ON THE ALEKSANDROV–RASSIAS PROBLEM AND THE HYERS–ULAM–RASSIAS STABILITY PROBLEM

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This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by C. Park

ABSTRACT. Let X and Y be normed linear spaces. A mapping $T : X \to Y$ is called preserving the distance r if for all x, y of X with $\|x - y\|_X = r$ then $\|T(x) - T(y)\| = r$. In this paper, we provide an overall account of the development of the Aleksandrov problem, the Aleksandrov–Rassias problem for mappings which preserve distances and details for the Hyers–Ulam–Rassias stability problem.

1. THE ALEKSANDROV–RASSIAS PROBLEM

The theory of isometry had its beginning in the important paper by Mazur and Ulam (cf. [22]) in 1932. Let X, Y be two metric spaces, d_1, d_2 the distances on X, Y. A mapping $f : X \to Y$ is defined to be an isometry if $d_2(f(x), f(y)) = d_1(x, y)$ for all elements x, y of X. Mazur and Ulam [22] proved that every isometry of a normed real linear space onto a normed real linear space is a linear mapping up to translation.

When the target space Y is a strictly convex real normed space, for into mappings, Baker [2] proved that every isometry of a normed real linear space into a

strictly convex normed real linear space is also a linear isometry up to translation. However, for into mappings, if Y is not a strictly convex normed space, an isometry $f : X \to Y$ may not be an affine. For example, $f : \mathbb{R} \to (\mathbb{R}^2, \| \cdot \|_\infty)$ defined by $f(x) = (x, \sin x)$ is an isometry under the normwise $\| \cdot \|_\infty$, but f is not an affine.

A mapping $T : X \to Y$ is called preserving the distance r if for all x, y of X with $d_1(x, y) = r$ then $d_2(T(x), T(y)) = r$. Aleksandrov (cf. [1]) has posed the following problem:

Aleksandrov Problem. *Whether the existence of a single conservative distance for some mapping T implies that T is an isometry .*

Let X, Y be two normed linear spaces. Consider the following conditions for $T : X \to Y$ introduced for the first time by Rassias and Šemrl [38]: *distance one preserving property* (**DOPP**) and *strongly distance one preserving property* (**SDOPP**).

(DOPP) *Let $x, y \in X$ with $\|x - y\|_X = 1$. Then $\|T(x) - T(y)\|_Y = 1$.*

(SDOPP) *Let $x, y \in X$. Then $\|T(x) - T(y)\|_Y = 1$ if and only if $\|x - y\|_X = 1$.*

Rassias Problem. *Let X and Y be normed linear spaces, and $T : X \to Y$ be a continuous and/or surjective mapping satisfying (DOPP). Is T necessary to be an isometry?*

Mappings satisfying the weaker assumption that they preserve unit distance in both directions are not very far from being isometries. Rassias and Šemrl [38] proved that

Theorem 1.1. *Let X and Y be real normed linear spaces such that one of them has dimension greater than one. Suppose that $f : X \to Y$ is a surjective mapping satisfying (SDOPP). Then f is an injective mapping satisfying*

$$| \, \|f(x) - f(y)\| - \|x - y\| \, | < 1, \quad (x, y \in X).$$

Moreover, f preserves distance n in both directions for any positive integer n.

In Theorem 1.1, the assumption that one of the spaces has dimension greater than one cannot be omitted. For example ([38]), let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x + 2 & \text{if } x \text{ is an integer} \\ x & \text{if } x \text{ is not an integer.} \end{cases}$$

The mapping f is bijective and preserves distance n in both directions for any positive integer n, but does not satisfy the inequality in Theorem 1.1. The condition (SDOPP) cannot be replaced by (DOPP). Let $g : [0, 1] \to [0, 1) \times \mathbb{R}$ be a bijective mapping and define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) = g(t - [t]) + ([t], 0)$, where $[t]$ denotes the integer part of t. f is a bijective mapping preserving distance n for all positive integers n. However, the inequality in Theorem 1.1 is not fulfilled. Furthermore, the inequality in Theorem 1.1 is sharp (cf. [38]).

A number of authors have discussed Aleksandrov problem under certain additional conditions for a given mapping satisfying DOPP in order to be an isometry and have posed several interesting and new open problems (cf. [3, 4, 5, 26, 29, 31,

32, 33, 35, 37]). Even if X, Y are normed linear spaces the above problem is not easy to answer. For example, the following question posed by Rassias (cf. [30]) has not been solved yet: *Is a mapping T from $\mathbb{R}^2$ to $\mathbb{R}^3$ preserving unit distance (simply DOPP) necessarily an isometry?*

Aleksandrov problem has been solved for Euclidean spaces $X = Y = \mathbb{R}^n$. If $n = 1$, Beckman and Quarles [3] pointed out that such a mapping T does not need to be an isometry. For example ([3]), let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$T(x) = \begin{cases} x + 1, & \text{if } x \text{ is an integer,} \\ x, & \text{otherwise.} \end{cases}$$

T satisfies (DOPP) but T is not an isometry. Rassias [31] gave a counterexample even if $T : \mathbb{R} \to \mathbb{R}$ is continuous, onto and satisfies (SDOPP): Let $T(x) = [x] + (x - [x])^2, \quad (x \in \mathbb{R})$, where $[x]$ denotes the integer part of x. It is easy to verify that T is continuous, injective and satisfies (SDOPP). Since $\mathbb{R}$ is a connected set and T is continuous, by the definition of T, T is surjective. But T is not an isometry.

If $2 \leq n < \infty$, T must be an isometry due to the theorem of Beckman and Quarles [3], Bishop [5] and in a special case Zvengrovski [24, Appendix to Chapter II] independently:

Theorem 1.2. *Let T be a transformation (possibly many-valued) of $\mathbb{R}^n (2 \leq n < \infty)$ into itself. Let $d(p, q)$ be the distance between points p and q of $\mathbb{R}^n$, and let $T(p), T(q)$ be any images of p and q, respectively. If there is a length $a > 0$ such that $d(T(p), T(q)) = a$ whenever $d(p, q) = a$, then T is a Euclidean transformation of $\mathbb{R}^n$ onto itself. That is, T is a linear isometry up to translation.*

For the Hilbert space ℓ^2, an example of a unit distance preserving mapping that is not an isometry has been given by Beckman and Quarles [3]: There is in ℓ^2 a denumerable and everywhere dense set of points which will be denoted by $\{y^k\}_{k=1}^{\infty}$. Define a mapping $g : \ell^2 \to \{y^k\}_{k=1}^{\infty}$ such that the distance $d(x, g(x)) < \frac{1}{2}$. Define $h : \{y^k\}_{k=1}^{\infty} \to X$ by $h(y^k) = a^k$, where $a^k = (a_1^k, a_2^k, \cdots)$ and $a_j^k = \delta_{jk}/\sqrt{2}$ (δ_{jk} is the Kronecker delta). Now let $T : \ell^2 \to \ell^2$ be the mapping $T = h \circ g$. It may be readily seen that T is a transformation of ℓ^2 into itself which preserves the unit distance. For, if $d(x^1, x^2) = 1$, then $g(x^1) \neq g(x^2), T(x^1) \neq T(x^2)$, and therefore $d(T(x^1), T(x^2)) = 1$. However T is not an isometry, for if x^1 and x^2 are any two points in ℓ^2, then $d(T(x^1), T(x^2))$ is either 0 or 1. Obviously, from the above definition of T, T need not to be one-to-one.

In case $n \neq m$, up to now, the Aleksandrov problem is only partially solved. In particular, in the most natural case ($n = 2$, $m = 3$) the answer is not yet known. In [27] Rassias gave some interesting counterexamples for $T : \mathbb{R}^2 \to \mathbb{R}^8$ and $T' : \mathbb{R}^2 \to \mathbb{R}^6$ that T and T' satisfy (DOPP) but T and T' are not isometries.

Example 1.3. ([27]) Let us take a unit 8-simplex P in $\mathbb{R}^8$. This simplex has 9 vertices. It is enough to find a mapping $f : \mathbb{R}^2 \to P$ which preserves unit distance.

Partition the plane into squares with unit diagonals as shown in Fig. 1.1. Each square contains the bottom left corner and two edges containing this corner. Now number the squares using the numbers from 1 to 9 in the way shown in Fig. 1.1. It is easy to see that if two points lie in squares of equal numbers, then these points have distance from 1. Now, number the vertices of the simplex P and map all the points of the plane to the vertices denoted by a suitable number.
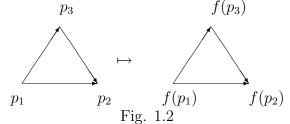
3	1	2	3	1
6	4	5	6	4
9	7	8	9	7
3	1	2	3	1

Fig. 1.1

If we use hexagons instead of squares we may construct such a mapping from $\mathbb{R}^2$ into $\mathbb{R}^6$. Generally, Rassias [27] proved the following result:

Theorem 1.4. *For any integer $n \geq 1$, there exists an integer m such that there is a map $T : \mathbb{R}^n \to \mathbb{R}^m$ which satisfies the condition (DOPP) but is not an isometry.*

A geometric interpretation is that for $f : \mathbb{R}^n \to \mathbb{R}^m$ and for arbitrary three points p_1, p_2, p_3 in $\mathbb{R}^n$ forming an equilateral triangle with unit length, if $f(p_1)$, $f(p_2)$ and $f(p_3)$ also form an equilateral triangle with unit length and $1 < m = n < \infty$,

Fig. 1.2

then f must be a linear isometry up to translation. In the case that $m = n = 1$ or $m = n = \infty$ or $m \neq n$, for arbitrary three points p_1, p_2, p_3 in $\mathbb{R}^n$, if $f(p_1)$, $f(p_2)$ and $f(p_3)$ form an equilateral triangle with unit length whenever p_1, p_2 and p_3 forming an equilateral triangle with unit length, f may be not an isometry.

Clearly, the mappings defined in the above counterexamples were not continuous or surjective. What happens if we impose conditions on T of continuity and/or surjectivity?

One answer was provided by Mielnik and Rassias [23] as follows

Theorem 1.5. *With the real classical Hilbert space denoted by $H(H = \mathbb{R}^n, 3 \leq n \leq \infty)$, let T be a homeomorphism of H onto H which preserves the distance $r > 0$. Then T is an isometry.*

Generally, for real normed linear spaces, Rassias and Šemrl [38] proved the following result based on Theorem 1.1:

Theorem 1.6. *Let X and Y be real normed linear spaces such that one of them has dimension greater than one. Suppose that $T : X \to Y$ is a Lipschitz mapping with $k = 1$:*

$$\|T(x) - T(y)\| \le \|x - y\| \quad (x, y \in X).$$

Assume also that T is a surjective mapping satisfying (SDOPP). Then T is a linear isometry up to translation.

Without the condition "$T : X \to Y$ is a Lipschitz mapping with $k = 1$", T need not be an isometry. For example, let $T : (\mathbb{R}^2, \| \cdot \|_\infty) \to (\mathbb{R}^2, \| \cdot \|_\infty)$ be a mapping defined by $T(x_1, x_2) = ([x_1] + (x_1 - [x_1])^2, x_2)$. It is easy to verify that T is a continuous surjective mapping satisfying (SDOPP) since $\lim_{x \to \pm\infty}\{[x] + (x - [x])^2\} = \pm\infty$ and $[x] + (x - [x])^2$ is continuous about x, but T is not an isometry.

According to Theorem 1.6, they [38] got the following corollaries:

Corollary 1.7. *Let X and Y be real normed linear spaces such that one of them has dimension greater than one. Assume also that one of them is strictly convex. Suppose that $T : X \to Y$ is a surjective mapping satisfying (SDOPP). Then T is a linear isometry up to translation.*

Corollary 1.8. *Let X and Y be real normed linear spaces, $\dim X > 1$, such that one of them is strictly convex. Suppose that $T : X \to Y$ is a homeomorphism satisfying (DOPP). Then T is a linear isometry up to translation.*

For special case $X = \mathbb{R}$, [39] derives

Corollary 1.9. *Suppose that $T : \mathbb{R} \to \mathbb{R}$ is a Lipschitz mapping with $k = 1$:*

$$\|T(x) - T(y)\| \le \|x - y\| \quad (x, y \in \mathbb{R}).$$

Assume that T is a mapping satisfying (DOPP). Then T is a linear isometry up to translation.

Since any real normed linear space with dimension one is linearly isometric to $\mathbb{R}$, according to Theorem 1.6 and Corollary 1.9, we obtained the following result without the condition $\dim X \ge 2$ in Theorem 1.6:

Corollary 1.10. *Let X and Y be real normed linear spaces. Suppose that $T : X \to Y$ is a Lipschitz mapping with $k = 1$:*

$$\|T(x) - T(y)\| \le \|x - y\| \quad (x, y \in X).$$

Assume also that T is a surjective mapping satisfying (SDOPP). Then T is a linear isometry up to translation.

The Rassias problem is still open for the special case that $T : \ell^2 \to \ell^2$ is a continuous mapping satisfying (DOPP).

The Aleksandrov problem was intensively considered by Ciesielski and Rassias in [8] for non-standard metrics. In $\mathbb{R}^n$ three classical metrics induce the same topology: $d_\Sigma(x, y) = \sum_{i=1}^n |x_i - y_i|$, $d_m = \max\{|x_i - y_i| : i = 1, 2, \cdots, n\}$ and the Euclidean metric d_E, where $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$.

Ciesielski and Rassias [8] gave some interesting examples about the Aleksandrov problem. For $(\mathbb{R}^n, d_m)$, $n > 1$, a mapping satisfying (DOPP) need not be an isometry. For this it is enough to consider the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$f(x_1, x_2, \cdots, x_n) = ([x_1], [x_2], \cdots, [x_n]),$$

where $[x]$ denotes the integer part of x.

For $(\mathbb{R}^2, d_\Sigma)$, a mapping satisfying (DOPP) need not be an isometry either. For this it is enough to consider the mapping $g : (\mathbb{R}^2, d_\Sigma) \to (\mathbb{R}^2, d_\Sigma)$ defined by $g = (\frac{1}{\sqrt{2}} \cdot R_{\pi/4}) \circ f \circ (\frac{1}{\sqrt{2}} \cdot R_{\pi/4}^{-1})$, where $f(x_1, x_2) = ([x_1], [x_2])$ and $R_{\pi/4}$ is the rotation:

$$(x, y) \to \left((x + y)/\sqrt{2}, (y - x)/\sqrt{2}\right).$$

The mapping g satisfies (DOPP) but is not an isometry. Furthermore, in [8] they proved the following result.

Theorem 1.11. *There is no mapping $f : (\mathbb{R}^2, d_m) \to (\mathbb{R}^2, d_E)$ satisfying the condition (DOPP).*

What happens if we require, instead of one conservative distance for a mapping between normed linear spaces, two conservative distances? An answer in a generalized form to this question was given by Benz and Berens [4] who proved the following theorem and pointed out that the condition that Y is strictly convex can not be relaxed.

Theorem 1.12. *Let X and Y be real normed linear spaces. Assume that $\dim X \geq 2$ and Y is strictly convex. Suppose $T : X \to Y$ satisfies that: T preserves the two distances ρ and $\lambda\rho$ for some integer $\lambda \geq 2$. That is, for all $x, y \in X$ with $\|x - y\| = \rho$, $\|T(x) - T(y)\| \leq \rho$; and for all $x, y \in X$ with $\|x - y\| = \lambda\rho$, $\|T(x) - T(y)\| \geq \lambda\rho$. Then T is a linear isometry up to translation.*

The Aleksandrov–Rassias problem. *If T preserves two distances with a noninteger ratio, and X and Y are real normed linear spaces such that Y is strictly convex and $\dim X \geq 2$, whether or not T must be an isometry (cf. [30]).*

The Aleksandrov–Rassias problem was extensively studied in Hilbert spaces [45, 46].

Theorem 1.13. *Let X, Y be Hilbert spaces and the dimension of X be greater than or equal to 2. Suppose that $f : X \to Y$ satisfies (DOPP) and the dimension of X is infinite. If one of the distances*

$$m\sqrt{n^2(2 \cdot 4^k - \frac{4^k - 1}{3})4^l - \frac{4^l - 1}{3}}$$

is preserved by f, $m, n = 1, 2, \ldots$ and $k, l = 0, 1, 2, \ldots$. Then f must be a linear isometry up to translation.

2. The Hyers–Ulam–Rassias stability problem

Problems connected with stability of isometries as well as perturbations of isometries have been extensively studied by [6, 7, 9, 14, 15, 16, 17, 18, 19, 28, 29, 34, 36].

Let E and F be real Banach spaces. For a fixed $\varepsilon > 0$, a mapping $T : E \to Y$ is called an $\varepsilon - isometry$ if

$$| \, \|T(x) - T(y)\| \; - \; \|x - y\| \, | \leq \varepsilon, \quad (x, y \in E).$$

The stability problem for isometries following Hyers and Ulam [15] is:

Does there exist a constant $M > 0$ depending only on X and Y with the following property: For each $\varepsilon > 0$ and each surjective ε-isometry f there is an isometry $U : X \to Y$ such that $\|T(x) - U(x)\| \leq M\varepsilon$ for each $x \in E$?

Hyers and Ulam [15] showed that the answer is affirmative in the case that $E = F$ is a Hilbert space. A possible value for M is 10. Bourgin [6] gave a positive answer whenever E and F belong to a class of uniformly convex Banach spaces including the $L_p(X, \Sigma, \mu)$ spaces $1 < p < \infty$.

In particular, Gruber [13] showed that if the problem of Hyers and Ulam has a positive solution for a pair E, F of Banach spaces, then one can assume $M \leq 5$.

In 1983, Gevirtz [12] based on a body of previous partial results extending over 38 years, proved that

Theorem 2.1. *There exist constants A and B such that if $T : E \to F$ is a surjective ε-isometry, then*

$$\|T((x_1 + x_0)/2) - (T(x_0) + T(x_1))/2\| \leq$$
$$A(\varepsilon\|x_0 - x_1\|)^{1/2} + B\varepsilon, \quad (x_0, x_1 \in E).$$

Omladic and Šemrl gave a sharp result answering the above question positively for $M = 2$.

Bourgin [7] and Tabor [42] considered the Hyers–Ulam problem for the special injective mapping called $\delta - onto$ mapping. Let S be a subset of E, and let $\delta \geq 0$ be arbitrary. A function $f : S \to F$ is called $\delta - onto$ (cf. [42]) if

$$\forall x \in F, \exists s \in S, \|x - f(s)\| \leq \delta.$$

Theorem 2.2. *Let E and F be Banach spaces, and let $f : E \to F$ be an ε-isometry which is δ-onto and such that $f(0) = 0$. Then there exists a unique linear isometry $U : E \to F$ such that $\|f(x) - U(x)\| \leq 2\varepsilon + 35\delta \quad (x \in E)$.*

For into mappings the Hyers–Ulam problem was seriously studied by Ding [9], Gruber [13], Wang [44], Xiang and Tan [47], Zhan [49], etc.

Theorem 2.3. *([13]) Let E be a real normed space and let $\phi : [0, \infty) \to [0, \infty)$ be a real function such that $\phi(\xi) = o(\xi)$ as $\xi \to +\infty$. Then there exists a real normed space F with the following property: For every $\varepsilon \in \mathbb{R}^+$ there exists an ε-isometry $T : E \to F$ such that there exists no isometry $I : E \to F$ for which $\|T(x) - I(x)\| = o(\phi(\|x\|))$ as $\|x\| \to \infty$ uniformly.*

Theorem 2.4. *([48]) Suppose that X is an infinite dimensional uniformly smooth Banach space, (Ω, Σ, μ) is a σ-finite measure space, and $T : X \to L^\infty(\Omega, \Sigma, \mu)$ is a linear bounded operator satisfying $(1-\varepsilon)\|x\| \leq \|Tx\| \leq \|x\|$ $(x \in X)$ for some positive number $\varepsilon < \frac{1}{2}$ with $\delta_{X^*}(2 - 2\varepsilon) > \frac{13}{14}$, then T is close to an isometry $U : X \to L^\infty(\Omega, \Sigma, \mu)$ such that*

$$\|T - U\| \leq 16(1 - \delta_{X^*}(2 - 2\varepsilon)) + \frac{1}{2}\varepsilon.$$

For the Hyers–Ulam stability problem there is much development such as the Hyers–Ulam–Rassias stability for additive mappings (cf. [17, 25]). For the stability of the Cauchy equation $f(x + y) = f(x) + f(y)$, Ulam [43] in 1940 raised the following question: *Under what conditions does there exist an additive mapping near an approximately additive mapping?*

Hyers [14] in 1941 proved that if $f : X \to Y$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon, \quad (x, y \in E),$$

then there exists a unique additive mapping $U : X \to Y$ such that

$$\|f(x) - U(x)\| \leq \varepsilon, \quad (x \in E).$$

In 1978, Rassias [25] generalized the theorem of Hyers significantly by considering the Cauchy difference to become unbounded and proved the following theorem:

Given $0 \leq p < 1$ and $\varepsilon \geq 0$, if a function $f : X \to Y$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \quad (x, y \in E),$$

then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p, \quad (x \in E).$$

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the function T is linear.

This result was later extended to all $p \neq 1$ and generalized by Czerwil, Gajda, Gavruta, Isac and Rassias, Jung, Šemrl, Lee and Jun, among others.

In the spirit of Hyers, Ulam and Rassias, Jung and Kim [21], and Dolinar [10] independently introduced an $(\varepsilon, p) - isometry$. A mapping $T : E \to F$ is called $(\varepsilon, p) - isometry$ if

$$|\|T(x) - T(y)\| - \|x - y\|| \leq \varepsilon\|x - y\|^p, \quad (x, y \in E).$$

Dolinar in [10] got the following results.

Theorem 2.5. *Let E and F be real Banach spaces and let $0 \leq p < 1$. There exists a constant $N(p)$, independent of E and F, such that for every surjective (ε, p)-isometry $T : E \to F$ with $T(0) = 0$ there exists a surjective isometry $U : E \to F$ satisfying*

$$\|T(x) - U(x)\| \leq \varepsilon N(p)\|x\|^p, \quad (x \in E).$$

Theorem 2.6. *Let E be a real Banach space and let F be a real Hilbert space. Suppose that $\varepsilon \geq 0$ and $0 < p < 1$. Then there exists a constant $K(\varepsilon, p)$, independent of E and F such that $\lim_{\varepsilon \to 0} K(\varepsilon, p) = 0$ and for every surjective (ε, p)-isometry $T : E \to F$ with $T(0) = 0$ there exists a surjective isometry $U : E \to F$ satisfying*

$$\|T(x) - U(x)\| \leq K(\varepsilon, p) \max\{\|x\|^p, \|x\|^{(1+p)/2}\} \quad (x \in E).$$

When the target space F is the $L_r(0, 1)$, Dolinar [10] obtained a similar result.

Theorem 2.7. *If $E = F = L_r(0, 1)$, where $1 < r < \infty$, $\varepsilon \geq 0$ and $0 < p < 1$ then there exists a constant $K(\varepsilon, p, r)$ such that $\lim_{\varepsilon \to 0} K(\varepsilon, p, r) = 0$ and for every surjective (ε, p)-isometry $T : E \to F$ with $T(0) = 0$ there exists a surjective isometry $U : E \to F$ satisfying*

$$\|T(x) - U(x)\| \leq K(\varepsilon, p, r) \max\{\|x\|^p, \|x\|^{1-(1-p)/s}\} \quad (x \in E)$$

where $s = \max\{r, \frac{r}{1-r}\}$.

It is somewhat surprising that if $p > 1$ then for finite-dimensional Banach spaces a superstability phenomenon occurs (cf. [10]). For $p = 1$, even very nice spaces are not stable. A counterexample was given by Dolinar for the mapping $T : R^2 \to R^2$ being $(\varepsilon, 1)$-isometry.

Generally, in [40, 46] we considered the Hyers–Ulam stability problem in the following case for into mapping $f : S \to F$ satisfying

$$| \, \|f(x) - f(y)\| - \|x - y\| \, | \leq \varepsilon \phi(x, y) \quad (x, y \in S)$$

when the target space F is a Hilbert space or when $F = L^q(\Omega, \Sigma, \mu)(1 < q < \infty)$, or generally in a class of uniformly convex Banach spaces, where $\phi : X \times X \to [0, +\infty)$ and S is a subset of E or $S = E$. Especially, we investigated the approximate isometry for the cases that $\phi(x, y) = \|x\|^p + \|y\|^p$ and $\phi(x, y) = \|x - y\|^p$ for $p \neq 1$. The pair (E, F) is stable too.

On the other hand, Swain [41] considered the stability of isometries on bounded metric spaces and proved the following result: Let S be a subset of a compact metric space (X, d) and let $\delta > 0$ be given. Then there exists an $\varepsilon > 0$ such that if $f : S \to X$ is an ε-isometry, then exists an isometry $U : S \to X$ with $d(f(x), U(x)) \leq \delta$ for any $x \in S$.

The stability problem of isometries on bounded subsets of $\mathbb{R}^n$ was studied by Fickett [11]: For $t \geq 0$, let us define $K_0(t) = K_1(t) = t$, $K_2(t) = 3\sqrt{3t}$, $K_i(t) = 27t^{m(i)}$, where $m(i) = 2^{1-i}$ for $i \geq 3$. Let S be a bounded subset of $\mathbb{R}^n$ with diameter $d(S)$, and suppose that $K_n(\delta/d(S)) \leq 1$ for some $\delta \geq 0$. If a function $f : S \to \mathbb{R}^n$ is an ε-isometry, then exists an isometry $U : S \to \mathbb{R}^n$ such that $\|f(x) - U(x)\| \leq d(S)K_{n+1}(\delta/d(S))$ for each $x \in S$.

Along with Swain and Fickett, Jung [20] studied the stability of isometries on restricted domains in Hilbert spaces and proved the result: Suppose that $f : S \to E$ satisfies the following

$$| \, \|f(x) - f(y)\| - \|x - y\| \, | \leq \delta + \varepsilon \|x - y\|^p \quad (x, y \in S)$$

where E is a Hilbert space and S is a subset of E. Then

(i) for $0 < p < 1$, d being a positive constant satisfying $2d - 2\varepsilon d^p - \delta \geq 0$ and $S = E - \{x \in E : \ 0 < \|x\| \leq d\}$, there exists a unique linear isometry $U : E \to E$ such that

$$\|f(x) - U(x) - f(0)\| \leq 2\delta + \frac{\sqrt{6\delta + 8\varepsilon}}{\sqrt{2 - 2^p}}\|x\|^{(1+p)/2} \quad (x \in S);$$

(ii) for $p > 1$, $\delta = 0$ and d being a positive constant, there exists a unique linear isometry $U : E \to E$ such that for $S = \{x \in E : \ 0 < \|x\| \leq d\}$

$$\|f(x) - U(x) - f(0)\| \leq \frac{2^{(1+p)/2}}{2^{(1+p)/2} - 1}\sqrt{\varepsilon}\|x\|^{(1+p)/2}, \quad (x \in S).$$

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