



## A FIXED POINT APPROACH TO THE STABILITY OF AN EQUATION OF THE SQUARE SPIRAL

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*This paper is dedicated to Professor Themistocles M. Rassias.*

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ABSTRACT. Cădariu and Radu applied the fixed point method to the investigation of Cauchy and Jensen functional equations. In this paper, we adopt the idea of Cădariu and Radu to prove the Hyers-Ulam-Rassias stability of a functional equation of the square root spiral,  $f(\sqrt{r^2 + 1}) = f(r) + \tan^{-1}(1/r)$ .

### 1. INTRODUCTION

In 1940, Ulam [16] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms: *Let  $G_1$  be a group and let  $G_2$  be a metric group with the metric  $d(\cdot, \cdot)$ . Given  $\varepsilon > 0$ , does there exist a  $\delta > 0$  such that if a function  $h : G_1 \rightarrow G_2$  satisfies the inequality  $d(h(xy), h(x)h(y)) < \delta$  for all  $x, y \in G_1$ , then there exists a homomorphism  $H : G_1 \rightarrow G_2$  with  $d(h(x), H(x)) < \varepsilon$  for all  $x \in G_1$ ?*

The case of approximately additive functions was solved by Hyers [6] under the assumption that  $G_1$  and  $G_2$  are Banach spaces. Indeed, he proved that each solution of the inequality  $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ , for all  $x$  and  $y$ , can be approximated by an exact solution, say an additive function. Rassias [14] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

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and generalized the result of Hyers. Since then, the stability of several functional equations has been extensively investigated.

The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds. The terminology can also be applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [4, 7, 8, 9, 10, 15].

Recently, Cădariu and Radu [2] applied the fixed point method to the investigation of the Cauchy additive functional equation (ref. [1, 13]). Using such a clever idea, they could present a short and simple proof for the stability of the Cauchy functional equation.

Assume that a function  $f : [1, \infty) \rightarrow \mathbb{R}$  is monotonically increasing and satisfies  $f(1) = 0$ . If  $f$  satisfies

$$f(\sqrt{r^2 + 1}) = f(r) + \tan^{-1} \frac{1}{r} \quad (1.1)$$

for all  $r \geq 1$ , then the resulting curve  $\theta = f(r)$  is a continuous square root spiral, where  $(r, \theta)$  are the polar coordinates. So the functional equation (1.1) is naturally called a functional equation of the square root spiral.

In 2000, Heuvers, Moak and Boursaw [5] investigated the general solution of Eq. (1.1) as follows:

**Theorem 1.1.** (Heuvers, et al.) *The general solution  $f : [1, \infty) \rightarrow \mathbb{R}$  of Eq. (1.1) is given by*

$$f(r) = p(r^2) + \sum_{i=0}^{\infty} \left( \tan^{-1} \frac{1}{\sqrt{1+i}} - \tan^{-1} \frac{1}{\sqrt{r^2+i}} \right),$$

where  $p$  is an arbitrary periodic function of period 1. If  $f$  is monotonically increasing, then  $p$  is a constant function. In particular, if  $f(1) = 0$  then  $p \equiv 0$  and the curve  $\theta = f(r)$  is the continuous square root spiral.

The Hyers-Ulam-Rassias stability of Eq. (1.1) was recently proved by using an elementary method (see [12]). In this paper, we will adopt the idea of Cădariu and Radu and apply a fixed point method for proving the Hyers-Ulam-Rassias stability of the same equation.

## 2. PRELIMINARIES

Let  $X$  be a set. A function  $d : X \times X \rightarrow [0, \infty]$  is called a generalized metric on  $X$  if and only if  $d$  satisfies

- (M<sub>1</sub>)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (M<sub>2</sub>)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (M<sub>3</sub>)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

We now introduce one of fundamental results of fixed point theory. For the proof, refer to [3].

**Theorem 2.1.** *Let  $(X, d)$  be a generalized complete metric space. Assume that  $\Lambda : X \rightarrow X$  is a strictly contractive operator with the Lipschitz constant  $L < 1$ . If there exists a nonnegative integer  $k$  such that  $d(\Lambda^{k+1}f, \Lambda^k f) < \infty$  for some  $f \in X$ , then the following properties are true:*

- (a) *The sequence  $\{\Lambda^n f\}$  converges to a fixed point  $F$  of  $\Lambda$ ;*
- (b)  *$F$  is the unique fixed point of  $\Lambda$  in*

$$X^* = \{g \in X \mid d(\Lambda^k f, g) < \infty\};$$

- (c) *If  $h \in X^*$ , then*

$$d(h, F) \leq \frac{1}{1-L}d(\Lambda h, h).$$

### 3. MAIN RESULTS

In the following theorem, by using the idea of Cădariu and Radu (see [1, 2]), we will prove the Hyers-Ulam-Rassias stability of the functional equation (1.1) for square root spirals.

**Theorem 3.1.** *Suppose  $\varphi : [1, \infty) \rightarrow [0, \infty)$  is a given function and there exists a constant  $L$ ,  $0 < L < 1$ , such that*

$$\varphi(\sqrt{r^2+1}) \leq L\varphi(r) \tag{3.1}$$

for all  $r \geq 1$ . If a function  $f : [1, \infty) \rightarrow \mathbb{R}$  satisfies the inequality

$$\left| f(\sqrt{r^2+1}) - f(r) - \tan^{-1} \frac{1}{r} \right| \leq \varphi(r) \tag{3.2}$$

for all  $r \geq 1$ , then there exists a unique solution  $F : [1, \infty) \rightarrow \mathbb{R}$  of Eq. (1.1), which satisfies

$$|F(r) - f(r)| \leq \frac{1}{1-L}\varphi(r) \tag{3.3}$$

for all  $r \geq 1$ .

*Proof.* We set  $X = \{h \mid h : [1, \infty) \rightarrow \mathbb{R} \text{ is a function}\}$  and introduce a generalized metric on  $X$  as follows,

$$d(g, h) = \inf\{C \in [0, \infty) \mid |g(r) - h(r)| \leq C\varphi(r) \text{ for all } r \geq 1\}.$$

First, we will verify that  $(X, d)$  is a complete space. Let  $\{g_n\}$  be a Cauchy sequence in  $(X, d)$ . According to the definition of Cauchy sequences, there exists, for any given  $\varepsilon > 0$ , a positive integer  $N_\varepsilon$  such that  $d(g_m, g_n) \leq \varepsilon$  for all  $m, n \geq N_\varepsilon$ . By considering the definition of the generalized metric  $d$ , we see that

$$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} \forall m, n \geq N_\varepsilon \forall r \geq 1 : |g_m(r) - g_n(r)| \leq \varepsilon\varphi(r) \tag{3.4}$$

If  $r \geq 1$  is fixed, (3.4) implies that  $\{g_n(r)\}$  is a Cauchy sequence in  $(\mathbb{R}, |\cdot|)$ . Since  $(\mathbb{R}, |\cdot|)$  is complete,  $\{g_n(r)\}$  converges in  $(\mathbb{R}, |\cdot|)$  for each  $r \geq 1$ . Hence we can define a function  $g : [1, \infty) \rightarrow \mathbb{R}$  by

$$g(r) = \lim_{n \rightarrow \infty} g_n(r).$$

If we let  $m$  increase to infinity, it follows from (3.4) that for any  $\varepsilon > 0$ , there exists a positive integer  $N_\varepsilon$  with  $|g_n(r) - g(r)| \leq \varepsilon\varphi(r)$  for all  $n \geq N_\varepsilon$  and all

$r \geq 1$ ; i.e., for any  $\varepsilon > 0$ , there exists a positive integer  $N_\varepsilon$  such that  $d(g_n, g) \leq \varepsilon$  for any  $n \geq N_\varepsilon$ . This fact leads us to the conclusion that  $\{g_n\}$  converges in  $(X, d)$ . Hence  $(X, d)$  is a complete space (cf. the proof of [11, Theorem 3.1] or [2, Theorem 2.5]).

We now define an operator  $\Lambda : X \rightarrow X$  by

$$(\Lambda h)(r) = h\left(\sqrt{r^2 + 1}\right) - \tan^{-1} \frac{1}{r} \quad (r \geq 1) \quad (3.5)$$

for any  $h \in X$ .

We assert that  $\Lambda$  is strictly contractive on  $X$ . Given  $g, h \in X$ , let  $C \in [0, \infty]$  be an arbitrary constant with  $d(g, h) \leq C$ , i.e.,

$$|g(r) - h(r)| \leq C\varphi(r)$$

for all  $r \geq 1$ . If in the last inequality we replace  $r$  by  $\sqrt{r^2 + 1}$  and make use of (3.1), then we have

$$|(\Lambda g)(r) - (\Lambda h)(r)| = \left| g\left(\sqrt{r^2 + 1}\right) - h\left(\sqrt{r^2 + 1}\right) \right| \leq C\varphi\left(\sqrt{r^2 + 1}\right) \leq LC\varphi(r)$$

for every  $r \geq 1$ , i.e.,  $d(\Lambda g, \Lambda h) \leq LC$ . Hence we conclude that  $d(\Lambda g, \Lambda h) \leq Ld(g, h)$  for any  $g, h \in X$ .

Next, we assert that  $d(\Lambda f, f) < \infty$ . In view of (3.2) and the definition of  $\Lambda$ , we get

$$|(\Lambda f)(r) - f(r)| \leq \varphi(r)$$

for each  $r \geq 1$ , i.e.,

$$d(\Lambda f, f) \leq 1. \quad (3.6)$$

By using mathematical induction, we now prove that

$$(\Lambda^n f)(r) = f\left(\sqrt{r^2 + n}\right) - \sum_{i=0}^{n-1} \tan^{-1} \frac{1}{\sqrt{r^2 + i}} \quad (3.7)$$

for all  $n \in \mathbb{N}$  and all  $r \geq 1$ . Since  $f \in X$ , the definition (3.5) implies that (3.7) is true for  $n = 1$ . Now, assume that (3.7) holds for some  $n \geq 1$ . It then follows from (3.5) and (3.7) that

$$\begin{aligned} (\Lambda^{n+1} f)(r) &= \Lambda(\Lambda^n f)(r) \\ &= (\Lambda^n f)\left(\sqrt{r^2 + 1}\right) - \tan^{-1} \frac{1}{r} \\ &= f\left(\sqrt{r^2 + n + 1}\right) - \sum_{i=0}^{n-1} \tan^{-1} \frac{1}{\sqrt{r^2 + i + 1}} - \tan^{-1} \frac{1}{r} \\ &= f\left(\sqrt{r^2 + n + 1}\right) - \sum_{i=0}^n \tan^{-1} \frac{1}{\sqrt{r^2 + i}}, \end{aligned}$$

which is the case where  $n$  is replaced by  $n + 1$  in (3.7).

Considering (3.6), if we set  $k = 0$  in Theorem 2.1, then Theorem 2.1 (a) implies that there exists a function  $F \in X$ , which is a fixed point of  $\Lambda$ , such that

$\Lambda^n f \rightarrow F$ ; more precisely,

$$\lim_{n \rightarrow \infty} \left[ f(\sqrt{r^2 + n}) - \sum_{i=0}^{n-1} \tan^{-1} \frac{1}{\sqrt{r^2 + i}} \right] = F(r) \quad (3.8)$$

for all  $r \geq 1$ .

Since  $k = 0$  (see (3.6)) and  $f \in X^* = \{g \in X \mid d(f, g) < \infty\}$  in Theorem 2.1, by Theorem 2.1 (c) and (3.6), we obtain

$$d(f, F) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{1}{1-L},$$

i.e., the inequality (3.3) is true for all  $r \geq 1$ .

By (3.8), we get

$$\begin{aligned} & \left| F(\sqrt{r^2 + 1}) - F(r) - \tan^{-1} \frac{1}{r} \right| \\ &= \lim_{n \rightarrow \infty} \left| f(\sqrt{r^2 + n + 1}) - \sum_{i=0}^{n-1} \tan^{-1} \frac{1}{\sqrt{r^2 + i + 1}} - f(\sqrt{r^2 + n}) \right. \\ & \quad \left. + \sum_{i=0}^{n-1} \tan^{-1} \frac{1}{\sqrt{r^2 + i}} - \tan^{-1} \frac{1}{r} \right| \\ &= \lim_{n \rightarrow \infty} \left| f(\sqrt{r^2 + n + 1}) - f(\sqrt{r^2 + n}) - \tan^{-1} \frac{1}{\sqrt{r^2 + n}} \right| \end{aligned}$$

for each  $r \geq 1$ . If in (3.2) we replace  $r$  by  $\sqrt{r^2 + n}$  and apply the resulting inequality to the last equality, then we have

$$\left| F(\sqrt{r^2 + 1}) - F(r) - \tan^{-1} \frac{1}{r} \right| \leq \lim_{n \rightarrow \infty} \varphi(\sqrt{r^2 + n})$$

for all  $r \geq 1$ .

On the other hand, by applying the mathematical induction to inequality (3.1), we can easily prove that

$$\varphi(\sqrt{r^2 + n}) \leq L^n \varphi(r) \rightarrow 0 \text{ as } n \rightarrow \infty$$

for any  $r \geq 1$ , which means that  $F$  is a solution of Eq. (1.1).

Assume that inequality (3.3) is also satisfied with another function  $G : [1, \infty) \rightarrow \mathbb{R}$  which is a solution of Eq. (1.1). (As  $G$  is a solution of Eq. (1.1),  $G$  satisfies  $G(r) = G(\sqrt{r^2 + 1}) - \tan^{-1}(1/r) = (\Lambda G)(r)$  for all  $r \geq 1$ . That is,  $G$  is a fixed point of  $\Lambda$ .) In view of (3.3) with  $G$  and the definition of  $d$ , we know that

$$d(f, G) \leq \frac{1}{1-L} < \infty,$$

i.e.,  $G \in X^* = \{g \in X \mid d(f, g) < \infty\}$ . Thus, Theorem 2.1 (b) implies that  $F = G$ . This proves the uniqueness of  $F$ .  $\square$

**Corollary 3.2.** For a constant  $a > 1$ , let  $\varphi : [1, \infty) \rightarrow [0, \infty)$  be given by

$$\varphi(r) = a^{-r^2} \quad (r \geq 1)$$

for some constant  $a > 1$ . If a function  $f : [1, \infty) \rightarrow \mathbb{R}$  satisfies inequality (3.2) for all  $r \geq 1$ , then there exists a unique solution  $F : [1, \infty) \rightarrow \mathbb{R}$  of Eq. (1.1) such that

$$|F(r) - f(r)| \leq \frac{a^{1-r^2}}{a-1}$$

for all  $r \geq 1$ .

*Proof.* Since

$$\varphi(\sqrt{r^2+1}) = a^{-r^2-1} = \frac{1}{a}\varphi(r)$$

for all  $r \geq 1$ , we can set  $L = 1/a$  and apply Theorem 3.1 to this case.  $\square$

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