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LINEAR MAPS RESPECTING UNITARY CONJUGATION

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ABSTRACT. We characterize linear maps on von Neumann algebras which leave every unital subalgebra invariant. We use this characterization to determine linear maps which respect unitary conjugation, answering a question of M. S. Moslehian.

1. Introduction

Let $\mathcal{H}$ be a complex, separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. It was asked by M.S. Moslehian (private communication) as to what are linear maps α on $\mathcal{B}(\mathcal{H})$ which satisfy

$$\alpha(UXU^*) = U\alpha(X)U^* \ \forall X \in \mathcal{B}(\mathcal{H}),$$

for every unitary U on $\mathcal{H}$. We answer this question by first proving a theorem characterizing linear maps on von Neumann algebras which leave all subalgebras invariant.

2. Maps leaving subalgebras invariant

Theorem 2.1. *Let $\mathcal{A}$ be a von Neumann algebra and let I denote the identity element in $\mathcal{A}$. Let $\alpha : \mathcal{A} \to \mathcal{A}$ be a norm continuous linear map. Then the following are equivalent:*

(i) $\alpha(\mathcal{B}) \subseteq \mathcal{B}$ for every von Neumann subalgebra $\mathcal{B}$ of $\mathcal{A}$ with $I \in \mathcal{B}$.

(ii) $\alpha(\mathcal{B}) \subseteq \mathcal{B}$ for every abelian von Neumann subalgebra $\mathcal{B}$ of $\mathcal{A}$ with $I \in \mathcal{B}$.

(iii) $\alpha(x) = cx + \psi(x)I$ for some $c \in \mathbb{C}$ and some norm continuous linear functional $\psi : \mathcal{A} \to \mathbb{C}$.

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Before we prove this Theorem in general, we prove a special case as a Lemma and recall Halmos decomposition for pairs of generic projections. In the following for any projection p, $p^\perp$ denotes the projection $(I-p)$.

Lemma 2.2. *Let $\mathcal{A}$ be the algebra $M_2(\mathbb{C})$ of 2×2 complex matrices. Suppose $\alpha : \mathcal{A} \to \mathcal{A}$ is a linear map which leaves every unital $*$-subalgebra of $M_2(\mathbb{C})$ invariant. Then $\alpha(x) = cx + \psi(x)I$ for some $c \in \mathbb{C}$ and some linear functional ψ on $M_2(\mathbb{C})$.*

Proof. To begin with we assume that $\text{trace}(\alpha(X)) = 0$ for all X. As $\{cI : c \in \mathbb{C}\}$ is a unital commutative $*$-subalgebra of $M_2(\mathbb{C})$, $\alpha(I) = bI$ for some $b \in \mathbb{C}$. Combined with the trace assumption made now, $\alpha(I) = 0$.

Similarly since any rank one projection p generates a unital commutative algebra consisting of linear combinations of $p, p^\perp$, we get $\alpha(p) = c_p(p - p^\perp)$ for some scalar c_p, for every rank one projection p. Hence $\alpha(p - p^\perp) = 2c_p(p - p^\perp)$. Equivalently, every self-adjoint trace zero element of $M_2(\mathbb{C})$ is an eigenvector for α. In particular, there exist constants $c_1, c_2, \dots, c_5$ such that $\alpha(A_i) = c_iA_i, 1 \le i \le 5$ where matrices A_i's are

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

respectively. Observing $A_1 + A_2 = A_4$ and $A_1 + A_3 = A_5$, linearity of α yields $c_1 = c_2 = c_4$ and $c_1 = c_3 = c_5$. Writing these matrices in the form $p - p^\perp$, we get a basis for $M_2(\mathbb{C})$ consisting of rank one projections and $\alpha(X) = c_1(X - \frac{1}{2}\text{trace}(X)I)$ for all X.

If α does not satisfy the assumption made above, consider β where,

$$\beta(X) = \alpha(X) - \frac{1}{2}\text{trace}(\alpha(X))I.$$

Proving the result for β is as good as proving the result for α. □

For any two projections p, q, denote the largest projection smaller than both p and q by $p \wedge q$. Recall that two projections p, q are said to be a generic pair if $p \wedge q = p \wedge q^\perp = p^\perp \wedge q = p^\perp \wedge q^\perp = 0$. The following result is well-known as Halmos decomposition ([1, 3]). If a pair of projections p, q on a Hilbert space $\mathcal{H}$ are generic, then $p(\mathcal{H})$ and $p^\perp(\mathcal{H})$ are isomorphic as Hilbert spaces and making use of this isomorphism, with respect to the decomposition $\mathcal{H} = p(\mathcal{H}) \oplus p^\perp(\mathcal{H})$, p and q have the form:

$$p = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad q = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

with $0 < c, s < I, s = (I - c^2)^{\frac{1}{2}}$.

Proof of Theorem 2.1 : (i)$\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are obvious. Now we show (ii) $\Rightarrow$ (iii).

If $\mathcal{A}$ has no non-trivial projection then $\mathcal{A}$ is isomorphic to $\mathbb{C}$ and there is nothing to show. Suppose p is a non-trivial projection in $\mathcal{A}$ and if only other non-trivial projection $\mathcal{A}$ has is $(I-p)$,then $\mathcal{A}$ is isomorphic to $\mathbb{C}^2$ and once again the result is obvious. In the following we exclude these two trivial cases.

Suppose p is a projection in $\mathcal{A}$ then the von Neumann algebra generated by p and I is $\{ap + bI : a, b \in \mathbb{C}\}$. It is abelian and hence left invariant by α. This shows that for any projection p in $\mathcal{A}$,

$$\alpha(p) = c_p p + d_p I \tag{2.1}$$

for some $c_p, d_p \in \mathbb{C}$. Note that scalars c_p, d_p are uniquely defined for non-trivial projections p. We wish to show that $c_p = c_q$ for any two non-trivial projections $p, q \in \mathcal{A}$.

Now suppose p_1, p_2, p_3 are three mutually orthogonal non-trivial projections in $\mathcal{A}$ such that $p_1 + p_2 + p_3 = I$. We have $\alpha(p_i) = c_{p_i} p_i + d_{p_i} I$ for $i = 1, 2, 3$. We also have $\alpha(p_1 + p_2) = c_{p_1+p_2}(p_1 + p_2) + d_{p_1+p_2} I$. But by linearity $\alpha(p_1 + p_2) = \alpha(p_1) + \alpha(p_2)$. So we get,

$$c_{p_1} p_1 + d_{p_1} I + c_{p_2} p_2 + d_{p_2} I = c_{p_1+p_2}(p_1 + p_2) + d_{p_1+p_2} I. \tag{2.2}$$

Multiplying this by p_3, yields, $d_{p_1} p_3 + d_{p_2} p_3 = d_{p_1+p_2} p_3$ or $d_{p_1} + d_{p_2} = d_{p_1+p_2}$. Substituting this back in (2.2) yields $c_{p_1} p_1 + c_{p_2} p_2 = c_{p_1+p_2}(p_1 + p_2)$, and then multiplications by p_1, p_2 show us $c_{p_1} = c_{p_2} = c_{p_1+p_2}$.

If p, q are two non-trivial projections in $\mathcal{A}$, such that $p \wedge q \neq 0$. Considering the triple $p \wedge q, p \ominus (p \wedge q), p^\perp$ we get $c_{p \wedge q} = c_p$, similarly $c_{p \wedge q} = c_q$, so $c_p = c_q$. It follows, that if p, q are non-trivial projections in $\mathcal{A}$, which are not in generic position and $q \neq p^\perp$, then $c_p = c_q$.

Suppose p, q are projections in $\mathcal{A}$ and are in generic position. If pqp is not a scalar multiple of p, then considering a non-trivial spectral projection p' of pqp, from the Halmos decomposition, we see that p', q are not in generic position as $(p')^\perp \wedge q \neq 0$. Hence $c_p = c_{p'} = c_q$. On the other hand, if p, q are in generic position and pqp is a scalar multiple of p, then by the Halmos decomposition it is clear that the algebra generated by p, q is $M_2(\mathbb{C})$ and we can apply Lemma 2.2 to get $c_p = c_q$.

Finally if $q = p^\perp$, on the one hand if there is a third non-trivial projection r different from p, q, we get $c_p = c_r = c_q$, and on the other hand if there is no such third projection then clearly $\mathcal{A}$ is isomorphic to $\mathbb{C}^2$ and we have already excluded this case.

This proves that for any two non-trivial projections in $\mathcal{A}$ we have $c_p = c_q$ (call this constant as c). Now if $p_1, p_2, \ldots, p_k$ are mutually orthogonal projections in $\mathcal{A}$ then for $x = \sum_{i=1}^k a_i p_i$ with scalars $a_1, a_2, \ldots, a_k$, $\alpha(x) = \sum_i a_i \alpha(p_i) = \sum_i a_i (c p_i + d_{p_i} I) = cx + d_x I$, for some scalar d_x. By spectral theorem every self-adjoint element of $\mathcal{A}$ can be approximated in norm by elements of the form $\sum_i a_i p_i$. It follows that, for every self-adjoint element $x \in \mathcal{A}$, α has the form,

$$\alpha(x) = cx + \psi(x) I.$$

for some $\psi(x) \in \mathbb{C}$. By continuity and linearity of α, it is clear that ψ is a continuous linear functional. □

Remark 2.3. No continuity assumption is needed in Theorem 2.1 in certain situations. For instance if the algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then as every bounded operator is a finite linear combination of projections (See [2, 4]), Theorem 2.1 follows without any continuity assumption (of course, then the functional ψ also need not be continuous).

Remark 2.4. It is a natural question as to whether in Theorem 2.1 (ii), we can replace 'abelian' by 'maximal abelian'. Clearly the answer is no, if the algebra $\mathcal{A}$ itself is abelian, as in this case every map α would satisfy (ii). However, this can be done if the algebra $\mathcal{A}$ is $\mathcal{B}(\mathcal{H})$. To see this consider any rank one projection p in $\mathcal{B}(\mathcal{H})$. Looking at maximal abelian subalgebras of $\mathcal{B}(p^\perp(\mathcal{H}))$, one has $\alpha(p) = c_p p + \beta_p$, where $c_p \in \mathbb{C}$ and β_p is in every maximal abelian subalgebra of $\mathcal{B}(p^\perp(\mathcal{H}))$. This of course, means that $\alpha(p)$ has the form (2.1). Now one can continue as in the proofs of Lemma 2.2 and Theorem 2.1 to get $c_p = c_q$ for every rank one projections p, q and that suffices to obtain (iii), under continuity assumption on α.

3. Unitary Conjugation

Finally, we have the result we were looking for.

Theorem 3.1. *Let $\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a linear map. Then the following are equivalent.*

(i) $\alpha(UXU^*) = U\alpha(X)U^*$ $\forall X, U$ in $\mathcal{B}(\mathcal{H})$ with $UU^* = U^*U = I$.

(ii) $\alpha(X) = cX + d$ trace $(X)I$ for some $c, d \in \mathbb{C}$ if $\mathcal{H}$ is finite dimensional, $\alpha(X) = cX$ for some $c \in \mathbb{C}$ if $\mathcal{H}$ is infinite dimensional.

Proof. Clearly (ii)⇒ (i). Now suppose $\mathcal{A}$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ with $I \in \mathcal{A}$. If U is a unitary in the commutant von Neumann algebra $\mathcal{A}'$ we get $\alpha(X) = \alpha(UXU^*) = U\alpha(X)U^*$ for $X \in \mathcal{A}$. So

$$\alpha(X)U = U\alpha(X).$$

However every element in a unital C^* is algebra is a linear combination of at most four unitaries. Hence

$$\alpha(X)Y = Y\alpha(X) \ \forall X \in \mathcal{A}, Y \in \mathcal{A}'.$$

Then by von Neumann's double commutant theorem $\alpha(X) \in \mathcal{A}$. Now with Remark 2.3, Theorem 2.1 is applicable, and we have $\alpha(X) = cX + \psi(X)I$ for some $c \in \mathbb{C}$ and some linear functional ψ. Further, by (i), for every unitary U,

$$cUXU^* + \psi(UXU^*)I = U[cX + \psi(X)I]U^*.$$

So, $\psi(UXU^*) = \psi(X)$ for all X. Taking $X = YU$, we get $\psi(UY) = \psi(YU)$ for every Y. Once again, since every operator is a linear combination of at most four unitaries, $\psi(XY) = \psi(YX)$. So ψ is a trace. It is well-known that if $\mathcal{H}$ is infinite dimensional $\mathcal{B}(\mathcal{H})$ does not admit a non-trivial finite trace. □

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