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# A 1-NORM BOUND FOR INVERSES OF TRIANGULAR MATRICES WITH MONOTONE ENTRIES

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Abstract. This paper provides some new bounds for 1−norms of positive triangular matrices with monotonic column entries. The main theorem refines a recent inequality of Vecchio and Mallik in the case of constant diagonal. The results are shown to be in a sense best possible under the given constraints. En route some partial order inequalities are obtained.

## 1. Introduction

This paper provides some new bounds for 1−norms of positive triangular matrices with monotonic column entries. The main theorem refines a recent inequality of Vecchio and Mallik [11] in the case of constant diagonal. We refer the reader to Vecchio [10] and Vecchio and Mallik [11] (and the reference therein) for discussion of applications particularly those to stability analysis of linear methods for solving Volterra integral equations. Other references on the topic include [3]–[7] and [9].

The matrices of interest here are $n \times n$ truncations of infinite lower triangular (real) matrices, i.e.

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$$A_n = \begin{bmatrix} a_{1,1} & & & & \\ a_{2,1} & a_{2,2} & & & \\ a_{3,1} & a_{3,2} & a_{3,3} & & \\ \cdot & \cdot & \cdot & \cdot & \\ a_{n,1} & \cdots & \cdot & a_{n,n-1} & a_{n,n} \end{bmatrix}. \tag{1.1}$$

The following result was proven in [11].

**Theorem 1.1.** *Assume that*

*(i)* $a_{i,j} \geq a > 0,\ j = 1, \ldots, i,\ i = 1, \ldots, n,$

*(ii)* $a_{i,i} \geq a_{i+1,i} \geq \cdots \geq a_{n,i},\ i = 1, \ldots, n,$

*and let*

$$a_{min} = \min_{i=1,\ldots,n} \{a_{i,i}\}, \tag{1.2}$$

*and* $B_n = [b_{i,j}]$ *be the inverse of the lower triangular matrix* $A_n$*. Then*

$$\|B_n\|_1 \leq \frac{1}{a_{min}} + \frac{2}{a}. \tag{1.3}$$

The result in (1.3) was first proven in the case of triangular Toeplitz matrices in [10] and improved to the following in [2].

**Theorem 1.2.** *Suppose that the sequence* $\{a_i\}_{i\geq 0}$ *satisfies*

$$a_0 \geq a_1 \geq a_2 \geq \cdots a_n \geq a > 0, \tag{1.4}$$

*for some constant* $a$ *and all* $n$ *and*

$$C_n = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ \cdot & \cdot & \cdot & \cdot & \\ a_n & \cdots & \cdot & a_1 & a_0 \end{bmatrix}. \tag{1.5}$$

*Then*

$$\|C_n^{-1}\|_1 \leq \frac{2}{a}\left(1 - \rho(a, a_0)^{\lceil \frac{n}{2} \rceil}\right) \tag{1.6}$$

*where* $\rho$ *is the inverse ratio defined via*

$$\rho(x, y) = 1 - x/y, \tag{1.7}$$

*and, in particular*

$$\|C_n^{-1}\|_1 \leq \frac{2}{a}, \tag{1.8}$$

*independent of* $a_0$ *and* $n$*.*

Here, we extend Theorem 1.2 (to non-Toeplitz matrices) and refine Theorem 1.1 in the case of constant diagonal. In particular we will prove the following.

**Theorem 1.3.** *Assume that the hypotheses of Theorem 1.1 are satisfied and in addition that*

$$a_{1,1} \leq a_{2,2} \leq \cdots \leq a_{n,n}. \tag{1.9}$$

*Then*

$$\|B_n\|_1 \leq \frac{2}{a}\left(\frac{a_{n,n}}{a_{1,1}}\right)\left(1 - \frac{\rho(a,a_{n,n})^{\lceil \frac{n}{2} \rceil} + \rho(a,a_{n,n})^{\lfloor \frac{n}{2} \rfloor}}{2}\right). \tag{1.10}$$

*In particular, if*

$$a_{1,1} = a_{2,2} = \cdots = a_{n,n} = a^*, \tag{1.11}$$

*then*

$$\|B_n\|_1 \quad \leq \quad \frac{2}{a}\left(1 - \frac{\rho(a,a^*)^{\lceil \frac{n}{2} \rceil} + \rho(a,a^*)^{\lfloor \frac{n}{2} \rfloor}}{2}\right), \tag{1.12}$$

*and hence*

$$\|B_n\|_1 < \frac{2}{a}, \tag{1.13}$$

*independent of* $a^*$*.*

Note that triangular matrices satisfying (1.11) arise in the study of linear groups (see for instance [8]) and are particularly important in the theory of matrix decompositions.

The inequality in (1.12) is in a sense best possible. In particular, for $0 < a < a^*$, set

$$\mathcal{A}_n(a,a^*) = \{A = [a_{i,j}]_{n \times n} \mid A \text{ satisfies (1.1), (i), (ii) and (1.11)}\}. \tag{1.14}$$

We have the following theorem regarding optimality.

**Theorem 1.4.** *For* $0 < a < a^*$*,*

$$\sup_{A \in \mathcal{A}_n(a,a^*)} \|A^{-1}\|_1 = \frac{2}{a}\left(1 - \frac{\rho(a,a^*)^{\lceil \frac{n}{2} \rceil} + \rho(a,a^*)^{\lfloor \frac{n}{2} \rfloor}}{2}\right). \tag{1.15}$$

*Proof.* We need to show that the bound in (1.12) is attained. To that end, suppose $a_{i,j} = a^* > 0$ for $i - j \in \{0, 1\}$ and $a_{i,j} \equiv a$ otherwise. It is easy to verify in this case, that for $1 \leq j \leq i \leq n$,

$$b_{i,j} = (-1)^{i-j} \frac{1}{a^*}\left(1 - \frac{a}{a^*}\right)^{\lfloor \frac{i-j}{2} \rfloor}, \tag{1.16}$$

and hence,

$$\begin{aligned}
\|A_n^{-1}\|_1 &= \sum_{i=1}^{n} |b_{i,1}| \\
&= \sum_{i=1}^{n} \frac{1}{a^*}\left(1-\frac{a}{a^*}\right)^{\lfloor \frac{i-1}{2} \rfloor} = \sum_{i=0}^{n-1} \frac{1}{a^*}\left(1-\frac{a}{a^*}\right)^{\lfloor \frac{i}{2} \rfloor} \\
&= \frac{1}{a^*}\left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor -1}\left(1-\frac{a}{a^*}\right)^{i} + \sum_{i=0}^{\lceil \frac{n}{2} \rceil -1}\left(1-\frac{a}{a^*}\right)^{i}\right) \\
&= \frac{2}{a}\left(1-\frac{\rho(a,a^*)^{\lceil \frac{n}{2} \rceil}+\rho(a,a^*)^{\lfloor \frac{n}{2} \rfloor}}{2}\right). \qquad (1.17)
\end{aligned}$$

□

Note also that

$$\sup_{A\in\bigcup_{n\geq 1}\mathcal{A}_n(a,a^*)} \|A^{-1}\|_1 = 2/a. \qquad (1.18)$$

The reader is referred to [1] for some discussion of bounds for inverses of matrices of the form in (1.1) when the condition of monotonicity within columns is replaced with that within rows.

## 2. Preliminaries and notation

In order to prove Theorem 1.3, consider the partial order on the set $\mathcal{V}_{b,a}$ of (arbitrary length) tuples $(a_1, a_2, \ldots, a_k)$ with

$$b \geq a_1 > a_2 > \cdots > a_k = a \qquad (2.1)$$

defined via

$$\mathbf{v} \prec \mathbf{z} \text{ if } \mathbf{z} \text{ is a } \textit{suffix of } \mathbf{v} \qquad (2.2)$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_k)$ is a suffix of $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ if $m > k$ and $\mathbf{v} = (v_1, \ldots, v_{m-k}, z_1, z_2, \ldots, z_k)$. For convenience, if $\mathbf{w} = (w_1, w_2, \ldots, w_r)$ we will write the $r+k-$tuple $(w_1, w_2, \ldots, w_r, z_1, z_2, \ldots, z_k)$ as $(\mathbf{w}; \mathbf{z})$. In addition, denote the length of $\mathbf{v}$ by $l(\mathbf{v}) = k$. The value $v_1$ will be referred to as the *initial* value of $\mathbf{v}$.

For a triangular double sequence $\{d_{i,j}\}_{j<i<n}$ satisfying $0 \leq d_{i,j} < 1$ for $j < i < n$ and

$$\sum_{i=j+1}^{n} d_{i,j} \leq x < 1,\ \ j = 1, 2, \ldots, n-1 \qquad (2.3)$$

define the function $D$ via

$$D(\mathbf{v}) = d_{v_k,v_{k-1}} \cdot d_{v_{k-1},v_{k-2}} \cdots d_{v_3,v_2} \cdot d_{v_2,v_1} \tag{2.4}$$

for $\mathbf{v} = (v_1, v_2, \ldots, v_k)$.

Note that it follows directly from the definition of $D$, the inequality in (2.3) and the non-negativity of $\{d_{i,j}\}$ that $D(\mathbf{v}) < D(\mathbf{z})$ for $\mathbf{v} \prec \mathbf{z}$.

**Lemma 2.1.** *Consider a set of tuples* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$*. If* $\mathbf{v}_i \prec \mathbf{z}$ *for* $1 \le i \le k$ *and* $\mathbf{v}_i \not\prec \mathbf{v}_j$ *for* $i \ne j$ *(i.e.* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *forms an antichain that is bounded above by* $\mathbf{z}$*) then*

$$D(\mathbf{z}) \ge D(\mathbf{v}_1) + D(\mathbf{v}_2) + \cdots + D(\mathbf{v}_k). \tag{2.5}$$

*Proof.* Let $\mathbf{z}_2$ be the least upper bound for $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, i.e. $\mathbf{z}_2 = \min\{\mathbf{w} \preceq \mathbf{z} : \mathbf{v}_i \preceq \mathbf{w}, 1 < i < k\}$. Clearly, $\mathbf{z}_2 \preceq \mathbf{z}$. We will show that

$$\sum_{i=1}^{k} D(\mathbf{v}_i) \le D(\mathbf{z}_2). \tag{2.6}$$

The result is immediate for $k = 1$. Hence suppose (2.6) holds for $1 \le k < K$.

Now, suppose that there exists a $\mathbf{z}_3 \prec \mathbf{z}_2$ and a set $S \subset \{1, 2, \ldots, K\}$ such that $2 \le \|S\| \le K - 1$, $\mathbf{v}_i \prec \mathbf{z}_3$ if $i \in S$, and $\mathbf{v}_i \not\prec \mathbf{z}_3$, if $i \in S^c$.
then by induction, we have

$$\sum_{i=1}^{K} D(\mathbf{v}_i) \le D(\mathbf{z}_3) + \sum_{i \in S^c} D(\mathbf{v}_i). \tag{2.7}$$

Considering the set $\{\mathbf{z}_3\} \cup \{\mathbf{v}_i : i \in S^c\}$ and applying induction again we have the inequality in (2.6).

Otherwise $\mathbf{v}_i$ is of the form $\mathbf{v}_i = (\mathbf{w}_i; (t_i); \mathbf{z}_3)$, $i = 1, 2, \ldots, k$, where $t_l \ne t_j$ for $l \ne j$ and $\mathbf{z}_3 = (z_{3,1}, \ldots, z_{3,l(\mathbf{z}_3)}) \preceq \mathbf{z}_2$. In this case, by (2.3),

$$\sum_{i=1}^{K} D(\mathbf{v}_i) \le \sum_{i=1}^{K} D((t_i); \mathbf{z}_3) = \sum_{i=1}^{K} d_{t_i,z_{3,1}} D(\mathbf{z}_3) = D(\mathbf{z}_3) \sum_{i=1}^{K} d_{t_i,z_{3,1}} \le D(\mathbf{z}_3) \tag{2.8}$$

and the proof is complete. ∎

The following lemma will be crucial.

**Lemma 2.2.** *For fixed* $s \ge 1$*, set* $S_{i,s} = 0$ *for* $i < s$*,* $S_{s,s} = 1$ *and for* $s + 1 \le m \le n$*, inductively,*

$$S_{m,s} = \sum_{i=s}^{m-1} d_{m,i} S_{i,s}. \tag{2.9}$$

*Then, for* $Q \subseteq \{s + 1, \ldots, n\}$*, we have*

$$\sum_{i \in Q} S_{i,s} \leq x_s + x_s^2 + \cdots + x_s^{\|Q\|}, \tag{2.10}$$

*where* $x_s = \max_{t=s,\ldots,n-1} \sum_{i=t+1}^{n} d_{i,t}$.

*Proof.* Note that it follows from straightforward induction that for $m > s$,

$$S_{m,s} = \sum_{\mathbf{v}=(m,\ldots,s) \in \mathcal{V}_{m,s}} D(\mathbf{v}). \tag{2.11}$$

Note that in (2.11), the summation is over all tuples $v = (v_1, v_2, \ldots, v_{l(\mathbf{v})})$ with

$$m = v_1 > v_2 > \cdots > v_{l(\mathbf{v})} = s. \tag{2.12}$$

Now, define

$$L_{m,s}^k = \sum_{\substack{\mathbf{v} \in \mathcal{V}_{m,s} \\ l(\mathbf{v})=k+1}} D(\mathbf{v}) \tag{2.13}$$

We will show inductively that

$$L_{m,s}^k \leq x_s^k. \tag{2.14}$$

First note that by (2.3) and the definition of $x_s$,

$$L_{m,s}^1 = d_{s+1,s} + d_{s+2,s} + d_{s+3,s} + \cdots + d_{m,s} \leq x_s. \tag{2.15}$$

Thus assume that (2.14) is true for $k < K$. Then, since $x_1 \geq x_2 \geq \cdots \geq x_n$,

$$L_{m,s}^K = \sum_{i=s+1}^{m} d_{i,s} L_{m,i}^{K-1} \leq \sum_{i=s+1}^{m} d_{i,s} x_i^{K-1} \leq x_s^{K-1} \sum_{i=s+1}^{m} d_{i,s} \leq x_s^K. \tag{2.16}$$

Now, define the sets

$$\mathcal{R}_1 = \{\mathbf{v} \in \mathcal{V}_{m,s} | 2 \leq l(\mathbf{v}) \leq \|Q\| + 1\} \tag{2.17}$$

and

$$\mathcal{R}_2 = \{\mathbf{v} \in \mathcal{V}_{m,s} | \mathbf{v} = (i, \ldots, s),\ i \in Q\} \tag{2.18}$$

and consider the quantity

$$\begin{aligned} H_Q &= \sum_{k=1}^{\|Q\|} L_{m,s}^k - \sum_{i \in Q} S_{i,s} && (2.19) \\ &= \sum_{\mathbf{v} \in \mathcal{R}_1} D(\mathbf{v}) - \sum_{\mathbf{v} \in \mathcal{R}_2} D(\mathbf{v}). && (2.20) \end{aligned}$$

We will prove that for all sets $Q$, $H_Q \geq 0$. The result will then follow from (2.20) and the inequality in (2.14).

We define the following scheme for matching elements $\mathbf{z}$ in $\mathcal{R}_1$ with (possibly empty) subsets $\mathcal{S}(\mathbf{z})$ of $\mathcal{R}_2$ such that $D(\mathbf{z}) \geq \sum_{\mathbf{v}\in S(\mathbf{z})} D(\mathbf{v})$ and $\{\mathcal{S}(\mathbf{z})|\mathbf{z} \in \mathcal{R}_1\}$ is a partition of $R_2$. In particular for $2 \leq t \leq n$, set

$$\mathcal{J}_t = \{\mathbf{v} \in \mathcal{R}_1 : l(v) = t\}, \tag{2.21}$$

and recursively in $t \geq 2$, for $\mathbf{z} \in \mathcal{J}_t$ let

$$S(\mathbf{z}) = \{\mathbf{v} \in \mathcal{R}_2 | \mathbf{v} \text{ is a maximal element in the set } \mathcal{W}(\mathbf{z})\}. \tag{2.22}$$

where

$$\mathcal{W}(\mathbf{z}) = \{\mathbf{w} \in \mathcal{R}_2 | \mathbf{w} \preceq \mathbf{z} \text{ and } \mathbf{w} \notin \bigcup_{\substack{\mathbf{v}\succ\mathbf{z} \\ \mathbf{v}\in\mathcal{R}_1}} \mathcal{S}(\mathbf{v})\}. \tag{2.23}$$

Here, again, the maximality in (2.22) is with respect to the given partial order on $\mathcal{V}_{m,s}$.

Now, fix $\mathbf{z} \in \mathcal{J}_t$ for some $2 \leq t \leq n$ and suppose $\{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathcal{S}(\mathbf{z})$ with $\mathbf{v}_1 \neq \mathbf{v}_2$. The fact that $\mathbf{v}_1 \not\prec \mathbf{v}_2$ and $\mathbf{v}_2 \not\prec \mathbf{v}_1$ follows from the maximality in (2.22). We then have that Lemma 2.1 is applicable and

$$D(\mathbf{z}) \geq \sum_{\mathbf{v}\in S(\mathbf{z})} D(\mathbf{v}), \tag{2.24}$$

as required. In addition, by the definition of $\mathcal{W}$ we have that the sets $\mathcal{S}(\mathbf{z})$, $\mathbf{z} \in \mathcal{R}_1$ are pairwise disjoint. To see that $\mathcal{R}_2 \subset \bigcup_{\mathbf{z}\in\mathcal{R}_1} \mathcal{S}(\mathbf{z})$, first suppose $\mathbf{v} \in \mathcal{R}_2$. Let $K_\mathbf{v}$ be a maximal chain in $\mathcal{V}_{m,s}$ such that $\mathbf{v} \in K_\mathbf{v}$ and set $\mathcal{T}_1 = K_\mathbf{v} \cap \mathcal{R}_1 = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_r\}$ and $\mathcal{T}_2 = K_\mathbf{v} \cap \mathcal{R}_2 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$, where $\mathbf{v}_1 \succ \mathbf{v}_2 \succ \cdots \succ \mathbf{v}_q$ and $\mathbf{z}_1 \succ \mathbf{z}_2 \succ \cdots \succ \mathbf{z}_r$. Note that $\|\mathcal{T}_1\| = \|Q\|$ and $\|\mathcal{T}_2\| \leq \|Q\|$ (since the only possible initial values for tuples are those in $Q$) and by (2.22), $\mathbf{v}_i \in \mathcal{S}(\mathbf{z}_i)$ for $1 \leq i \leq r$ and in particular $\mathbf{v} \in \bigcup_{\mathbf{z}\in\mathcal{R}_1} \mathcal{S}(\mathbf{z})$. Since $\bigcup_{\mathbf{z}\in\mathcal{R}_1} \mathcal{S}(\mathbf{z}) \subset \mathcal{R}_2$ by (2.22), the result is proven. □

## 3. Proof of the main theorem

In this section we prove Theorem 1.3.

First note that the lower triangular matrix $B_n = [b_{i,j}] = A_n^{-1}$ satisfies $b_{s,s} = 1/a_{s,s}$ and

$$b_{m,s} = \sum_{j=s}^{m-1} -\alpha_{m,j} b_{j,s}, \tag{3.1}$$

for $1 \leq s < m \leq n$, where $\alpha_{m,j} = (a_{m,j}/a_{m,m})$ for $1 \leq j \leq m \leq n$ (see for instance [1]).

Define $h_{i,j} = a_{j,j} b_{i,j}$ for $1 \leq j \leq i \leq n$, so that $h_{s,s} = 1$ and for $1 \leq s < m \leq n$,

$$h_{m,s} = \sum_{j=s}^{m-1} -\alpha_{m,j} h_{j,s}. \tag{3.2}$$

We have the following lemma (contrast with Equation (2.3) in [11]).

**Lemma 3.1.** *Suppose that* $[a_{i,j}]$ *satisfies the hypotheses of Theorem 1.3. Then*

$$h_{i,j} = S_{i,j} - S_{i,j+1}, \tag{3.3}$$

*for* $1 \leq j \leq i \leq n$*, where* $\{S_{i,j}\}$ *is as in (2.9) for the nonnegative double sequence* $\{d_{i,j}\}$ *defined via*

$$d_{m,j} = \alpha_{m-1,j} - \alpha_{m,j}, \tag{3.4}$$

*for* $1 \leq j < m \leq n$*. In addition, (2.3) is satisfied with*

$$x_s = \max_{t=s,\ldots,n-1} \sum_{i=t+1}^{n} d_{i,t} \leq 1 - \frac{a}{a_{n,n}} = x. \tag{3.5}$$

*Proof.* First, note that by (3.4), (ii) and (1.9)

$$d_{m,j} = \alpha_{m-1,j} - \alpha_{m,j} = \frac{a_{m-1,j}}{a_{m-1,m-1}} - \frac{a_{m,j}}{a_{m,m}} \geq 0, \tag{3.6}$$

and

$$\sum_{m=j+1}^{n} d_{m,j} = \alpha_{j,j} - \alpha_{n,j} = 1 - \frac{a_{n,j}}{a_{n,n}} \leq 1 - \frac{a}{a_{n,n}} < 1. \tag{3.7}$$

In addition, for $s+2 \leq m \leq n$, $s = 1, 2\ldots, n$,

$$\begin{aligned} h_{m,s} - h_{m-1,s} &= \sum_{j=s}^{m-1} -\alpha_{m,j} h_{j,s} + \sum_{j=s}^{m-2} \alpha_{m-1,j} h_{j,s} \\ &= \sum_{j=s}^{m-2} (\alpha_{m-1,j} - \alpha_{m,j}) h_{j,s} - \alpha_{m,m-1} h_{m,m-1}, \end{aligned} \tag{3.8}$$

and hence since $d_{m,m-1} = \alpha_{m-1,m-1} - \alpha_{m,m-1} = 1 - \alpha_{m,m-1}$,

$$h_{m,s} = \sum_{j=s}^{m-2} d_{m,j} h_{j,s} + (1 - \alpha_{m,m-1}) h_{m,m-1} = \sum_{j=s}^{m-1} d_{m,j} h_{j,s}. \tag{3.9}$$

In addition,

$$\begin{aligned} S_{m,s} - S_{m,s+1} &= \sum_{i=s}^{m-1} d_{m,i} S_{i,s} - \sum_{i=s+1}^{m-1} d_{m,i} S_{i,s+1} \\ &= d_{m,s} S_{s,s} + \sum_{i=s+1}^{m-1} d_{m,i} (S_{i,s} - S_{i,s+1}) \\ &= d_{m,s} (S_{s,s} - S_{s,s+1}) + \sum_{i=s+1}^{m-1} d_{m,i} (S_{i,s} - S_{i,s+1}) \\ &= \sum_{i=s}^{m-1} d_{m,i} (S_{i,s} - S_{i,s+1}), \end{aligned} \tag{3.10}$$

since $S_{s,s+1} = 0$.

Comparing (3.9) and (3.10) and noting that $h_{s,s} = 1 = S_{s,s} - S_{s,s+1}$ and $h_{s+1,s} = -\alpha_{s+1,s} = (1-\alpha_{s+1,s}) - 1 = d_{s+1,s}S_{s,s} - 1 = S_{s+1,s} - S_{s+1,s+1}$, the result follows. ☐

We are now in a position to prove Theorem 1.3.

*Proof of Theorem 1.3.* Employing Lemma 3.1 and the definition of $\{h_{i,j}\}$, we have

$$\begin{aligned}\|A_n^{-1}\|_1 = \|B_n\|_1 &= \max_{1\le j\le n}\sum_{i=j}^{n}|b_{i,j}| \\ &= \max_{1\le j\le n}\sum_{i=j}^{n}\left|\frac{1}{a_{j,j}}(S_{i,j}-S_{i,j+1})\right|. \qquad (3.11)\end{aligned}$$

Now, fix $1 \le j \le n$. We have, by the nonnegativity of $\{S_{i,j}\}$, that

$$\begin{aligned}\sum_{i=j}^{n}|S_{i,j}-S_{i,j+1}| &= |S_{j,j}-S_{j,j+1}| + |S_{j+1,j}-S_{j+1,j+1}| + \\ &\quad \sum_{i\in Q_1}(S_{i,j}-S_{i,j+1}) + \sum_{i\in Q_1^c}(S_{i,j+1}-S_{i,j}) \\ &\le |S_{j,j}-S_{j,j+1}| + |S_{j+1,j}-S_{j+1,j+1}| + \sum_{i\in Q_1}S_{i,j} + \sum_{i\in Q_1^c}S_{i,j+1}, \qquad (3.12)\end{aligned}$$

where $Q_1 = \{j+2 \le i \le n | S_{i,j} > S_{i,j+1}\}$.

Noting that $S_{j,j} = 1$, $S_{j,j+1} = 0$, $S_{j+1,j} = d_{j+1,j} < 1$ and $S_{j+1,j+1} = 1$, we have from (3.12) that

$$\sum_{i=j}^{n}|S_{i,j}-S_{i,j+1}| \le 2 + \sum_{i\in Q_1}S_{i,j} + \sum_{i\in Q_1^c}S_{i,j+1}. \qquad (3.13)$$

.

Letting $y = \|Q_1\| \le n-j-1$, recalling $x_j \le 1 - a/a_{n,n} = x < 1$ and employing Lemma 2.2 gives

$$\begin{aligned}\sum_{i=j}^{n}|S_{i,j}-S_{i,j+1}| &\le (1+x+\cdots+x^y) + (1+x+\cdots+x^{n-j-1-y}) \\ &\le \frac{1-x^{y+1}}{1-x} + \frac{1-x^{n-(y+1)}}{1-x} \le \frac{2-(x^{y+1}+x^{n-(y+1)})}{1-x} \\ &. \qquad (3.14)\end{aligned}$$

By the convexity of the function $f$ defined via $f(t) = x^t$, we have that $x^{y+1} + x^{n-(y+1)} \geq x^{\lfloor n/2 \rfloor} + x^{\lceil n/2 \rceil}$. Thus, returning to (3.11), we obtain

$$\begin{aligned} \|A_n^{-1}\|_1 &\leq \frac{2}{\min_i\{a_{i,i}\}} \frac{1 - \frac{x^{\lfloor n/2 \rfloor} + x^{\lceil n/2 \rceil}}{2}}{1-x} \\ &= \frac{2}{a_{1,1}} \left( \frac{1 - \frac{(1-a/a_{n,n})^{\lceil \frac{n}{2} \rceil} + (1-a/a_{n,n})^{\lfloor \frac{n}{2} \rfloor}}{2}}{a/a_{n,n}} \right). \end{aligned} \tag{3.15}$$

In the case when $a_{i,i} = a^*$ for all $i$, (3.15) gives

$$\|A_n^{-1}\|_1 \leq \frac{2}{a} \left( 1 - \frac{(1-a/a^*)^{\lceil \frac{n}{2} \rceil} + (1-a/a^*)^{\lfloor \frac{n}{2} \rfloor}}{2} \right), \tag{3.16}$$

as required. □

## References


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