*

i. *$Tx_0 = \lambda_T x_0$ for each $T \in \mathcal{T}$;*
ii. *$\phi(Tx) = \lambda_T \phi(x)$ for all $T \in \mathcal{T}$ and $x \in X$.*

Then $\mathcal{T}$ does not determine the norm of X.

Proof. Pick $\alpha \in \mathbb{C} \setminus \{0, \phi(x_0)\}$. It is easily seen that the map $\Phi\colon X \to X$ defined by $\Phi(x) = \alpha x - \phi(x)x_0$ is a discontinuous invertible linear map. Consequently, the map $|\cdot|$ defined on X by $|x| = \|\Phi(x)\|$ for each $x \in X$ is a complete norm on X which is not equivalent to the given norm $\|\cdot\|$. On the other hand, for every $T \in \mathcal{T}$ we have

$$|T(x)| = \|\alpha T(x) - \phi(Tx)x_0\| = \|\alpha T(x) - \phi(x)\lambda_T x_0\| =$$
$$\|\alpha T(x) - \phi(x)T(x_0)\| = \|T(\Phi(x))\| \leq \|T\|\|\Phi(x)\| = \|T\||x|$$

for each $x \in X$, which shows that T is continuous with respect to the norm $|\cdot|$. □

Lemma 3.3. *Let (k_n) be a strictly increasing sequence of positive integers and let (ρ_n) be a sequence of real numbers such that $\rho_n \geq 0$ and $\sum_{n=1}^{\infty} \rho_n = \infty$. Then there exists a discontinuous linear functional on $L^2(\mathbb{T})$ which vanishes on $\left\{f \in L^2(\mathbb{T})\colon \ \sum_{n=1}^{\infty} \sqrt{\rho_n}|\widehat{f}(k_n)| < \infty\right\}$, where $\widehat{f}(k)$ stands for the Fourier transform of f at k for each $k \in \mathbb{Z}$.*

Proof. Write $M = \left\{x \in \ell^2(\mathbb{Z})\colon \ \sum_{n=1}^{\infty} \sqrt{\rho_n}|x_{k_n}| < \infty\right\}$. It is clear that M is dense in $\ell^2(\mathbb{Z})$. If it were $M = \ell^2(\mathbb{Z})$, a standard application of the uniform boundedness theorem (see [14, Example 2 in Section 7.6]) would give $\sum_{n=1}^{\infty} \rho_n < \infty$, contrary to our assumption. Thus $M \neq \ell^2(\mathbb{Z})$ and there exists a discontinuous linear functional ϕ on $\ell^2(\mathbb{Z})$ vanishing at M. Of course, the linear functional $f \mapsto \phi(\widehat{f}\,)$ defined on $L^2(\mathbb{T})$ satisfies our requirements. □

Let us recall that a set $E \subset \mathbb{R}$ (or rather, $E \subset \mathbb{R}/2\pi\mathbb{Z}$ thereof) is said to be a *N-set* if there exists a trigonometric series $\sum[a_n \cos(n\theta) + b_n \sin(n\theta)]$ which converges absolutely in E but not everywhere (see [15, Vol. I, Chapter VI]). It is well known that countable sets are N-sets. Let $p > 0$, let (k_n) be a sequence of positive integers, and let (a_n) be a sequence of real numbers with $a_n \geq 0$ for each $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n = \infty$. Then the set

$$\left\{\theta \in \mathbb{R}\colon \ \sum_{n=1}^{\infty} a_n \left|1 - e^{ik_n\theta}\right|^p < \infty\right\}$$

is shown in [7, Section 2.5] to be a Borel subgroup of $\mathbb{R}$ which is a N-set. Furthermore, this later group is uncountable in the case where $\sum_{n=1}^{\infty} a_n(k_n/k_{n+1})^2 < \infty$ and $p = 2$. It is well known that N-sets have Lebesgue measure zero. Nevertheless, it is also known that there are N-sets supporting continuous probability measures. It is even known that there are N-sets which have Hausdorff dimension 1.

Theorem 3.4. *Let E be a N-set of $\mathbb{R}/2\pi\mathbb{Z}$. Then the set of rotation operators on $L^2(\mathbb{S}^1)$ corresponding to rotations of $\mathbb{S}^1$ by angles from E does not determine the norm topology of $L^2(\mathbb{S}^1)$.*

Proof. We first observe that $\tau_2(z)(1) = 1$ for each $z \in \mathbb{T}$.

It is well known (see [15, Vol. I, Chapter VI]) that there exists a series of sines $\sum_{n=1}^{\infty} \rho_n \sin(n\theta)$ converging absolutely on E but with $\rho_n \geq 0$ ($n \in \mathbb{N}$) and $\sum_{n=1}^{\infty} \rho_n = \infty$.

Let $\theta_1, \ldots, \theta_J \in E$, $f_1, \ldots, f_J \in L^2(\mathbb{T})$, and let $f = \sum_{j=1}^{J} (\tau_2(z_j) - I)(f_j)$, where we are writing $z_j = e^{i\theta_j}$ for $j = 1, \ldots, J$. Then

$$\widehat{f}(2n) = \sum_{j=1}^{J} (z_j^{-2n} - 1)\widehat{f}_j(2n)$$

and so

$$\sqrt{\rho_n}\left|\widehat{f}(2n)\right| \leq \sum_{j=1}^{J} \sqrt{\rho_n}\left|z_j^{2n} - 1\right| \left|\widehat{f}_j(2n)\right|.$$

Therefore

$$\sum_{n=1}^{\infty} \sqrt{\rho_n}\left|\widehat{f}(2n)\right| \leq \sum_{j=1}^{J}\left[\left(\sum_{n=1}^{\infty} \rho_n \left|1 - z_j^{2n}\right|^2\right)^{1/2} \left(\sum_{n=1}^{\infty}\left|\widehat{f}_j(2n)\right|^2\right)^{1/2}\right] \leq$$

$$\sum_{j=1}^{J} \|f_j\|_2 \left(\sum_{n=1}^{\infty} \rho_n 2\left|1 - z_j^{2n}\right|\right)^{1/2} = \sum_{j=1}^{J} \|f_j\|_2 \left(\sum_{n=1}^{\infty} \rho_n 4\left|\sin(n\theta_j)\right|\right)^{1/2} < \infty.$$

By Lemma 3.3, there is a discontinuous linear functional on $L^2(\mathbb{T})$ which vanishes on $\sum_{\theta\in E}(\tau_2(e^{i\theta}) - I)(L^2(\mathbb{T}))$ and so it satisfies condition (ii) in Lemma 3.2, which completes the proof. ☐

A closed subset $E \subset \mathbb{R}/2\pi\mathbb{Z}$ is said to be a *Dirichlet set* if there is a strictly increasing sequence (k_n) of positive integers such that

$$\lim_{n\to\infty} \sup_{\theta\in E} \left|1 - e^{ik_n\theta}\right| = 0.$$

We refer the reader to [9] for a thorough discussion about such sets. Dirichlet sets are N-sets and our next goal is to show that those thinner sets are useless for determining the norm topology even when considering the convolution operators corresponding to all the measures carried by such sets. Note that the convolution operators corresponding to the point mass measures are nothing but the operators corresponding to the rotations.

Theorem 3.5. *Let E be a Dirichlet subset of $E \subset \mathbb{R}/2\pi\mathbb{Z}$. Then the set*

$$\left\{f \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{-i\theta}\,\cdot\,\right)\, d\mu(\theta)\right\}$$

of convolution operators on $L^2(\mathbb{S}^1)$ corresponding to all bounded complex-valued regular Borel measures μ of $\mathbb{R}/2\pi\mathbb{Z}$ which are concentrated on E does not determine the norm of $L^2(\mathbb{S}^1)$.

Proof. Let Θ denote the set of all measures described in the theorem and let $\mathcal{T} = \{T_\mu\colon\ \mu \in \Theta\}$ be the set of convolution operators corresponding to measures of Θ.

Note that $T_\mu(1) = \widehat{\mu}(0)1$ for each $\mu \in \Theta$.

On account of the definition of Dirichlet set, we can construct a strictly increasing sequence (k_n) of positive integers such that $\left|1 - e^{ik_n\theta}\right| < 4^{-n}$ for all

$\theta \in E$ and $n \in \mathbb{N}$. Let $\mu_1, \ldots, \mu_J \in \Theta$, $f_1, \ldots, f_J \in L^2(\mathbb{S}^1)$, and let $f = \sum_{j=1}^{J}(T_{\mu_j} - \widehat{\mu}_j(0)I)(f_j)$. Then

$$\widehat{f}(k_n) = \sum_{j=1}^{J}(\widehat{\mu}_j(k_n) - \widehat{\mu}_j(0))\widehat{f}_j(k_n)$$

and so

$$\left|\widehat{f}(k_n)\right| \leq \sum_{j=1}^{J} |\widehat{\mu}_j(k_n) - \widehat{\mu}_j(0)| \; \left|\widehat{f}_j(k_n)\right|.$$

On the other hand, if $\mu \in \Theta$, then

$$|\widehat{\mu}(k_n) - \widehat{\mu}(0)| = \left|\int_E \left(e^{-ik_n\theta} - 1\right) \, d\mu(\theta)\right| \leq \int_E \left|e^{ik_n\theta} - 1\right| \, d|\mu|(\theta) \leq 4^{-n}\|\mu\|$$

for each $n \in \mathbb{N}$. Consequently, we deduce that

$$\left|\widehat{f}(k_n)\right| \leq \sum_{j=1}^{J} 4^{-n}\|\mu_j\| \left|\widehat{f}_j(k_n)\right|,$$

which clearly implies that $\sum_{n=1}^{\infty} 2^n|\widehat{f}(k_n)|^2 < \infty$. Lemma 3.3 yields a discontinuous linear functional on $L^2(\mathbb{T})$ vanishing on $\sum_{\mu\in\Theta}(T_\mu - \widehat{\mu}(0))(L^2(\mathbb{T}))$. It is straightforward to check that such a functional satisfies condition (ii) in Lemma 3.2 and therefore $\mathcal{T}$ does not determine the norm topology of $L^2(\mathbb{S}^1)$. □

Remark 3.6. One question still unanswered is whether or not there is a subgroup of $SO(2)$ of Lebesgue measure zero such that the set consisting of the corresponding rotation operators on $L^2(\mathbb{S}^1)$ determines the norm topology of that space.

References

[1] J. Alaminos, J. Extremera and A.R. Villena, *Applications of Kazhdan's property (T) to automatic continuity*, J. Math. Anal. Appl., **307** (2005), 167–180.

[2] N. Bourbaki, *Éléments de mathématique. Intégration*, Hermann, Paris, 1963.

[3] J.B. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics, 96, Springer-Verlag, New York, 1985.

[4] V.G. Drinfeld, *Finitely additive measure on S^2 and S^3, invariant with respect to rotations*, Funct. Anal. Appl., **18** (1984), 245–246.

[5] G.B. Folland, *A Course in Abstract Harmonic Analysis*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[6] P. de la Harpe and A. Valette, *La propriété (T) pour les groupes localement compacts*, Astérisque 175, Soc. Math. de France, 1989.

[7] B. Host, J.F. Méla, and F. Parreau, *Non singular transformations and spectral analysis of measures*, Bull. Soc. Math. France, **119** (1991), 33–90.

[8] K. Jarosz, *Uniqueness of translation invariant norms*, J. Funct. Anal., **174** (2000), 417–429.

[9] T.W. Korner, *Some results on Kronecker, Dirichlet and Helson sets*, Ann. Inst. Fourier (Grenoble), **20** fasc. 2 (1970), 219–324 .

[10] G.A. Margulis, *Some remarks on invariant means*, Mh. Math., **90** (1980), 233–235.

[11] A.M. Sinclair, *Automatic Continuity of Linear Operators*, Cambridge Univ. Press, Cambridge, UK, 1976.

[12] D. Sullivan, *For $n > 3$ there is only finitely-additive rotationally invariant measure on the n-sphere defined on all Lebesgue measurable sets*, Bull. Amer. Math. Soc., **4** (1981), 121–123.

[13] A.R. Villena, *Invariant functionals and the uniqueness of invariant norms*, J. Funct. Anal., **215** (2004), 365–398.

[14] A. Wilansky, *Functional Analysis*, Blaisdell Publishing Co., Ginn and Co., New York-Toronto-London, 1964.

[15] A. Zygmund, *Trigonometric Series. Vol. I, II*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002.

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