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NON-CONTINUOUS LINEAR FUNCTIONALS ON TOPOLOGICAL VECTOR SPACES

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This paper is dedicated to Professor Mienie De Kock

Submitted by H. Dedania

Abstract. In this article we study the existence of non-continuous linear functionals on topological vector spaces. Both sufficient and necessary conditions for the existence of such maps are found.

1. Introduction

We know that, on every finite dimensional T_2 topological vector space, all linear functionals are always continuous (recall that by linear functional we mean linear maps from a vector space into the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.) In infinite dimensions, or when we do not have the T_2 hypothesis, very different things occur, as we shall show in this paper. We begin this introduction with the following basic results, that can be found in any basic reference for topological vector spaces, for instance, [1] and [2].

Theorem 1.1. *Let X be a vector space. Then:*

1. *The trivial topology on X is always a vector topology, that makes all the non-zero linear functionals on X non-continuous.*
2. *The topology on X generated by the sub-basis*

$$\left\{ f^{-1}(U) : f : X \longrightarrow \mathbb{K} \text{ is linear and } U \subseteq \mathbb{K} \text{ is open} \right\}$$

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is a vector topology that makes all the linear functionals on X continuous.
(3) *The discrete topology on X is never a vector topology unless $X = 0$.*

Throughout this note we use the notation

$$N = \{x \in X : x \text{ belongs to every neighborhood of } 0\},$$

for a given topological vector space X.

Theorem 1.2. *Let X be a topological vector space. Let us consider the set N described above. Then:*
(1) *The set N is a closed vector subspace of X whose relative topology is the trivial topology.*
(2) *The space X is T_2 if and only if $N = \{0\}$.*

Notice that, according to Theorem 1.2, if a topological vector space X is not T_2 then the corresponding closed vector subspace N is different from $\{0\}$. Therefore, by Theorem 1.1 any non-zero linear functional on N is not continuous. As a consequence, there are non-continuous linear functionals on X.

Corollary 1.3. *Let X be a topological vector space. Let $f : X \longrightarrow \mathbb{K}$ be a linear functional. Then:*
(1) *If f is continuous then $N \subseteq \ker(f)$, where N is the closed vector subspace of X described above.*
(2) *If N is topologically complemented and M is a topological complement for N in X, then f is continuous if and only if $N \subseteq \ker(f)$ and $f|_M$ is continuous. In particular, if X is finite dimensional then f is continuous if and only if $N \subseteq \ker(f)$.*

Corollary 1.4. *Let X be a topological vector space. If X is not T_2 then there are non-continuous linear functionals on X.*

Notice that all of these are opposite to what we might expect if we work in the category of all Banach spaces.

2. Motivating results

The motivation in this paper comes from the following results.

Theorem 2.1. *Let X be a finite dimensional T_2 topological vector space. Let M be a convex subset of X such that $\operatorname{span}(M) = X$. If $0 \in M$ then M has non-empty interior.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a Hamel basis for X contained in M. Then, the convex hull $\operatorname{co}\{0, e_1, \ldots, e_n\} \subseteq M$. Since X is T_2 we have that the vector topology on X is induced by the norm

$$\|\lambda_1 e_1 + \cdots + \lambda_n e_n\| = \max\{|\lambda_1|, \ldots, |\lambda_n|\},$$

for every $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$. We show that the usual closed ball

$$\mathsf{B}_X\left(\frac{1}{n+1}e_1 + \cdots + \frac{1}{n+1}e_n, \frac{1}{n(n+1)}\right)$$

of center $\frac{1}{n+1}e_1+\cdots+\frac{1}{n+1}e_n$ and radius $\frac{1}{n(n+1)}$ is contained in the set $\operatorname{co}\{0, e_1, \ldots, e_n\}$. Let

$$\lambda_1 e_1 + \cdots + \lambda_n e_n \in \mathsf{B}_X \left(\frac{1}{n+1} e_1 + \cdots + \frac{1}{n+1} e_n, \frac{1}{n\,(n+1)} \right).$$

We have

$$\left| \frac{1}{n+1} - \lambda_k \right| \leq \frac{1}{n\,(n+1)}$$

for each $k \in \{1, \ldots, n\}$, which means that

$$0 \leq \lambda_k \leq \frac{1}{n}$$

for each $k \in \{1, \ldots, n\}$. In other words,

$$\lambda_1 e_1 + \cdots + \lambda_n e_n \in \operatorname{co}\{0, e_1, \ldots, e_n\}.$$

□

Corollary 2.2. *Let X be a topological vector space. If X is finite dimensional and* T_2*, then all convex balanced absorbing subsets M of X have non-empty interior.*

Notice that from the previous corollary the following question arises naturally.

Question 2.3. Let X be a topological vector space. Assume that all convex balanced absorbing subsets M of X have non-empty interior. Is X finite dimensional and T_2?

We will try to answer this question in the next section. In this one, we provide a natural approach to a positive answer to Question 2.3.

Lemma 2.4. *Let X be a topological vector space. Let M be a convex balanced subset of X. Then, M is absorbing if and only if* $\operatorname{span}(M) = X$.

Proof. It is well known that every absorbing set is a generator system, that is, its linear span is the whole space. Conversely, assume that M is a generator system. Let $x \in X \setminus \{0\}$ and consider $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ and $m_1, \ldots, m_n \in M$ such that $x = \lambda_1 m_1 + \cdots + \lambda_n m_n$. Because $x \neq 0$ we have that $|\lambda_1| + \cdots + |\lambda_n| > 0$, and thus we can consider

$$\lambda = \frac{1}{|\lambda_1| + \cdots + |\lambda_n|}.$$

Now, take any $\alpha \in \mathbb{K}$ with $|\alpha| \leq \lambda$. We have that $\alpha x = (\alpha\lambda_1)\, m_1 + \cdots + (\alpha\lambda_n)\, m_n$ and $|\alpha\lambda_1| + \cdots + |\alpha\lambda_n| \leq 1$, therefore since M is absolutely convex we have $\alpha x \in M$. □

Theorem 2.5. *Let X be a topological vector space. If there exists a Hamel basis B for X that is not closed, then X possesses a convex balanced absorbing subset M with empty interior.*

Proof. We can assume that $0 \in \operatorname{cl}(B) \setminus B$. Let M be the absolutely convex hull of B. By Lemma 2.4, we have that M is absorbing. We shall show that M has empty interior. Assume to the contrary. Then there exists a neighborhood U of 0 contained in M. Next, pick another neighborhood V of 0 such that $V + V \subseteq U$. There exists $b \in B \cap V$. Then, $2b \in M$, but this is impossible. □

3. Main results

In this section we provide a partial positive answer to Question 2.3.

Theorem 3.1. *Let X be a topological vector space. If there exists a non-continuous linear functional f on X, then X possesses a convex balanced absorbing subset M with empty interior.*

Proof. Let us take $M = f^{-1}(\{t \in \mathbb{K} : |t| \leq 1\})$. We have that M is convex, balanced, and absorbing. Let us show that M has empty interior. Otherwise, since M is absolutely convex (convex and balanced) we have that 0 belongs to the interior of M. Therefore, we can find a balanced and absorbing neighborhood U of 0 contained in M. It suffices to show that f is continuous at 0. So, let $\varepsilon > 0$. Then, εU is a neighborhood of 0 and $f(\varepsilon U) \subseteq \{t \in \mathbb{K} : |t| \leq \varepsilon\}$. Hence f is continuous at 0, and so on X. This completes the proof. $\square$

Theorem 3.2. *Let X be a topological vector space. If there exists a Hamel basis B for X that is not closed, then there exists a non-continuous linear functional f on X.*

Proof. Again, we can assume that $0 \in \operatorname{cl}(B) \setminus B$. Then, we can consider a linear functional f on X such that $f(B) = \{1\}$. Clearly, f is not continuous on X. $\square$

Theorem 3.3. *Let X be a topological vector space. If X has local basis $\mathcal{U}$ of neighborhoods of 0 such that the cardinal of $\mathcal{U}$ is less than or equal to the algebraic dimension of X, then X possesses a Hamel basis B that is not closed.*

Proof. Let B' be a Hamel basis for X. Since $\operatorname{card}(\mathcal{U}) \leq \operatorname{card}(B')$ there exists an injective mapping

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & B' \\ U & \longmapsto & b_U. \end{array}$$

Now, since every element of $\mathcal{U}$ is an absorbing set, for every $U \in \mathcal{U}$ we can find $\lambda_U \in \mathbb{K} \setminus \{0\}$ such that $\lambda_U b_U \in U$. Then, $(\lambda_U b_U)_{U \in \mathcal{U}}$ is a null net such that $\{\lambda_U b_U : U \in \mathcal{U}\}$ is a free system (linearly independent). In accordance to the Zorn Lemma, there exists a Hamel basis B for X containing the set $\{\lambda_U b_U : U \in \mathcal{U}\}$. Obviously, $0 \in \operatorname{cl}(B) \setminus B$. So B is not closed. $\square$

Theorem 3.4. *Let X be a topological vector space. If X either has the trivial topology and is not zero or is infinite dimensional and first countable, then X has a local basis $\mathcal{U}$ of neighborhoods of 0 such that the cardinal of $\mathcal{U}$ is less than or equal to the algebraic dimension of X.*

Proof. If X has the trivial topology and is not zero then the cardinal of any local basis of neighborhoods of 0 is 1 and the algebraic dimension of X is greater than or equal to 1. Therefore, assume that X is infinite dimensional and first countable. There exists a local basis $\mathcal{U}$ of neighborhoods of 0 such that $\operatorname{card}(\mathcal{U}) = \aleph_0$. Now, let us consider any Hamel basis B for X. Since X has infinite dimension we have that

$$\operatorname{card}(B) \geq \aleph_0 = \operatorname{card}(\mathcal{U}).$$

$\square$

Now, we are ready to state and prove a partial positive solution to Question 2.3.

Theorem 3.5. *Let X be a topological vector space. Assume that all convex balanced absorbing subsets M of X have non-empty interior. Then:*

(1) *All linear functionals f on X are continuous. In particular, X is T_2.*
(2) *If the topology on X coincides with the topology generated by the sub-basis*

$$\left\{ f^{-1}(U) : f : X \longrightarrow \mathbb{K} \text{ is linear and } U \subseteq \mathbb{K} \text{ is open} \right\},$$

then X is finite dimensional.

Proof.

(1) According to Theorem 3.1 all linear functionals on X must be continuous. Therefore, by Corollary 1.4 we deduce that X is T_2.

(2) Let B be any Hamel basis of X. Let M be the absolutely convex hull of B. By hypothesis we have that M has non-empty interior. Therefore, 0 belongs to the interior of M and hence we can find $f_1, \ldots, f_n$ linear functionals on X and $U_1, \ldots, U_n \subseteq \mathbb{K}$ open neighborhoods of 0 such that

$$f_1^{-1}(U_1) \cap \cdots \cap f_n^{-1}(U_n) \subseteq M.$$

Thus, we also have that

$$\ker(f_1) \cap \cdots \cap \ker(f_n) \subseteq M.$$

Now, suppose that we can find $0 \neq x \in \ker(f_1) \cap \cdots \cap \ker(f_n)$. We can uniquely write $x = \lambda_1 b_1 + \cdots + \lambda_m b_m$ with $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ and $b_1, \ldots, b_m \in B$. Observe that, since $x \in M$, we have that $\sum_{i=1}^m |\lambda_i| \leq 1$. Next, let us take

$$\alpha > \frac{1}{\sum_{i=1}^m |\lambda_i|}.$$

Then,

$$\alpha x \in \ker(f_1) \cap \cdots \cap \ker(f_n) \subseteq M,$$

but $\alpha x = (\alpha\lambda_1) b_1 + \cdots + (\alpha\lambda_m) b_m$ and $\sum_{i=1}^m |\alpha\lambda_i| > 1$, which is impossible. Therefore

$$\ker(f_1) \cap \cdots \cap \ker(f_n) = 0.$$

In other words, the algebraic dual of X is finite dimensional and so is X. □

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