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A STUDY ON SOME NEW TYPES OF HARDY–HILBERT'S INTEGRAL INEQUALITIES

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ABSTRACT. Some new kinds of Hardy–Hilbert's integral inequality in which the weight function is homogeneous function are given. Other results are also obtained.

1. INTRODUCTION

Let $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)dt < \infty \text{ and } 0 < \int_0^\infty g^2(t)dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left(\int_0^\infty f^2(t)dt \int_0^\infty g^2(t)dt \right)^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible (cf. Hardy et al. [2]). Inequality (1.1) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy [1] as follows: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)dt < \infty \text{ and } 0 < \int_0^\infty g^q(t)dt < \infty,$$

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then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{sin(\pi/p)} \left(\int_0^\infty f^p(t)dt\right)^{1/p} \left(\int_0^\infty g^q(t)dt\right)^{1/q}, \quad (1.2)$$

where the constant factor $\frac{\pi}{sin(\pi/p)}$ is the best possible. Inequality (1.2) is called Hardy–Hilbert's integral inequality and is important in analysis and applications (cf. Mitrinovic et al. [3]).

B. Yang gave the following extension of (1.2) as follows :

Theorem [4]. If $\lambda > 2 - \min\{p, q\}$ and $f, g \geq 0$ satisfy

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)dt < \infty \text{ and } 0 < \int_0^\infty t^{1-\lambda} g^q(t)dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < k_\lambda(p) \left(\int_0^\infty t^{1-\lambda} f^p(t)dt\right)^{1/p} \left(\int_0^\infty t^{1-\lambda} g^q(t)dt\right)^{1/q},$$

where the constant factor $k_\lambda(p) = B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})$ is the best possible B is the beta function. The function $f(x, y)$ is said to be homogeneous of degree λ, if

$$f(tx, ty) = t^\lambda f(x, y) \qquad (t > 0).$$

The object of this paper is that to give some new inequalities similar to that of Hardy–Hilbert's integral inequality.

2. Main result

Lemma 2.1. *Let $K(t, 1), K(1, t)$ be positive increasing functions, $0 < \mu + 1 \leq \alpha$. Set for $s \geq 1$,*

$$f(s) = s^{-\alpha} \int_0^s \frac{t^\mu}{K(1,t)}dt, \qquad g(s) = s^{-\alpha} \int_0^s \frac{t^\mu}{K(t,1)}dt.$$

Then

$$f(s) \leq f(1), g(s) \leq g(1).$$

Proof. We have

$$\begin{aligned} f'(s) &= s^{-\alpha}\frac{s^\mu}{K(1,s)} - \alpha s^{-\alpha-1} \int_0^s \frac{t^\mu}{K(1,t)}dt \\ &\leq \frac{s^{\mu-\alpha}}{K(1,s)} - \frac{\alpha s^{-\alpha-1}}{K(1,s)} \int_0^s t^\mu dt \\ &= \frac{s^{\mu-\alpha}}{K(1,s)} \left(1 - \frac{\alpha}{\mu+1}\right) \leq 0. \end{aligned}$$

This shows that f is nonincreasing and hence $f(s) \leq f(1)$. The other part has a similar proof. □

The following is our main result

Theorem 2.2. *Let $f,g\geq 0$, $K(u,v)$ be positive, increasing, homogeneous function of degree λ, $0<\lambda\leq\min\{(1-b)q/p,(1-a)p/q\}$, $a,b>0$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. Set*

$$F(u)=\int_0^u f(t)dt,\qquad G(v)=\int_0^v g(t)dt.$$

Then

$$\int_0^T\int_0^T\frac{F(u)G(v)}{K(u,v)}dudv \leq T^{\alpha}\sqrt[p]{pK_1}\sqrt[q]{qK_2}\left(\int_0^T(T-t)F^{p-1}(t)f(t)dt\right)^{1/p}\times\left(\int_0^T(T-t)G^{q-1}(t)g(t)dt\right)^{1/q},$$

where

$$K_1=\int_0^1\frac{t^{a-1}}{K(1,t)}dt, K_2=\int_0^1\frac{t^{b-1}}{K(t,1)}dt.$$

Proof.

$$\begin{aligned}\int_0^T\int_0^T\frac{F(u)G(v)}{K(u,v)}dudv &= \int_0^T\int_0^T\frac{F(u)v^{\frac{a-1}{p}}}{u^{\frac{b-1}{q}}K^{1/p}(u,v)}\times\frac{G(v)u^{\frac{b-1}{q}}}{v^{\frac{a-1}{p}}K^{1/q}(u,v)}dudv\\ &\leq\left(\int_0^T\int_0^T\frac{F^p(u)v^{a-1}}{u^{(b-1)p/q}K(u,v)}dudv\right)^{1/p}\\ &\times\left(\int_0^T\int_0^T\frac{G^q(v)u^{b-1}}{v^{(a-1)q/p}K(u,v)}dudv\right)^{1/q}\\ &= M^{1/p}N^{1/q}.\end{aligned}$$

We first consider

$$M=\int_0^T u^{(1-b)p/q}F^p(u)du\int_0^T\frac{v^{a-1}}{K(u,v)}dv.$$

Observe that on putting $v=uy, dv=udy, 0\leq y\leq t/u$, we have, in view of Lemma 2.1, by writing $\alpha=a+(1-b)p/q-\lambda$,

$$\begin{aligned}\int_0^T\frac{v^{a-1}}{K(u,v)}dv &= \int_0^{T/u}\frac{(uy)^{a-1}u}{K(u,uy)}dy=u^{a-\lambda}\int_0^{T/u}\frac{y^{a-1}}{K(1,y)}dy\\ &= u^{a-\lambda}\left(\frac{T}{u}\right)^{\alpha}\left(\frac{T}{u}\right)^{-\alpha}\int_0^{T/u}\frac{y^{a-1}}{K(1,y)}dy\\ &\leq T^{\alpha}u^{a-\lambda-\alpha}\int_0^1\frac{y^{a-1}}{K(1,y)}dy=T^{\alpha}K_1u^{a-\lambda-\alpha}.\end{aligned}$$

By above we obtain

$$\begin{aligned}M &\leq T^{\alpha}K_1\int_0^T u^{a+(1-b)p/q-\lambda-\alpha}F^p(u)du\\ &= T^{\alpha}K_1\int_0^T F^p(u)du.\end{aligned}$$

As

$$F^p(u)=\int_0^u\left(F^p(t)\right)'dt=p\int_0^u F^{p-1}(t)f(t)dt,$$

we have

$$\begin{aligned} M &\leq pT^{\alpha}K_1 \int_0^T \int_0^u F^{p-1}(t)f(t)dtdu \\ &= pT^{\alpha}K_1 \int_0^T (T-t)F^{p-1}(t)f(t)dt. \end{aligned}$$

Similarly, the other part follows by using Lemma 2.1, replacing α by β, where $\beta = b + (1-a)q/p - \lambda$ to obtain

$$N \leq qT^{\alpha}K_2 \int_0^T (T-t)G^{q-1}(t)g(t)dt.$$

This completes the proof of the theorem. □

3. Applications

Corollary 3.1. *By an application of Theorem 2.2, for the special case $a = b = \lambda/2$, we have*

$$\int_0^T \int_0^T \frac{F(u)G(v)}{K(u,v)} dudv \leq T^{\alpha} \sqrt[p]{pK_1} \sqrt[q]{qK_3} \left(\int_0^T (T-t)F^{p-1}(t)f(t)dt \right)^{1/p} \times \left(\int_0^T (T-t)G^{q-1}(t)g(t)dt \right)^{1/q},$$

where

$$K_3 = \int_1^{\infty} \frac{t^{\frac{\lambda}{2}-1}}{K(1,t)} dt,$$

Furthermore, when $K(u,v) = (u+v)^{\lambda}$, we have

$$\int_0^T \int_0^T \frac{F(u)G(v)}{(u+v)^{\lambda}} dudv \leq T^{\alpha} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \sqrt[p]{p} \sqrt[q]{q} \left(\int_0^T (T-t)F^{p-1}(t)f(t)dt \right)^{1/p} \times \left(\int_0^T (T-t)G^{q-1}(t)g(t)dt \right)^{1/q}.$$

Proof. For $a = b = \lambda/2$, we have $K_2 = K_3$ as

$$\int_0^1 \frac{t^{\frac{\lambda}{2}-1}}{K(t,1)} = \int_0^1 \frac{t^{\frac{\lambda}{2}-1}}{K(t,tt^{-1})} dt = \int_0^1 \frac{t^{-\frac{\lambda}{2}-1}}{K(1,t^{-1})} dt = \int_1^{\infty} \frac{t^{\frac{\lambda}{2}-1}}{K(1,t)} dt.$$

The other part follows from the fact that for $K(1,t) = (1+t)^{\lambda}$,

$$K_1 = K_2 = K_3 = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right).$$

□

Corollary 3.2. *By an application of Theorem* 2.2 *with* $K(u,v) = u^\lambda + v^\lambda$, *we have*

$$\int_0^T \int_0^T \frac{F(u)G(v)}{u^\lambda + v^\lambda} dudv \leq T^\alpha \sqrt[p]{pK_a} \sqrt[q]{qK_b} \left(\int_0^T (T-t) F^{p-1}(t) f(t) dt \right)^{1/p} \times \left(\int_0^T (T-t) G^{q-1}(t) g(t) dt \right)^{1/q},$$

where

$$K_r = \int_0^1 \frac{t^{r-1}}{1+t^\lambda} dt \qquad (r \in \{a,b\}).$$

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