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UNIVALENCE CONDITIONS FOR SOME GENERAL INTEGRAL OPERATORS

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Abstract. We consider two general integral operators where are the extensions of the Kim–Merkes operator and Pfaltzgraff operator. We will proved in this paper the univalent conditions for these operators when we make some restrictions about the functions from your definitions.

1. Introduction and preliminaries

We consider the unit open disk denoted by $\mathcal{U}$, $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ the class of functions which are the form $f(z) = z + a_2 z^2 + \cdots$.Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of the functions $f \in \mathcal{A}$ which are univalent in $\mathcal{U}$.

J. Pfatzgraff in [5] and Y.J. Kim and E.P. Merkes in [3] had obtained the next results.

Theorem 1.1. *If the function $f \in \mathcal{S}$ and α is a complex number with $|\alpha| \le \frac{1}{4}$, then the integral operator G_α given by*

$$G_\alpha(z) = \int_0^z [f'(t)]^\alpha \, dt$$

is in the class $\mathcal{S}$.

Theorem 1.2. *If the function $f \in \mathcal{S}$ and γ is a complex number with $|\alpha| \le \frac{1}{4}$ then the integral operator F_γ given by*

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$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt$$

is in the class $\mathcal{S}$.

In [4] V. Pescar and S. Owa had obtained the next results:

Theorem 1.3. *Let* $f \in \mathcal{S}$, γ *be a complex number and* M *be a positiv real number. If* f *satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M, \;\; z \in \mathcal{U},$$

then, for

$$|\gamma| \leq \frac{3\sqrt{3}}{2M},$$

the integral operator

$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt$$

is in the class $\mathcal{S}$.

Theorem 1.4. *Let* $f \in \mathcal{S}$, α *be a complex number and* M *be a positiv real number. If* f *satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < M \qquad (z \in \mathcal{U}),$$

then, for

$$|\alpha| \leq \frac{3\sqrt{3}}{2M},$$

the integral operator

$$G_\alpha(z) = \int_0^z \left(f'(t)\right)^\alpha dt$$

belongs to $\mathcal{S}$.

Also, we consider the general Schwarz Lemma:

Lemma 1.5. *Let the function* $f(z)$ *be regular in the disk* $\mathcal{U}_R = \{z \in \mathbb{C}; |z| < R\}$, *with* $|f(z)| < M$ *for fixed* M. *If* $f(z)$ *has one zero with multiply* $\geq m$ *for* $z = 0$, *then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \;\; z \in \mathcal{U}_R.$$

The equality can hold only if $f(z) = e^{i\theta} \frac{M}{R^m} z^m$, *where* θ *is constant.*

To discuss our problems for the general integral operators, we need the following theorem:

Theorem 1.6. *If $f(z) = z + a_2 z^2 + \ldots$ is analytic in $\mathcal{U}$ and*

$$\left(1 - |z|^2\right) \left| \frac{z f''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$, then the function $f(z)$ is univalent in $\mathcal{U}$.

We consider two general integral operators. First integral operator was introduced by D. Breaz and N. Breaz in the paper [1]. This operator have the form

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t} \right)^{\alpha_n} dt \tag{1.1}$$

and for $n = 1$ we obtain the operator of Kim and Merkes and for $n = 1$ and $\alpha_1 = 1$ we obtain the integral operator of Alexander.

The next operator was introduced from D. Breaz, S. Owa and N. Breaz in [2] and has the form:

$$G(z) = \int_0^z \left(f_1'(t)\right)^{\alpha_1} \cdots \left(f_n'(t)\right)^{\alpha_n} dt. \tag{1.2}$$

Observe that for $n = 1$ we obtain the integral operator of Pfaltzgraff.

2. Main results

Theorem 2.1. *Let $f_i \in \mathcal{S}$, α_i be complex numbers and M_i be positive real numbers for $i \in \{1, \cdots, n\}$. If f_i satisfies the conditions*

$$\left| \frac{z f_i'(z)}{f_i z} - 1 \right| < M_i, z \in \mathcal{U}, i \in \{1, \cdots, n\} \tag{2.1}$$

and

$$M_1 |\alpha_1| + \cdots + M_n |\alpha_n| \leq \frac{3\sqrt{3}}{2} \tag{2.2}$$

then the integral operator defined in (1.1) is univalent.

Proof. Let

$$h(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdots \alpha_n|} \cdot \frac{z F''(z)}{F'(z)}.$$

According with the formulla (1.1) we obtain:

$$h(z) = \sum_{i=1}^{n} \frac{\alpha_i}{|\alpha_1 \cdot \alpha_2 \cdots \alpha_n|} \cdot \left(\frac{z f_i'(z)}{f_i(z)} - 1 \right)$$

and

$$|h(z)| \leq \sum_{i=1}^{n} \frac{|\alpha_i|}{|\alpha_1 \cdot \alpha_2 \cdots \alpha_n|} \cdot \left| \frac{z f_i'(z)}{f_i(z)} - 1 \right|. \tag{2.3}$$

Applying in the formula (2.3) the inequality (2.1) we obtain that:

$$|h(z)| \leq \frac{M_1 |\alpha_1| + \cdots + M_n |\alpha_n|}{|\alpha_1 \cdot \alpha_2 \cdots \alpha_n|}$$

for all $z \in \mathcal{U}$. The last equation and since $h(0) = 0$ applying Schwarz Lemma we obtain that:

$$\frac{1}{|\alpha_1 \cdot \alpha_2 \cdots \alpha_n|} \cdot \left| \frac{zF''(z)}{F'(z)} \right| \leq \frac{M_1 |\alpha_1| + \cdots + M_n |\alpha_n|}{|\alpha_1 \cdot \alpha_2 \cdots \alpha_n|} \cdot |z|,$$

for all $z \in \mathcal{U}$. This last inequality is equivalent with:

$$\left(1 - |z|^2\right) \cdot \left| \frac{zF''(z)}{F'(z)} \right| \leq (M_1 |\alpha_1| + \cdots + M_n |\alpha_n|) \cdot \left(1 - |z|^2\right) \cdot |z|. \tag{2.4}$$

But

$$\max_{|z| \leq 1} \left(1 - |z|^2\right) \cdot |z| = \frac{2}{3\sqrt{3}}. \tag{2.5}$$

From (2.2), (2.4) and (2.5) we obatin that:

$$\left(1 - |z|^2\right) \cdot \left| \frac{zF''(z)}{F'(z)} \right| \leq 1.$$

We apply the Theorem 1.6 and obtain that the integral operator (1.1) is univalent. □

Remark 2.2. In Theorem 2.1, for $n = 1$ and $\alpha_1 = \gamma$ we obtain Theorem 1.3.

Corollary 2.3. *Let $f \in \mathcal{S}$. If f satisfies*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{3\sqrt{3}}{2}, \;\; z \in \mathcal{U},$$

the integral operator of Alexander

$$H(z) = \int_0^z \frac{f(t)}{t} dt$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2.1 we consider $n = 1$, and $\alpha_1 = 1$. □

Theorem 2.4. *Let $f_i \in \mathcal{S}$, α_i be complex numbers and M_i be pozitive real numbers for $i \in \{1, \cdots, n\}$. If f_i satisfies the conditions*

$$\left| \frac{z f_i''(z)}{f_i'(z)} \right| < M_i, z \in \mathcal{U}, i \in \{1, \cdots, n\} \tag{2.6}$$

and

$$|\alpha_1| \cdot M_1 + \cdots + |\alpha_n| \cdot M_n \leq \frac{3\sqrt{3}}{2} \tag{2.7}$$

the integral operator defined in (1.2) *are univalent.*

Proof. We consider the function

$$p(z) = \frac{1}{|\alpha_1 \cdots \alpha_n|} \cdot \frac{zG''(z)}{G'(z)}. \tag{2.8}$$

We have:

$$\frac{zG''(z)}{G'(z)} = \alpha_1 \frac{z f_1''(z)}{f_1'(z)} + \cdots + \alpha_1 \frac{z f_n''(z)}{f_n'(z)} \tag{2.9}$$

From (2.8) and (2.9) we have:

$$p(z) = \frac{1}{|\alpha_1 \cdots \alpha_n|} \left(\alpha_1 \frac{z f_1''(z)}{f_1'(z)} + \cdots + \alpha_1 \frac{z f_n''(z)}{f_n'(z)} \right)$$

From (2.6) and last equality obtain that:

$$|p(z)| \leq \frac{|\alpha_1| M_1 + \cdots + |\alpha_n| M_n}{|\alpha_1 \cdots \alpha_n|}$$

for all $z \in \mathcal{U}$.

Since $p(0) = 0$ and applying Schwarz Lemma we have:

$$\frac{1}{|\alpha_1 \cdots \alpha_n|} \cdot \left| \frac{zG''(z)}{G'(z)} \right| \leq \frac{|\alpha_1| M_1 + \cdots + |\alpha_n| M_n}{|\alpha_1 \cdots \alpha_n|} \cdot |z|,$$

for all $z \in \mathcal{U}$.

This inequality are equivalent with

$$\left(1 - |z|^2\right) \left| \frac{zG''(z)}{G'(z)} \right| \leq (|\alpha_1| M_1 + \cdots + |\alpha_n| M_n) \cdot |z| \cdot \left(1 - |z|^2\right), \tag{2.10}$$

for all $z \in \mathcal{U}$.

But

$$\max_{|z| \leq 1} \left(1 - |z|^2\right) \cdot |z| = \frac{2}{3\sqrt{3}}. \tag{2.11}$$

So, from (2.7), (2.10) and (2.11) we obtain that

$$\left(1-|z|^2\right)\left|\frac{zG''(z)}{G'(z)}\right| \leq 1$$

for all $z \in \mathcal{U}$.

Applying Theorem 1.6 we obtain that the integral operator defined in (1.2) is univalent. □

Remark 2.5. In the Theorem 2.4, for $n=1$ we obtain Theorem 1.4.

Corollary 2.6. *Let $f \in \mathcal{A}$. If f satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3\sqrt{3}}{2} \quad (z \in \mathcal{U}),$$

then, $f \in \mathcal{S}$.

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