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ON SINGLE-VALUEDNESS OF SET-VALUED MAPS SATISFYING LINEAR INCLUSIONS

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ABSTRACT. In this paper we give some results on single-valuedness of set-valued maps satisfying linear inclusions.

1. Introduction and preliminaries

Some basic notions of set-valued analysis, as linearity, affinity, convexity, additivity, are defined by linear inclusions (cf., e.g., [2]—[5], [8]–[12],[14]–[18]). Under appropriate conditions, such set-valued maps with the property that their value at a point is a singleton, are single-valued maps. We recall some known results of this type. Let X, Y be real vector spaces. We denote by $\mathcal{P}_0(Y)$ the collection of all nonempty subsets of Y. A set-valued map $F : X \to \mathcal{P}_0(Y)$ is called:

1) **additive**, if for all $x, y \in X$

$$F(x+y) = F(x) + F(y);$$

2) **convex**, if for all $x, y \in X$ and all $\lambda \in [0, 1]$

$$(1-\lambda)F(x) + \lambda F(y) \subseteq F((1-\lambda)x + \lambda y); \tag{1.1}$$

3) p-**convex**, if (1.1) holds for all $x, y \in X$ and a fixed $\lambda = p \in (0, 1)$;

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4) **midconvex**, if (1.1) holds for all $x, y \in X$ and $\lambda = \frac{1}{2}$;

5) **affine** if (1.1) holds for all $x, y \in X$ and all $\lambda \in \mathbb{R}$;

6) **convex process**, if for all $x, y \in X$ and all $\alpha, \beta \geq 0$

$$\alpha F(x) + \beta F(y) \subseteq F(\alpha x + \beta y); \tag{1.2}$$

7) (α, β)**-convex process** if (1.2) holds for all $x, y \in X$ and a fixed pair (α, β), $\alpha > 0$, $\beta > 0$;

8) **linear** if (1.2) holds for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{R}$.

For a set-valued map $F : X \to \mathcal{P}_0(Y)$ with singleton values, we will write $F(x) = y$ instead of $F(x) = \{y\}$ and we will identify such set-valued map with the function $F : X \to Y$.

Let X, Y be real vector spaces and $F : X \to \mathcal{P}_0(Y)$ be a set-valued map. We denote by 0_X and 0_Y the zero vectors of X and Y, respectively.

Theorem 1.1. (Berge [3, p. 134]) *If $F : X \to \mathcal{P}_0(Y)$ is linear and $F(0_X) = \{0_Y\}$, then F is single-valued.*

Theorem 1.2. (Godini [5]) *If $F : \mathbb{R} \to \mathcal{P}_0(\mathbb{R})$ is additive and $F(0) = \{0\}$, then F is single-valued.*

Theorem 1.3. (Tan [18]) *If $F : X \to \mathcal{P}_0(Y)$ is affine and $F(x_0)$ is singleton for some $x_0 \in X$, then F is single-valued.*

Theorem 1.4. (Deutsch–Singer [4]) *If $F(x_0)$ is a singleton for some $x_0 \in X$, then $F : X \to \mathcal{P}_0(Y)$ is convex if and only if F is single-valued and affine.*

Theorem 1.5. (Rockafellar [14, p. 414]) *If $F : \mathbb{R}^m \to \mathcal{P}_0(\mathbb{R}^n)$ is a convex process and $F(0_{\mathbb{R}^m})$ is bounded, then F is single-valued and linear.*

Theorem 1.6. (Nikodem [9]) *Suppose that Y is a topological vector space, $F : X \to \mathcal{P}_0(Y)$ is additive and $F(x_0)$ is bounded for some $x_0 \in X$. Then F is single-valued.*

Theorem 1.7. (Popa [11]) *If $F : X \to \mathcal{P}_0(Y)$ is p-convex and $F(x_0)$ is a singleton for some $x_0 \in X$, then there exist an additive function $a : X \to Y$ and a constant $c \in Y$ such that $F(x) = a(x) + c, \quad x \in X$.*

Theorem 1.8. (Nikodem–Papalini–Vercillo [10]) *If $D \subseteq X$ is a convex and algebraically open set, $F : D \to \mathcal{P}_0(Y)$ is a set-valued map, $F(x_0)$ is a singleton for some $x_0 \in D$, then F is midconvex if and only if there exist an additive function $a : X \to Y$ and a constant $c \in Y$ such that*

$$F(x) = a(x) + c, \quad x \in D.$$

In this paper we will study the single-valuedness of the solutions of a general linear inclusion of the form

$$\alpha F(x) + \beta F(y) \subseteq F(\gamma x + \delta y), \quad x, y \in X, \tag{1.3}$$

where $F : X \to \mathcal{P}_0(Y)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are fixed or ranges over a given subset of $\mathbb{R}$. For particular values of $\alpha, \beta, \gamma, \delta$ we get all the classes of set-valued maps defined above. The linear inclusion (1.3) can be also regarded as an extension of the general linear equation studied, among others, by Aczél, Daróczy and Losonczi (cf. [1] and the references therein) to set-valued maps. A representation of set-valued solutions of the general linear equation is given by D. Popa and N. Vornicescu in [12].

We give also a result on single-valuedness for convex processes and (α, β)-convex processes, where the condition that $F(x_0)$ is a singleton is replaced by the assumption that $F(x_0)$ is a bounded subset of a topological vector space Y.

2. Main results

The first of our main results concerns single-valuedness of the solutions of a general linear inclusion.

Theorem 2.1. *Let X, Y be real vector spaces and $\alpha, \beta, \gamma, \delta \in \mathbb{R} \setminus \{0\}$. If a set-valued map $F : X \to \mathcal{P}_0(Y)$ satisfies the general linear inclusion*

$$\alpha F(x) + \beta F(y) \subseteq F(\gamma x + \delta y), \quad x, y \in X, \tag{2.1}$$

and $F(x_0)$ is a singleton for some $x_0 \in X$, then F is single-valued. Moreover, F may only be be of the form

$$F(x) = a(x) + c, \quad x \in X, \tag{2.2}$$

with an additive map $a : X \to Y$ and a constant $c \in Y$.

Proof. Take an arbitrary $x \in X$ and put $y = \dfrac{x_0 - \gamma x}{\delta}$ in (2.1) to get

$$\alpha F(x) + \beta F\left(\frac{x_0 - \gamma x}{\delta}\right) \subseteq F(x_0).$$

Since $F(x_0)$ is a singleton, it follows that $F(x)$ is also a singleton. Therefore F is single-valued. Consequently, the relation (2.1) becomes

$$\alpha F(x) + \beta F(y) = F(\gamma x + \delta y), \quad x, y \in X. \tag{2.3}$$

It is known that if F satisfies (2.3), then it must be of the form (2.2) (cf. [1, p. 66]). However, we will present a proof of this fact for the sake of completeness. For $y = 0_X$ in (2.3) one gets

$$\alpha F(x) + \beta F(0_X) = F(\gamma x), \quad x \in X, \tag{2.4}$$

and for $x = 0_X$ in (2.3) one gets

$$\alpha F(0_X) + \beta F(y) = F(\delta y), \quad y \in X. \tag{2.5}$$

Now let u, v be arbitrary elements in X. By (2.3) one obtains

$$F(u + v) = \alpha F\left(\frac{u}{\gamma}\right) + \beta F\left(\frac{v}{\delta}\right) \tag{2.6}$$

and by (2.4) and (2.5) it follows

$$\begin{aligned} \alpha F\left(\frac{u}{\gamma}\right) &= F(u) - \beta F(0_X), \\ \beta F\left(\frac{v}{\delta}\right) &= F(v) - \alpha F(0_X). \end{aligned} \tag{2.7}$$

The relations (2.6) and (2.7) lead to

$$\begin{aligned} F(u+v) &= F(u) + F(v) - \alpha F(0_X) - \beta F(0_X) \\ &= F(u) + F(v) - F(\delta \cdot 0_X + \beta \cdot 0_X) \\ &= F(u) + F(v) - F(0_X). \end{aligned} \tag{2.8}$$

Define the map $a : X \to Y$, $a(x) = F(x) - F(0_X)$, $x \in X$. By (2.8) it follows

$$a(x+y) = a(x) + a(y), \quad x, y \in X,$$

therefore a is an additive map. Denoting $c = F(0_X)$ we get

$$F(x) = a(x) + c, \quad x \in X, \tag{2.9}$$

which was to be proved. □

Remark 2.2. Taking appropriate coefficients in (2.1) we get the quoted above Theorems 1.1–1.4, 1.7 and 1.8 (in the case where $D = X$) as simple consequences of Theorem 2.1.

Remark 2.3. Let us note that functions of the form (2.9) need not, in general, satisfy the equation (2.3). This depends essentially on the coefficients in (2.3) (cf. Aczél [1]). For instance, if $\alpha, \gamma \in (0,1)$, $\beta = 1 - \alpha$, $\delta = 1 - \gamma$ and α, γ are conjugate (i.e. they are both transcendental or they are algebraic and have the same minimal polynomial with rational coefficients), then there exist (non-zero) additive solutions of (2.3). However, if α, γ are not conjugate, then the only solutions of (2.3) are constant functions (cf. [6, 7]). It is also well known that F satisfies (2.3) with $\alpha = \beta = \gamma = \delta = 1/2$ (i.e. F is a Jensen function) if and only if it is of the form (2.9)

Now we give a result on single-valuedness of convex processes, in which the condition that $F(x_0)$ is a singleton for some $x_0 \in X$ is replaced by the boundedness of $F(x_0)$.

Theorem 2.4. *Let X be a real vector space, Y a real topological vector space and $F : X \to \mathcal{P}_0(Y)$ a convex-process with the property that there exists an $x_0 \in X$ such that $F(x_0)$ is a bounded set. Then F is single-valued and linear.*

Proof. By the relation (1.2) for $\alpha = \beta = 1$ it follows

$$F(x_0) + F(0_X) \subseteq F(x_0).$$

Hence $F(0_X)$ is a bounded set, in view of the boundedness of $F(x_0)$. For $\alpha = \beta = 1$, $x = y = 0_X$ in (1.2) one gets

$$F(0_X) + F(0_X) \subseteq F(0_X).$$

By induction it follows that

$$nF(0_X) \subseteq F(0_X) + \cdots + F(0_X) \subseteq F(0_X),$$

for every positive integer n. Hence, the boundedness of $F(0_X)$ implies

$$F(0_X) = \{0_Y\}.$$

Now let $x \in X$. Since

$$F(x) + F(-x) \subseteq F(x + (-x)) = \{0_Y\},$$

$F(x)$ is a singleton. Thus F is single-valued and therefore it satisfies the relation

$$\alpha F(x) + \beta F(y) = F(\alpha x + \beta y)$$

for all $\alpha, \beta \geq 0$ and all $x, y \in X$. Hence it follows that F is additive and positively homogeneous. But being additive it is odd and, consequently, it is homogeneous. Thus F is linear. □

Remark 2.5. The above result improves Theorems 1.5 and 1.1 quoted at the beginning of this paper and [4, Corollary 2.2].

Now we will present a more general version of the previous result for (α, β)-convex processes.

Theorem 2.6. *Let X be a real vector space, Y a real topological vector space and $F : X \to \mathcal{P}_0(Y)$ an (α, β)-convex process with $\alpha + \beta > 1$. If $F(x_0)$ is bounded for some $x_0 \in X$, then F is single-valued and additive. Moreover, F is α-homogeneous and β-homogeneous.*

Proof. By the relation (1.2) we get

$$\alpha F\left(\frac{x_0}{\alpha}\right) + \beta F(0_X) \subseteq F(x_0),$$

whence $F(0_X)$ is a bounded set.

For $x = y = 0_X$ in (1.2) we have

$$\alpha F(0_X) + \beta F(0_X) \subseteq F(0_X),$$

and taking account of $(\alpha + \beta)F(0_X) \subseteq \alpha F(0_X) + \beta F(0_X)$ we get

$$(\alpha + \beta)F(0_X) \subseteq F(0_X).$$

Hence, by induction,

$$(\alpha + \beta)^n F(0_X) \subseteq F(0_X),$$

for every positive integer n.

The boundedness of $F(0_X)$ and the condition $\alpha + \beta > 1$ leads to

$$F(0_X) = \{0_Y\}. \tag{2.10}$$

Now, by Theorem 2.1 F is single-valued and has the form

$$F(x) = a(x) + c, \tag{2.11}$$

where $a : X \to Y$ is an additive map and $c \in Y$ is a constant. Putting $x = 0_X$ in (2.11) and using once more (2.10), we get $c = 0_Y$. Thus $F = a$ is additive. Being a single-valued (α, β)-convex process, F satisfies

$$\alpha F(x) + \beta F(y) = F(\alpha x + \beta y). \tag{2.12}$$

Now, putting in (2.12) $y = 0_X$ and $x = 0_X$ in turn, we get

$$F(\alpha x) = \alpha F(x) \text{ and } F(\beta x) = \beta F(x).$$

This finishes the proof. □

Remark 2.7. The above result generalizes Theorems 1.5 and 1.6 quoted at the beginning of this paper.

Remark 2.8. If $\alpha + \beta \le 1$ then the result obtained in Theorem 2.6 does not hold. Indeed:

1) If $\alpha + \beta = 1$ and M is a nonempty convex and bounded subset of Y then the set-valued map F given by $F(x) = M$ for every $x \in X$ is an (α, β)-convex process but it is not single valued if M has at least two elements.

2) If $\alpha + \beta < 1$ and M is a nonempty convex bounded and balanced subset of Y then the set-valued map F is given by $F(x) = M$ for every $x \in X$ an (α, β)-convex process but it is not single valued if M has at least two elements.

In the previous theorems, the condition that the domain of F is the whole space X is essential.

Example 2.9. The set-valued map $F : [0, \infty) \to \mathcal{P}_0(\mathbb{R})$

$$F(x) = [0, x], \quad x \in [0, \infty)$$

is a convex process, $F(0) = \{0\}$, $F(x)$ is bounded for all $x \in [0, \infty)$, but F is not single valued.

Example 2.10. Let $\alpha, \beta \ge 1$. The set-valued map $F : [0, \infty) \to \mathcal{P}_0(\mathbb{R})$

$$F(x) = [0, x^2], \quad x \in [0, \infty)$$

is an (α, β)-convex process, $F(0) = \{0\}$ but it is not additive.

Finally we give a remark on the definition of affine set-valued maps. Some authors define the notion of affine set-valued maps using equality instead of inclusion in (1.1) (cf., e.g., [18]; see also [4]) and in general the domain of F is an affine set. We will prove that both definitions are equivalent.

Proposition 2.11. *Let X, Y be real vector spaces, D an affine subset of X and $F : D \to \mathcal{P}_0(Y)$. The following relations are equivalent:*

1) $(1 - \lambda)F(x) + \lambda F(y) \subseteq F((1 - \lambda)x + \lambda y)$ for all $x, y \in D$ and all $\lambda \in \mathbb{R}$;

2) $(1 - \lambda)F(x) + \lambda F(y) = F((1 - \lambda)x + \lambda y)$ for all $x, y \in D$ and all $\lambda \in \mathbb{R}$.

Proof. Only the implication 1) $\Rightarrow$ 2) needs a proof.

Let $\lambda \in \mathbb{R}$, $x, y \in D$ and $z = (1-\lambda)x + \lambda y$. We have to prove

$$F(z) \subseteq (1-\lambda)F(x) + \lambda F(y). \tag{2.13}$$

For $\lambda = 1$ the relation (2.13) is obvious. Let $\lambda \neq 1$. We have:

$$x = \frac{1}{1-\lambda}z + \frac{-\lambda}{1-\lambda}y$$

and since

$$\frac{1}{1-\lambda} + \frac{-\lambda}{1-\lambda} = 1$$

one obtains

$$\frac{1}{1-\lambda}F(z) + \frac{(-\lambda)}{1-\lambda}F(y) \subseteq F(x).$$

Hence

$$F(z) - \lambda F(y) \subseteq (1-\lambda)F(x)$$

and, consequently,

$$F(z) \subseteq (1-\lambda)F(x) + \lambda F(y).$$

□

Remark 2.12. The above result is analogous to the well known characterization of affine single-valued functions (see, e.g., [13, p. 214]). Note, however, that in contrast with the single-valued case, condition (1.1) assumed for all $\lambda \in \mathbb{R}$ (i.e. the affinity of F) is not equivalent to the same condition assumed only for $\lambda \in [0,1]$. For instance, if A is a convex non-singleton set then the set-valued map $F(x) = A$, $x \in \mathbb{R}$, satisfies (1.1) for all $\lambda \in [0,1]$, but it is not affine.

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