Geometric dynamics of calcium oscillations ODEs systems

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Abstract

From a differential geometric point of view, this paper expresses in time dependent least squares Lagrangian terms that the solutions of any DEs systems of order one are harmonic curves on 1-jet spaces. Natural time dependent electromagnetic fields, together with their generalized Maxwell equations, are derived from the given DEs systems and suitable geometric structure. Important applications to biological DEs systems governing the intracellular calcium oscillations in a model involving degradation of inositol triphosphate or calcium oscillations in a model that takes into account three stored in the cell (endoplasmic reticulum, mitochondria and cytosolic proteins), together with some natural biologic-electromagnetic Yang-Mills energies of geometric-physical type, are established. Some derived geometric-biological interpretations are exposed as well.

Mathematics Subject Classification: 53C43, 53C07, 37J35, 83C22.

Key words: 1-jet spaces, harmonic curves, generalized Maxwell equations, calcium oscillations in living cells.

1 Introduction

According to Olver's opinion expressed in [23] and private discussions, we consider that the 1-jet spaces of kind $J^1(T, M)$, where T is a smooth p-dimensional *"multi-time"* manifold and M is a smooth n-dimensional *"spatial"* manifold, are basic objects in the study of a lot of applicative problems coming from many branches of Theoretical Physics: continuum mechanics [15], quantum field theories [23], generalized multi-time field theories [17]. All preceding applicative studies were required a profound analysis of the differential geometry of 1-jet spaces. Consequently, many different geometrical methods on 1-jet spaces were intensively studied by a lot of authors: Saunders [26] (Riemannian geometrical methods on jet spaces of arbitrary orders $J^r(T, M)$, $r \geq 1$), Vondra [34] (Lagrangian geometrical methods on $J^1(\mathbb{R}, M)$), Giachetta, Mangiarotti and Sardanashvily [9] (Hamiltonian polysymplectic geometrical methods on dual of $J^1(T, M)$), Marsden, Pekarsky, Shkoller and West [15] (Hamiltonian multisymplectic

Balkan Journal of Geometry and Its Applications, Vol.9, No.2, 2004, pp. 36-67.

geometrical methods on dual of $J^1(T,M)$), Asanov [2] (Riemann-Finslerian Gauge geometrized methods on $J^1(T,M)$) or Neagu [19], [17] (Riemann-Lagrange geometrical methods on $J^1(T,M)$).

In this paper we will present some Riemann-Lagrange geometrical results on 1-jet spaces of kind $J^1(\mathbb{R},\mathbb{R}^n)$, that allow the description of some interesting applications in Theoretical Biology. In this direction, we recall that the 1-jet spaces are good mathematical models for fruitful geometrical studies of DEs or PDEs. We refer to the point of view related on the Poincaré problem [24] and its generalization due to Udrişte [30], concerning the possibility of finding of a geometrical structure that to convert the *trajectories* of a given vector field X into *"geodesics"* or, more general, the *solutions* of a given PDEs system of order one into *"harmonic maps" or "potential maps"*. We would like to point out that, via a natural notion of *generalized harmonic map* on $J^1(T,M)$ introduced by first author of this paper in his Ph.D. Thesis [19] the generalized Poincaré problem and its generalization are now geometrically solved by Neagu and Udrişte in [22].

The final answer at the Poincaré problems is given in [22], using that so-called *the least squares variational calculus method for PDEs systems of order one.* Briefly speaking, the least squares variational calculus method for PDEs systems of order one consists in a natural extension of the following well known and simple idea: *In any euclidian vector space* $(V,<,>)$ *the equivalence* $v=0_V \Leftrightarrow ||v||=0$ *holds always good.* In order to extend this simple idea from linear algebra to the study of PDEs systems of order one, let us consider a PDEs system of order one on the 1-jet space $J^1(T,M)$, expressed locally by

$$\text{(1.1.1.1)}\quad \begin{aligned} &x^i_\alpha = X^{(i)}_{(\alpha)}(t^\gamma,x^k), \qquad \forall i=\overline{1,n},\ \forall\alpha=\overline{1,p},\ \Leftrightarrow\\ &x^i_\alpha - X^{(i)}_{(\alpha)}(t^\gamma,x^k)=0, \quad \forall i=\overline{1,n},\ \forall\alpha=\overline{1,p},\end{aligned}$$

where $(t^\gamma)\to(x^i(t^\gamma))$ is the unknown map, $x^i_\alpha=\partial x^i/\partial t^\alpha$ and $X^{(i)}_{(\alpha)}(t^\gamma,x^k)$ is a given and known d-tensor on $J^1(T,M)$.

Obviously, using now two "a priori" fixed Riemannian metrics $h_{\alpha\beta}(t^\gamma)$ on T and $\varphi_{ij}(x^k)$ on M that produce the vertical metrical d-tensor $h^{\alpha\beta}(t^\gamma)\varphi_{ij}(x^k)$ on $J^1(T,M)$, we conclude that the PDEs system (1.1.1.1) is equivalent with (see the preceding idea from linear algebra)

$$\text{(1.1.1.2)}\qquad \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{p} h^{\alpha\beta}\varphi_{ij}\left(x^i_\alpha - X^{(i)}_{(\alpha)}\right)\left(x^j_\beta - X^{(j)}_{(\beta)}\right)=0.$$

We emphasize that the equivalent form (1.1.1.2) of the initial PDEs system (1.1.1.1) is a more convenient one for our geometrical studies. This is because the solutions of the PDEs system (1.1.1.2) are exactly the global minimum points of the quadratic multi-time Lagrangian of electrodynamics kind

$$\mathcal{PDE}s\mathcal{ED}=\left\{h^{\alpha\beta}(t^\gamma)\varphi_{ij}(x^k)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t^\gamma,x^k)x^i_\alpha + \Phi(t^\gamma,x^k)\right\}\sqrt{h},$$

where $h=\det(h_{\alpha\beta})$, $U^{(\alpha)}_{(i)} = -2h^{\alpha\mu}\varphi_{im}X^{(m)}_{(\mu)}$ and $\Phi = h^{\mu\nu}\varphi_{rs}X^{(r)}_{(\mu)}X^{(s)}_{(\nu)}$, whose Riemann-Lagrange geometrization, in the sense of derived nonlinear connections, linear d-connections, d-torsions, d-curvatures and generalized Maxwell and Einstein equations, is now completely done in the papers [20] and [22].

Taking into account yet that only the nonlinear connections, the d-torsions and the generalized Maxwell equations are dependent efectively on the PDEs system (1.1.1.1), that is in their local expressions appear explicitely the tensorial components $X^{(i)}_{(\alpha)}(t^\gamma, x^k)$ that define the PDEs system (1.1.1.1) or the covariant partial derivatives of $X^{(i)}_{(\alpha)}(t^\gamma, x^k)$, we will study only these geometrical objects and properties. We point out that the other geometrical objects derived from the multi-time Lagrangian $\mathcal{PDE}s\mathcal{ED}$, like the linear d-connections, the d-curvatures and the generalized Einstein equations, are dependent only on the pair of metrics $(h_{\alpha\beta}(t^\gamma), \varphi_{ij}(x^k))$. For more geometrical details, the reader is invited to consult the work [22].

In this geometrical context, the aim of this paper is to apply the preceding least squares variational calculus method to several important biological nonlinear DEs equations on 1-jet spaces $J^1(T, M)$, where $T \subset \mathbb{R}$ and $M \subset \mathbb{R}^n$, $n \geq 2$. We will prove that the solutions of these biological DEs systems are *harmonic curves* on jet space $J^1(T, M)$, that is these solutions minimize a least squares relativistic time dependent energy functional. Moreover, the Riemann-Lagrange geometry of these biological DEs systems, in the sense of the derived nonlinear connections, d-torsions and time dependent biological evolution, will be described. Particularly, interesting and unpublished geometric-biological interpretations for calcium oscillations in the biological living cells are obtained.

Remark 1.1. *i) In other ways, using as a pattern the already classical Lagrangian geometrical methods developed on the tangent bundle* $\mathbf{T}M$ *by Miron and Anastasiei in [16], interesting geometrical results about DEs systems from Biology or Biodynamics are given by Antonelli and Miron [1].*

ii) The special curves (v-paths, h-paths and geodesics) on the 1-jet spaces $J^1(T, M)$*, together with few particular and interesting computer-drawn Maple-V plots, are studied by Balan in [3]. From our point of view, these special curves on 1-jet spaces may be intimately connected by that we have called harmonic curves.*

Taking into account that we would like to develope the Riemann-Lagrange geometry of the DEs systems that govern the calcium oscillations in a large variety of living cells, we would like to expose few biological properties of these oscillations. So, we recall that the oscillations of cytosolic calcium concentration, known as calcium oscillations, play a vital role in providing the intracellular signalling and a lot of biological processes are controlled by the oscillatory changes of cytosolic calcium concentration. For more details, please see the paper of Rottingen and Iversen [25]. Since the 1980's, when the self-sustained calcium oscillations were found experimentally by Cuthbertson and Cobbold [6] or by Woods, Cuthbertson and Cobbold [35], a lot of experimental works have been published. For review in this topics, please see the book of Goldbeter [10] or the paper of Berridge, Lipp and Bootman [4]. Also, interesting results may be found in the paper of Shuttleworth and Thompson [27].

Various models have been constructed to simulate calcium oscillations in living cells. In this paper, we will consider and will geometrically study only two of these models. These mathematical models were proposed in the course of investigations of plausible mechanisms capable of generating complex calcium oscillations.

The first one was proposed by Borghans, G. Dupont and A. Goldbeter in [5] and was deeply mathematically analysed by Houart, Dupont and Goldbeter in [13]. This first mathematical model, describing the cytosolic calcium oscillations, relies on the interplay between $CICR^*$ (calcium-induced calcium release) and the Ca^{2+}-stimulated degradation of $InsP_3$.

Alternatively, the second mathematical model was introduced by Marhl, Haberichter, Brumen and Heinrich in [14] and was intensively studied from a mathematical point of view in [12]. This model, refering also to the calcium oscillations, is based on the interplay between three calcium stores in the living cells: endoplasmic reticulum, mitochondria and cytosolic proteins.

2 From DEs systems of order one to geometric dynamics on 1-jet fibre bundles

In order to approach the DEs systems that govern the calcium oscillations in biological living cells from a differential geometric point of view, let us consider the jet fibre bundle of order one $J^1(T, M) \to T \times M$ associated to the Riemannian submanifolds

$$(T, h_{11}(t)) \subset (\mathbb{R}, h_{11}(t))$$

and

$$(M, \varphi_{ij}(x^k)) \subset (\mathbb{R}^n, \varphi_{ij}(x^k)).$$

Remark 2.1. *i) The inverse d-tensors of the preceding Riemannian metrics are denoted by $h^{11}(t)$ and $\varphi^{ij}(x^k)$. We recall that these must verify the formulas $h^{11}(t) = 1/h_{11}(t)$ and $\sum_{m=1}^{n} \varphi^{im}(x^k)\varphi_{mj}(x^k) = \delta^i_j$.*

ii) From a physical point of view, it is well known that the Riemannian metrics are modelling the **gravitational potentials** *of a space of events [28]. By a natural extension to the biological phenomenas, we may assert that the Riemannian metrics may model abstract microscopic* **biologic-gravitational potentials** *intrinsically produced by biological matter.*

We recall that the local coordinates of the 1-jet space $J^1(T, M)$ are (t, x^i, x^i_1), $i = \overline{1, n}$, and transform by the rules

$$\widetilde{t} = \widetilde{t}(t), \quad \widetilde{x}^i = \widetilde{x}^i(x^j), \quad \widetilde{x}^i_1 = \frac{\partial \widetilde{x}^i}{\partial x^j}\frac{dt}{d\widetilde{t}}x^j_1. \tag{2.2.2.1}$$

From a physical point of view, the manifold T is regarded as the *relativistic physical time*, the manifold M is viewed as a *space of theoretical physics events with n-freedom degrees*, and the coordinates x^i_1 have the physical meaning of *velocity* or *direction,* which is intimately related on the well known physical concept of *anisotropy.*

The Riemann-Lagrange geometry of the 1-jet spaces $J^1(\mathbb{R}, M)$, where $(M, \varphi_{ij}(x^k))$ is a general Riemannian manifold, is now completely done in Neagu's paper [21]. In order to develope our applicative geometrical least squares variational calculus method for the DEs systems from Biology, let us recall the main geometrical concepts and properties with physical meaning used in [21].

2.1 Time dependent sprays, harmonic curves, nonlinear connections and h-normal Γ-linear connections on the 1-jet space $J^1(\mathbb{R},\mathbb{R}^n)$

One of the most important geometrical concepts used on 1-jet spaces, which is connected to the Euler-Lagrange equations of the time dependent Lagrangians $\mathcal{L} = L\sqrt{h_{11}(t)}$, $h_{11}(t) > 0$, is the notion of *time dependent spray.*

Definition 2.1. *A pair $\mathcal{S} = (H^{(i)}_{(1)1}, G^{(i)}_{(1)1})$ of local functions on the 1-jet space $J^1(T,M)$, which transform by the rules*

$$(2.2.2.2)\qquad \begin{aligned} 2\widetilde{H}^{(k)}_{(1)1} &= 2H^{(j)}_{(1)1}\left(\frac{dt}{d\widetilde{t}}\right)^2\frac{\partial\widetilde{x}^k}{\partial x^j} - \frac{dt}{d\widetilde{t}}\frac{\partial\widetilde{x}^k_1}{\partial t},\\ 2\widetilde{G}^{(k)}_{(1)1} &= 2G^{(j)}_{(1)1}\left(\frac{dt}{d\widetilde{t}}\right)^2\frac{\partial\widetilde{x}^k}{\partial x^j} - \frac{\partial x^i}{\partial\widetilde{x}^j}\frac{\partial\widetilde{x}^k_1}{\partial x^i}\widetilde{x}^j_1,\end{aligned}$$

is called a **time dependent spray** *on $J^1(T,M)$. The local functions $H^{(j)}_{(1)1}$ (resp. $G^{(j)}_{(1)1}$) are called the* **temporal** *(resp.* **spatial***)* **components** *of the time dependent spray $\mathcal{S}$ of the 1-jet space $J^1(T,M)$.*

Remark 2.2. *The introducing of the transformation rules (2.2.2.2) was suggested by the local transformations laws of the second derivatives $\ddot{x}^i(t) \overset{not}{=} x^i_{11}$ of the components $x^i(t)$ of an arbitrary smooth curve $c: T \to M$, $c(t) = (x^1(t), x^2(t), ..., x^n(t))$.*

Example 2.1. *Let us consider the Christoffel symbols $H^1_{11}(t)$ and $\gamma^i_{jk}(x^l)$ of the Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x^k)$, where*

$$H^1_{11}(t) = \frac{h^{11}}{2}\frac{dh_{11}}{dt},\quad \gamma^i_{jk}(x^l) = \frac{\varphi^{im}}{2}\left(\frac{\partial\varphi_{mj}}{\partial x^k} + \frac{\partial\varphi_{mk}}{\partial x^j} - \frac{\partial\varphi_{jk}}{\partial x^m}\right).$$

Taking into account the transformation laws of the preceding Christoffel symbols, together with the form of the group of transformations (2.2.2.1), by local computations, we must conclude that the components

$$\mathring{H}^{(i)}_{(1)1} = -\frac{1}{2}H^1_{11}x^i_1,\quad \mathring{G}^{(i)}_{(1)1} = \frac{1}{2}\gamma^i_{jm}x^m_1 x^j_1,$$

represent a time dependent spray $\mathcal{S}_0$ on the 1-jet space $J^1(T,M)$. This is called **the canonical time dependent spray on $J^1(T,M)$ produced by the Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x^k)$.**

Definition 2.2. *Let $\mathcal{S} = (H^{(i)}_{(1)1}, G^{(i)}_{(1)1})$ be a time dependent spray on the 1-jet space $J^1(T,M)$. A smooth curve $c \in C^\infty(T,M)$, locally expressed by $c(t) = (x^i(t))$, whose components verify the second order DEs system*

$$(2.2.2.3)\qquad h^{11}(t)\left\{x^i_{11} + 2G^{(i)}_{(1)1} + 2H^{(i)}_{(1)1}\right\} = 0,\quad \forall i = \overline{1,n},$$

is called a **harmonic curve on $J^1(T,M)$ of the time-dependent spray $\mathcal{S}$, with respect to the Riemannian metric $h_{11}(t)$.**

Remark 2.3. *i) The harmonic curves on $J^1(T,M)$ provided by the particular time dependent spray $\mathcal{S}_0 = (\mathring{H}^{(i)}_{(1)1}, \mathring{G}^{(i)}_{(1)1})$ are exactly the classical* **harmonic maps** *between the Riemannian manifolds $(T, h_{11}(t))$ and $(M, \varphi_{ij}(x^k))$. We recall that these harmonic maps are minimizing the energy action functional*

$$\mathbb{E} : C^\infty(T,M) \to \mathbb{R}_+, \quad \mathbb{E}(c(t)) = \int_T h^{11}(t)\varphi_{ij}(x^k(t))\dot{x}^i(t)\dot{x}^j(t)\sqrt{h_{11}(t)}dt,$$

where $\dot{x}^i(t) \overset{not}{=} x^i_1(t) = dx^i/dt$. In other words, the Euler-Lagrange equations of the energy action functional $\mathbb{E}$ are exactly

$$h^{11}(t)\left\{x^i_{11} - H^1_{11}x^i_1 + \gamma^i_{jm}x^m_1 x^j_1\right\} = 0, \quad \forall i = \overline{1,n}. \tag{2.2.2.4}$$

A deep study of classical harmonic maps between general Riemannian manifolds may be found in the work of Eells and Lemaire [8].

ii) More particularly, when the temporal manifold T is the usual physical time represented by the Riemannian manifold $(\mathbb{R}, h_{11}(t) = 1)$, the harmonic curves become exactly the classical **geodesics** *of the Riemannian manifold $(M, \varphi_{ij}(x^k))$. In other words, these verify the DEs system of order two*

$$h^{11}(t)\left\{x^i_{11} + \gamma^i_{jm}x^m_1 x^j_1\right\} = 0, \quad \forall i = \overline{1,n}. \tag{2.2.2.5}$$

iii) Arithmetically speaking, it is obvious that the second order DEs system (2.2.2.3), that define the harmonic curves of the time dependent spray, are equivalent with the DEs system

$$x^i_{11} + 2G^{(i)}_{(1)1} + 2H^{(i)}_{(1)1} = 0, \quad \forall i = \overline{1,n}. \tag{2.2.2.6}$$

From a geometrical point of view connected to the Einstein's principles for General Relativity, we point out that the equations (2.2.2.3), that is the equations (2.2.2.6) multiplied with the term $h^{11}(t)$, have a global geometrical character on the 1-jet space $J^1(T,M)$ (i. e., the equations have the same form in all physical systems of references), while the equations (2.2.2.6) have not a such global geometrical-physics character. For more details upon the relativistic aspects of the differential geometry, please see the paper of Sóos [28].

iv) We point out also that the harmonic curves on 1-jet spaces naturally generalize the classical **paths of sprays** *used in the Lagrangian geometry of Miron and Anastasiei [16].*

In conclusion, we consider that the statements i), ii), iii) and iv) emphasize the naturalness of Definition 2.2.

In Riemann-Lagrange geometrical studies of 1-jet spaces a very important role is played by the *nonlinear connections* $\Gamma = (M^{(i)}_{(1)1}, N^{(i)}_{(1)j})$ that allow the construction of some *convenient derivative operators* or, alternatively, of some *adapted basis* of vector fields $\{\delta/\delta t, \delta/\delta x^i, \partial/\partial x^i_1\}$. These adapted covariant derivative operators are characterized by simple tensorial laws of transformation and are defined by

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - M^{(j)}_{(1)1}\frac{\partial}{\partial x^j_1} \quad \text{and} \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{(j)}_{(1)i}\frac{\partial}{\partial x^j_1}. \tag{2.2.2.7}$$

Because of preceding reason, let us introduce the definition of a nonlinear connection and let us describe few interesting geometrical properties of them.

Definition 2.3. *A pair* $\Gamma = (M^{(i)}_{(1)1}, N^{(i)}_{(1)j})$ *of local functions on the 1-jet space* $J^1(T, M)$*, which transform by the rules*

$$\begin{aligned} \widetilde{M}^{(j)}_{(1)1}\frac{d\tilde{t}}{dt} &= M^{(k)}_{(1)1}\frac{dt}{d\tilde{t}}\frac{\partial \tilde{x}^j}{\partial x^k} - \frac{\partial \tilde{x}^j_1}{\partial t}, \\ \widetilde{N}^{(j)}_{(1)k}\frac{\partial \tilde{x}^k}{\partial x^i} &= N^{(k)}_{(1)i}\frac{dt}{d\tilde{t}}\frac{\partial \tilde{x}^j}{\partial x^k} - \frac{\partial \tilde{x}^j_1}{\partial x^i}, \end{aligned} \tag{2.2.2.8}$$

is called a **nonlinear connection** *on* $J^1(T, M)$*. The local functions* $M^{(i)}_{(1)1}$ *(resp.* $N^{(i)}_{(1)j}$*) are called the* **temporal** *(resp.* **spatial***)* **components** *of the nonlinear connection* Γ *of the 1-jet space* $J^1(T, M)$*.*

Example 2.2. *Studying the local transormation rules of the local components*

$$\mathring{M}^{(i)}_{(1)1} = -H^1_{11}x^i_1 \quad \textit{and} \quad \mathring{N}^{(i)}_{(1)j} = \gamma^i_{jm}x^m_1,$$

we conclude that the pair $\Gamma_0 = (\mathring{M}^{(i)}_{(1)1}, \mathring{N}^{(i)}_{(1)j})$ *is a nonlinear connection on* $J^1(T, M)$*. This is called the* **canonical nonlinear connection produced by the Riemannian metrics** $h_{11}(t)$ **and** $\varphi_{ij}(x^k)$*.*

Taking into account the local transformations laws (2.2.2.2) and (2.2.2.8), it is easy to prove the following important result that connects the geometrical concepts of time dependent spray and nonlinear connection on 1-jet spaces.

Proposition 2.1. *i) If the pair* $\mathcal{S} = (H^{(i)}_{(1)1}, G^{(i)}_{(1)1})$ *is a time dependent spray on* $J^1(T, M)$*, then the pair* $\Gamma_{\mathcal{S}} = (M^{(i)}_{(1)1}, N^{(i)}_{(1)j})$*, where*

$$M^{(i)}_{(1)1} = 2H^{(i)}_{(1)1} \quad \textit{and} \quad N^{(i)}_{(1)j} = \frac{\partial G^{(i)}_{(1)1}}{\partial x^j_1},$$

is a nonlinear connection on the 1-jet space $J^1(T, M)$*.*

ii) Conversely, if the pair $\Gamma = (M^{(i)}_{(1)1}, N^{(i)}_{(1)j})$ *is a nonlinear connection on* $J^1(T, M)$*, then the pair* $\mathcal{S}_\Gamma = (H^{(i)}_{(1)1}, G^{(i)}_{(1)1})$*, where*

$$H^{(i)}_{(1)1} = \frac{1}{2}M^{(i)}_{(1)1} \quad \textit{and} \quad G^{(i)}_{(1)1} = \frac{1}{2}N^{(i)}_{(1)m}x^m_1,$$

is a time dependent spray on the 1-jet space $J^1(T, M)$*.*

It is well known from the classical Riemannian geometry the importance of covariant derivatives (linear connections $\bigtriangledown$) in the geometrical studies of a space of physical events. The importance of linear connections is coming from the possibility of construction of the torsion and curvature tensors which locally characterize the form of the space. For that reason, let us study some convenient special covariant derivatives on the 1-jet space $J^1(T,M)$, whose local behavior is described at level of adapted components (i. e., components derived using the adaptad basis of vector fields $\{\delta/\delta t, \delta/\delta x^i, \partial/\partial x_1^i\}$ of a fixed nonlinear connection Γ. We recall that these special linear connections $\bigtriangledown\Gamma$ produce some local covariant derivatives "$_{/1}$" (the T-horizontal covariant derivative), "$_{|i}$" (the M-horizontal covariant derivatives) and "$|_{(i)}^{(1)}$" (the vertical derivatives), which are convenient to use in order to differentiate the geometrical objects of the 1-jet space $J^1(T,M)$. For a deep study of the local covariant derivatives produced by special linear connections $\bigtriangledown\Gamma$ on 1-jet spaces, the reader is invited to consult the papers [2, 21, 18]. In the preceeding direction, let us recall the following geometrical definition.

Definition 2.4. *A set of local functions* $\bigtriangledown\Gamma = (H_{11}^1, G_{i1}^k, L_{ij}^k, C_{i(j)}^{k(1)})$ *on* $J^1(T,M)$, *whose adapted components transform by the rules [21]*

$$G_{i1}^k = \widetilde{G}_{j1}^m \frac{\partial x^k}{\partial \widetilde{x}^m}\frac{\partial \widetilde{x}^j}{\partial x^i}\frac{d\widetilde{t}}{dt},$$

$$L_{ij}^m \frac{\partial \widetilde{x}^r}{\partial x^m} = \widetilde{L}_{pq}^r \frac{\partial \widetilde{x}^p}{\partial x^i}\frac{\partial \widetilde{x}^q}{\partial x^j} + \frac{\partial^2 \widetilde{x}^r}{\partial x^i \partial x^j},$$

$$C_{i(j)}^{k(1)} = \widetilde{C}_{p(r)}^{s(1)} \frac{\partial x^k}{\partial \widetilde{x}^s}\frac{\partial \widetilde{x}^p}{\partial x^i}\frac{\partial \widetilde{x}^r}{\partial x^j}\frac{dt}{d\widetilde{t}},$$

is called an h**-normal Γ-linear connection** *on the 1-jet space* $J^1(T,M)$.

Example 2.3. *Using the group of local transformations (2.2.2.1), by local computations, we deduce that the set of local functions*

$$B\Gamma_0 = (H_{11}^1, 0, \gamma_{ij}^k, 0),$$

defines an h*-normal* Γ_0*-linear connection on* $J^1(T,M)$*. This is called the* **generalized Berwald connection produced by the pair of Riemannian metrics** $h_{11}(t)$ **and** $\varphi_{ij}(x^k)$.

In Riemann-Lagrange geometry of the 1-jet space $J^1(T,M)$, the local covariant derivatives of the generalized Berwald connection $B\Gamma_0$ produced by the pair of metrics (h,φ) are ususally denoted by "$_{//1}$", "$_{||i}$" and "$||_{(i)}^{(1)}$". In order to reach the aim of our paper, let us consider now $X = \left(X_{(1)}^{(i)}(t,x^k)\right)$ as an arbitrary d-tensor field on the 1-jet space $J^1(T,M)$, whose local components are independent on the directions x_1^j. Taking into account the independence of directions of the d-tensor components $X_{(1)}^{(i)}(t,x^k)$ and some general formulas from [21], by direct local computations, we easily find the following important geometrical result.

Proposition 2.2. *The T-horizontal and M-horizontal local covariant derivatives "$_{/1}$" and "$_{||j}$" of the generalized Berwald connection $B\Gamma_0$ produce the following covariant derivatives of the d-tensor field $X_{(1)}^{(i)}(t, x^k)$:*

$$X_{(1)/1}^{(i)} = \frac{\partial X_{(1)}^{(i)}}{\partial t} - X_{(1)}^{(i)} H_{11}^1, \quad X_{(1)||j}^{(i)} = \frac{\partial X_{(1)}^{(i)}}{\partial x^j} + X_{(1)}^{(m)} \gamma_{mj}^i. \tag{2.2.2.9}$$

2.2 Solutions of DEs systems of order one as harmonic curves on 1-jet spaces. Canonical nonlinear connections

In the sequel, let us show that the solutions of class C^2 of an arbitrary DEs system of order one may be naturally regarded as harmonic curves on 1-jet spaces. So, let us consider an unknown curve $c = (x^i(t))$ and an arbitrary given d-tensor field $X = \left(X_{(1)}^{(i)}(t, x^k)\right)$, that define on the 1-jet space $J^1(T, M)$ the following DEs system of order one:

$$\begin{aligned} x_1^i &= X_{(1)}^{(i)}(t, x^k(t)), \qquad \forall i = \overline{1, n} \Leftrightarrow \\ x_1^i &- X_{(1)}^{(i)}(t, x^k(t)) = 0, \quad \forall i = \overline{1, n}, \end{aligned} \tag{2.2.2.10}$$

where $x_1^i = \dot{x}^i(t) = dx^i/dt$.

Recall that we are working with two *a priori* Riemannian manifolds $(T, h_{11}(t)) \subset (\mathbb{R}, h_{11}(t))$ and $(M, \varphi_{ij}(x^k)) \subset (\mathbb{R}^n, \varphi_{ij}(x^k))$. Moreover, suppose that the relativistic time-manifold T is a connected, compact and orientable manifold (i. e., like a closed interval of real numbers $[a, b]$). Under these geometrical assumptions, we can prove the following interesting qualitative-energetic result.

Theorem 2.1. *All solutions of class C^2 of the DEs system (2.2.2.10) are harmonic curves on the 1-jet space $J^1(T, M)$ of the time dependent spray $\mathcal{S}_{DEs} = (H_{(1)1}^{(i)}, G_{(1)1}^{(i)})$, whose components are given by the formulas*

$$H_{(1)1}^{(i)} = -\frac{1}{2} H_{11}^1 x_1^i, \quad G_{(1)1}^{(i)} = \frac{1}{2} \gamma_{jk}^i x_1^j x_1^k + h_{11} F^i(t, x^k, x_1^r),$$

where

$$F^i(t, x^k, x_1^r) = \frac{h^{11}}{2} \left\{ \varphi^{il} X_{(1)||l}^{(s)} \varphi_{sr} \left[X_{(1)}^{(r)} - x_1^r \right] + X_{(1)||m}^{(i)} x_1^m + X_{(1)/1}^{(i)} \right\}.$$

In other words, the C^2 solutions of the initial first order DEs system (2.2.2.10) are verifying the second order DEs system (2.2.2.3) of harmonic curves for the time dependent spray $\mathcal{S}_{DEs}$.

Proof. Via the least squares variational calculus geometric method for general PDEs systems of order one discussed at the beginning of this paper, note that the initial first order DEs system (2.2.2.10) is an equivalent one with the first order DEs system

$$\sum_{i,j=\overline{1,n}} \left\{ h^{11}(t) \varphi_{ij}(x^k) \left(x_1^i - X_{(1)}^{(i)}(t, x^k(t)) \right) \left(x_1^j - X_{(1)}^{(j)}(t, x^k(t)) \right) \right\} = 0. \tag{2.2.2.11}$$

This is because, we underline this fact, the Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x^k)$ are positive definite ones and produce a d-metric on $J^1(T, M)$. It is obvious now that the local computations in the preceding DEs system lead to the following new form of the equations (2.2.2.11):

$$h^{11}(t) \sum_{i,j=1}^{n} \varphi_{ij}(x^k) x_1^i x_1^j + \sum_{1}^{n} U_{(i)}^{(1)}(t, x^k) x_1^i + \Phi(t, x^k) = 0, \tag{2.2.2.12}$$

where $U_{(1)}^{(i)}(t, x^k) = -2h^{11} \sum_{m=1}^{n} \varphi_{im} X_{(1)}^{(m)}$ and $\Phi(t, x^k) = h^{11} \sum_{r,s=1}^{n} \varphi_{rs} X_{(1)}^{(r)} X_{(1)}^{(s)}$.

Using the Einstein convention for summations, let us consider now the least squares time dependent Lagrangian of electrodynamics kind, which is given by

$$\begin{aligned} \mathcal{DE}s\mathcal{ED} &= \|\mathbf{C} - \mathbf{X}\|^2 \sqrt{h_{11}(t)} = \\ &= \left\{ h^{11}(t) \varphi_{ij}(x^k) \left[x_1^i - X_{(1)}^{(i)} \right] \left[x_1^j - X_{(1)}^{(j)} \right] \right\} \sqrt{h_{11}(t)} = \\ &= \left\{ h^{11}(t) \varphi_{ij}(x^k) x_1^i x_1^j + U_{(i)}^{(1)}(t, x^k) x_1^i + \Phi(t, x^k) \right\} \sqrt{h_{11}(t)}, \end{aligned}$$

where $\mathbf{C} = x_1^i (\partial / \partial x_1^i)$, $\mathbf{X} = X_{(1)}^{(i)} (\partial / \partial x_1^i)$. Let us consider also the least squares energy action functional $\mathbb{E}_{\mathcal{DE}s\mathcal{ED}} : C^2(T, M) \to \mathbb{R}_+$, given by

$$\mathbb{E}_{\mathcal{DE}s\mathcal{ED}}(c) = \int_T \mathcal{DE}s\mathcal{ED} dt = \int_T \|\mathbf{C} - \mathbf{X}\|^2 \sqrt{h_{11}(t)} dt \geq 0.$$

It is obvious now that a smooth curve $c \in C^2(T, M)$, locally expressed by $c(t) = (x^1(t), x^2(t), ..., x^n(t))$, is a solution of the DEs system (2.2.2.12) if and only if the curve c vanishes the time dependent Lagrangian $\mathcal{DE}s\mathcal{ED}$. In other words, c is a solution of the DEs system (2.2.2.12) if and only if the curve c is a global minimum point for the least squares energy action functional $\mathbb{E}_{\mathcal{DE}s\mathcal{ED}}$. Therefore, every curve $c = (x^i(t))$ of class C^2 is a solution of the initial DEs system (2.2.2.10) if and only if it verifies the Euler-Lagrange equations

$$\frac{\partial [\mathcal{DE}s\mathcal{ED}]}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial [\mathcal{DE}s\mathcal{ED}]}{\partial x_1^i} \right) = 0, \quad \forall i = \overline{1, n}. \tag{2.2.2.13}$$

Taking into account the expression of the time dependent least squares Lagrangian $\mathcal{DE}s\mathcal{ED}$ and some local differential computations in the Euler-Lagrange equations (2.2.2.13), we claim that the equations (2.2.2.13) can be rewritten in the form (2.2.2.3) of the second order DEs system of the harmonic curves of a time dependent spray

$$\mathcal{S}_{DEs} = (H_{(1)1}^{(i)}, G_{(1)1}^{(i)}), \tag{2.2.2.14}$$

whose temporal components are given by

$$H_{(1)1}^{(i)} = -\frac{1}{2} H_{11}^1 x_1^i.$$

Moreover, the spatial components of the time-dependent $\mathcal{S}_{DEs}$ are expressed by

$$G^{(i)}_{(1)1} = \frac{1}{2}\gamma^i_{jk}x^j_1x^k_1 + \frac{h_{11}\varphi^{il}}{4}\left[U^{(1)}_{(l)m}x^m_1 + \frac{\partial U^{(1)}_{(l)}}{\partial t} + U^{(1)}_{(l)}H^1_{11} - \frac{\partial \Phi}{\partial x^l}\right],$$

where

$$U^{(1)}_{(i)j} = \frac{\partial U^{(1)}_{(i)}}{\partial x^j} - \frac{\partial U^{(1)}_{(j)}}{\partial x^i}.$$

In what follows, using the expressions which give $U^{(1)}_{(i)}$ and Φ, together with direct local computations, we find

$$U^{(1)}_{(i)j} = -2h^{11}\left[\varphi_{im}X^{(m)}_{(1)||j} - \varphi_{jm}X^{(m)}_{(1)||i}\right],$$

$$\frac{\partial U^{(1)}_{(i)}}{\partial t} + U^{(1)}_{(i)}H^1_{11} = -2h^{11}\varphi_{im}X^{(m)}_{(1)/ /1},$$

$$\frac{\partial \Phi}{\partial x^l} = 2h^{11}\varphi_{mr}X^{(m)}_{(1)}X^{(r)}_{(1)||l}.$$

In conclusion, all our preceding constructions imply what we were looking for. For a more clear geometrical understanding of the proof of this Theorem, the authors of this paper invite the reader to study the proofs of the Theorems 2.3.1 and 2.3.2 from the paper [21], pp 156-157. □

Definition 2.5. *The time dependent spray $\mathcal{S}_{DEs} = (H^{(i)}_{(1)1}, G^{(i)}_{(1)1})$ given by Theorem 2.1 is be called the* **canonical time-depedent spray produced by the DEs system (2.2.2.10) and the pair of Riemannian metrics** $(h_{11}(t), \varphi_{ij}(x^k))$.

Remark 2.4. *Note that the Theorem 2.1 holds also good if we use a semi-Riemannian metric $\varphi_{ij}(x^k)$ (not necessarily a Riemannian one) on the spatial manifold of physical events M. This is because the Euler-Lagrange equations (2.2.2.13) of the electrodynamic Lagrangian $\mathcal{DE}s\mathcal{ED}$ have the same geometrical form and properties in the semi-Riemannian case, too.*

Taking into account the geometrical connection between the time dependent sprays and the nonlinear connections on 1-jet spaces, given by the Proposition 2.1, we easily deduce the following important geometrical corollary and definition.

Corollary 2.1.1. *The* **canonical nonlinear connection** $\Gamma_{\mathcal{S}_{DEs}} = (M^{(i)}_{(1)1}, N^{(i)}_{(1)j})$ **produced by the DEs system (2.2.2.10) and the pair of Riemannian metrics** $(h_{11}(t), \varphi_{ij}(x^k))$ *on the 1-jet space $J^1(T, M)$ has the components*

$$M^{(i)}_{(1)1} = -H^1_{11}x^i_1 \quad \textit{and} \quad N^{(i)}_{(1)j} = \gamma^i_{jk}x^k_1 - \mathcal{H}^{(i)}_{(1)j}, \tag{2.2.2.15}$$

where

$$\mathcal{H}^{(i)}_{(1)j} = \frac{1}{2}\left[X^{(i)}_{(1)||j} - \varphi^{ir}X^{(s)}_{(1)||r}\varphi_{sj}\right].$$

2.3 Time dependent geometric dynamics of DEs systems of order one. Generalized Maxwell equations

The geometrization of the time dependent Lagrangians on 1-jet spaces of form $J^1(\mathbb{R}, M)$, in the sense of natural construction of canonical nonlinear connections, generalized Cartan connections, torsion and curvature d-tensors from these Lagrangians, is now completely developed in the paper [21]. Some generalized gravitational and electromagnetic field theories produced by such time dependent Lagrangians, which are characterized by natural and interesting generalized Einstein and Maxwell equations, are described there too.

In this section, we will particularize the main geometrical and physical results from [21], [17], and [20] for our particular time dependent electrodynamics Lagrangian $\mathcal{DE}s\mathcal{ED}$ on the 1-jet space $J^1(T, M)$, where $(T, h_{11}(t)) \subset (\mathbb{R}, h_{11}(t))$ and $(M, \varphi_{ij}(x^k)) \subset (\mathbb{R}^n, \varphi_{ij}(x^k))$. We recall that the least squares Lagrangian $\mathcal{DE}s\mathcal{ED}$ from Theorem 2.1 is derived from the DEs system (2.2.2.10) and the pair of Riemannian metrics $(h_{11}(t), \varphi_{ij}(x^k))$.

Definition 2.6. *The pair $(J^1(T, M), \mathcal{DE}s\mathcal{ED})$, endowed with the nonlinear connection $\Gamma_{\mathcal{S}_{DEs}}$ given by (2.2.2.15), is denoted by $\mathcal{DE}s\mathcal{EDL}^n_1$ and is called the* **canonical relativistic rheonomic Lagrange space produced by the DEs system (2.2.2.10) and the pair of metrics** $(h_{11}(t), \varphi_{ij}(x^k))$.

In this geometrical context, using general formulas from [20], [21] and [22], together with local computations, we find the following important differential geometric results:

Theorem 2.2. *i) The canonical generalized Cartan connection $C\Gamma_{\mathcal{S}_{DEs}}$ of the relativistic rheonomic Lagrange space $\mathcal{DE}s\mathcal{EDL}^n_1$ has the adapted components*

$$C\Gamma_{\mathcal{S}_{DEs}} = (H^1_{11}, 0, \gamma^i_{jk}, 0).$$

where γ^i_{jk} are the Christoffel symbols of the semi-Riemannian metric φ_{ij}.

ii) The torsion d-tensor $\mathbf{T}$ of the canonical generalized Cartan connection $C\Gamma_{\mathcal{S}_{DEs}}$ of the space $\mathcal{DE}s\mathcal{EDL}^n_1$ is determined by two adapted local d-tensors:

$$R^{(i)}_{(1)1j} = \frac{1}{2}\left[X^{(i)}_{(1)||j//1} - \varphi^{ir}X^{(s)}_{(1)||r//1}\varphi_{sj}\right],$$

$$R^{(i)}_{(1)jk} = r^i_{jkm}x^m_1 - \frac{1}{2}\left[X^{(i)}_{(1)||j||k} - \varphi^{ir}X^{(s)}_{(1)||r||k}\varphi_{sj}\right],$$

where r^l_{ijk} are the local curvature tensors of semi-Riemannian metric φ_{ij} and the second covariant derivatives of the d-tensor $X^{(i)}_{(1)}(t, x^k)$ are given by

$$X^{(i)}_{(1)||j//1} = \frac{\partial X^{(i)}_{(1)||j}}{\partial t} - X^{(i)}_{(1)||j}H^1_{11},$$

$$X^{(i)}_{(1)||j||k} = \frac{\partial X^{(i)}_{(1)||j}}{\partial x^k} - X^{(m)}_{(1)||j}\gamma^i_{mk} - X^{(i)}_{(1)||m}\gamma^m_{jk}.$$

iii) The curvature d-tensor $\mathbf{R}$ *of the canonical generalized Cartan connection* $C\Gamma_{\mathcal{S}_{DEs}}$ *of the space* $\mathcal{DE}s\mathcal{EDL}_1^n$ *is determined only by the adapted components* $R_{ijk}^l = r_{ijk}^l$*, that is exactly the components of the curvature tensor of the semi-Riemannian metric* φ_{ij}*.*

Remark 2.5. *Both generalized Cartan* $C\Gamma_{\mathcal{S}_{DEs}}$ *and Berwald* $B\Gamma_0$ *linear connections on the 1-jet space* $J^1(T, M)$ *have the same adapted components. However, they are two distinct linear connections. This is because the generalized Cartan connection is a* $\Gamma_{\mathcal{S}_{DEs}}$*-linear connection while the generalized Berwald connection is a* Γ_0*-linear one. For more details, please see the works [18] and [19].*

In the sequel, following the geometrical and abstract physical ideas from the papers [22] and [21], by direct local computations, we can construct a canonical 2-form of the space which is similar to the electromagnetic field of the space $\mathcal{DE}s\mathcal{EDL}_1^n$ and describe its generalized Maxwell equations.

Theorem 2.3. *i) The time dependent electromagnetic field* F *of the relativistic rheonomic Lagrange space* $\mathcal{DE}s\mathcal{EDL}_1^n$ *is expressed by the distinguished 2-form (gyroscopic field)*

$$F = F_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where $\delta x_1^i = dx_1^i + M_{(1)1}^{(i)} dt + N_{(1)j}^{(i)} dx^j$ *and*

$$F_{(i)j}^{(1)} = \frac{h^{11}}{2} \left[\varphi_{im} X_{(1)||j}^{(m)} - \varphi_{jm} X_{(1)||i}^{(m)} \right].$$

ii) The adapted components $F_{(i)j}^{(1)}$ *of the gyroscopic field are governed by the following generalized Maxwell equations*

$$\begin{cases} F_{(i)j//1}^{(1)} = \dfrac{1}{4} \mathcal{A}_{\{i,j\}} \left\{ h^{11} \varphi_{im} \left[X_{(1)||j//1}^{(m)} - \varphi^{mr} X_{(1)||r//1}^{(s)} \varphi_{sj} \right] \right\} \\ \sum_{\{i,j,k\}} F_{(i)j||k}^{(1)} = 0 \\ \sum_{\{i,j,k\}} F_{(i)j}^{(1)} \Big|\Big|_{(k)}^{(1)} = 0, \end{cases}$$

where $\mathcal{A}_{\{i,j\}}$ *represents an alternate sum and* $\sum_{\{i,j,k\}}$ *means a cyclic sum.*

Remark 2.6. *We did not describe the time dependent gravitational theory of the relativistic rheonomic Lagrange space* $\mathcal{DE}s\mathcal{EDL}_1^n$ *because its time dependent gravitational field* G *and its attached generalized Einstein equations are independent on the tensorial components* $X_{(1)}^{(i)}(t, x^k)$ *that define DEs system (2.2.2.10). In fact, the geometric time dependent gravitational entities are depending only on the pair of metrics* $(h_{11}(t), \varphi_{ij}(x^k))$*.*

3 Geometric dynamics produced on 1-jet spaces by DEs systems of order one and pair of Euclidian metrics $\Delta = (1, \delta_{ij})$

In order to use the preceding geometrical results for the study of some DEs systems of order one coming from Theoretical Biology, let us consider the particular Euclidian metrics as the pair of Riemannian metrics:

$$\Delta = (h_{11}(t) = 1, \varphi_{ij}(x^k) = \delta_{ij}),$$

where

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j, \end{cases}$$

is the classical Kronecker symbol.

In this particular situation, we are placed on the 1-jet space $J^1(T, M)$ produced by the Riemannian submanifolds $(T, h_{11}(t) = 1) \subset (\mathbb{R}, 1)$ and $(M, \varphi_{ij}(x^k) = \delta_{ij}) \subset (\mathbb{R}^n, \delta_{ij})$. Consequently, the Theorem 2.1 asserts now that all solutions of C^2 class of the DEs system (2.2.2.10) may be regarded as the harmonic curves of the time-dependent spray $\mathcal{S}_{DEs}(\Delta) = (H^{(i)}_{(1)1}, G^{(i)}_{(1)1})$, whose components are given by the formulas

$$H^{(i)}_{(1)1} = 0, \quad G^{(i)}_{(1)1} = \frac{1}{2}\left\{X^{(r)}_{(1)||i}\left[X^{(r)}_{(1)} - x^r_1\right] + X^{(i)}_{(1)||m}x^m_1 + X^{(i)}_{(1)//1}\right\},$$

where

$$X^{(i)}_{(1)//1} = \frac{\partial X^{(i)}_{(1)}}{\partial t} \quad \text{and} \quad X^{(i)}_{(1)||j} = \frac{\partial X^{(i)}_{(1)}}{\partial x^j}.$$

In other words, following the Proof of Theorem 2.1, we conclude that the solutions of the DEs system (2.2.2.10) are minimizing the least squares time dependent Lagrangian

$$\mathcal{LSQ}(t, x^k, x^k_1) = \sum_{i=1}^{n}\left(x^i_1 - X^{(i)}_{(1)}(t, x^k)\right)^2,$$

which is obviously the time dependent electrodynamics Lagrangian $\mathcal{DE}s\mathcal{ED}$ for the particular pair of Euclidian metrics $\Delta = (1, \delta_{ij})$.

In what follows, let us denote the rheonomic Lagrange space produced by the DEs system (2.2.2.10) and the pair of Euclidian metrics $\Delta = (1, \delta_{ij})$ by

$$\mathcal{DE}s\mathcal{ED}L^n_1(\Delta) = (J^1(T, M), \mathcal{LSQ}).$$

By simple computations, the geometrical results from preceding sections may be expressed on the particular space $\mathcal{DE}s\mathcal{ED}L^n_1(\Delta)$.

Theorem 3.1. *i) The canonical nonlinear connection* $\Gamma_{\mathcal{S}_{DEs}(\Delta)} = (M^{(i)}_{(1)1}, N^{(i)}_{(1)j})$ *of the space* $\mathcal{DE}s\mathcal{ED}L^n_1(\Delta)$ *is given by the components*

$$M^{(i)}_{(1)1} = 0 \quad and \quad N^{(i)}_{(1)j} = \frac{1}{2}\left[\frac{\partial X^{(i)}_{(1)}}{\partial x^j} - \frac{\partial X^{(j)}_{(1)}}{\partial x^i}\right].$$

ii) The canonical generalized Cartan connection $C\Gamma_{\mathcal{S}_{DEs}(\Delta)}$ of the relativistic rheonomic Lagrange space $\mathcal{DE}s\mathcal{EDL}^n_1(\Delta)$ has all its adapted components as null ones.

iii) The torsion d-tensor $\mathbf{T}$ of the canonical generalized Cartan connection $C\Gamma_{\mathcal{S}_{DEs}(\Delta)}$ of the space $\mathcal{DE}s\mathcal{EDL}^n_1(\Delta)$ is determined by two adapted local d-tensors:

$$R^{(i)}_{(1)1j} = \frac{1}{2}\left[\frac{\partial^2 X^{(i)}_{(1)}}{\partial x^j \partial t} - \frac{\partial^2 X^{(j)}_{(1)}}{\partial x^i \partial t}\right] \quad and \quad R^{(i)}_{(1)jk} = -\frac{1}{2}\left[\frac{\partial^2 X^{(i)}_{(1)}}{\partial x^j \partial x^k} - \frac{\partial^2 X^{(j)}_{(1)}}{\partial x^i \partial x^k}\right].$$

iv) All adapted components of the curvature d-tensor $\mathbf{R}$ of the canonical generalized Cartan connection $C\Gamma_{\mathcal{S}_{DEs}(\Delta)}$ of the space $\mathcal{DE}s\mathcal{EDL}^n_1(\Delta)$ vanish.

v) The 2-form F of the space $DEsEDL^n_1(\Delta)$ is expressed by

$$F = F^{(1)}_{(i)j}\delta x^i_1 \wedge dx^j,$$

where

$$\delta x^i_1 = dx^i_1 + \frac{1}{2}\left[\frac{\partial X^{(i)}_{(1)}}{\partial x^j} - \frac{\partial X^{(j)}_{(1)}}{\partial x^i}\right] dx^j \quad and \quad F^{(1)}_{(i)j} = \frac{1}{2}\left[\frac{\partial X^{(i)}_{(1)}}{\partial x^j} - \frac{\partial X^{(j)}_{(1)}}{\partial x^i}\right].$$

vi) The components $F^{(1)}_{(i)j}$ of the relativistic rheonomic Lagrange space $\mathcal{DE}s\mathcal{EDL}^n_1(\Delta)$ are governed by the following generalized Maxwell equations:

$$\begin{cases} F^{(1)}_{(i)j//1} = \dfrac{1}{4}\mathcal{A}_{\{i,j\}}\left\{\left[\dfrac{\partial^2 X^{(i)}_{(1)}}{\partial x^j \partial t} - \dfrac{\partial^2 X^{(j)}_{(1)}}{\partial x^i \partial t}\right]\right\} = \dfrac{1}{2}\left[\dfrac{\partial^2 X^{(i)}_{(1)}}{\partial x^j \partial t} - \dfrac{\partial^2 X^{(j)}_{(1)}}{\partial x^i \partial t}\right] \\ \sum_{\{i,j,k\}} F^{(1)}_{(i)j||k} = 0, \end{cases}$$

where $\mathcal{A}_{\{i,j\}}$ represents an alternate sum, $\sum_{\{i,j,k\}}$ means a cyclic sum and

$$F^{(1)}_{(i)j//1} = \frac{\partial F^{(1)}_{(i)j}}{\partial t}, \quad F^{(1)}_{(i)j||k} = \frac{\partial F^{(1)}_{(i)j}}{\partial x^k}.$$

In many applicative problems from Theoretical Biology we meet d-tensors $X^{(i)}_{(1)}$ which are not depending on the time coordinate $t \in T \subset \mathbb{R}$. In other words, in a lot of biological problems we are working with DEs systems of order one given by d-tensors X on $J^1(T, M)$, having the components of the particular form $X^{(i)}_{(1)} = X^{(i)}_{(1)}(x^k)$. In these time independent situations, it is obvious that many geometrical objects studied by us disappear, in the sense that they are vanishing. In fact, for time independent d-tensors $X^{(i)}_{(1)} = X^{(i)}_{(1)}(x^k)$ the Riemann-Lagrange geometrical structure produced by the DEs system (2.2.2.10) and the pair of Riemannian metrics Δ is characterized by the following Corollary, in which the geometrical space of biological events is denoted $\mathcal{ADE}s\mathcal{EDL}_1^n(\Delta)$. We use this notation because we have in study that so called an *autonomous Lagrange space* or, in other words, a time-independent Lagrange space (i. e., a non-rheonomic Lagrange space).

Corollary 3.1.1. *i) The canonical nonlinear connection* $\Gamma_{\mathcal{S}_{DEs}(\Delta)} = (M^{(i)}_{(1)1}, N^{(i)}_{(1)j})$ *of the autonomous Lagrange space* $\mathcal{ADE}s\mathcal{EDL}_1^n(\Delta)$ *is given by the components*

$$M^{(i)}_{(1)1} = 0 \quad and \quad N^{(i)}_{(1)j} = \frac{1}{2}\left[\frac{\partial X^{(i)}_{(1)}}{\partial x^j} - \frac{\partial X^{(j)}_{(1)}}{\partial x^i}\right].$$

ii) The torsion d-tensor $\mathbf{T}$ *of the null canonical generalized Cartan connection* $C\Gamma_{\mathcal{S}_{DEs}(\Delta)}$ *of the autonomous Lagrange space* $\mathcal{ADE}s\mathcal{EDL}_1^n(\Delta)$ *is determined only by the adapted components*

$$R^{(i)}_{(1)jk} = -\frac{1}{2}\left[\frac{\partial^2 X^{(i)}_{(1)}}{\partial x^j \partial x^k} - \frac{\partial^2 X^{(j)}_{(1)}}{\partial x^i \partial x^k}\right].$$

iii) The gyroscopic field F *is a time independent one and is given by the same formulas as in the Theorem 3.1.*

iv) The components $F^{(1)}_{(i)j}$ *are governed by the following more simple generalized Maxwell equations*

$$\sum_{\{i,j,k\}} F^{(1)}_{(i)j||k} = 0.$$

4 Geometric Yang-Mills energy

Let us consider in this Section that the DEs system (2.2.2.10) governs some phenomenas coming from Theoretical Biology. In a such biological context, our geometrical methods have proved that the given DEs system, together with the pair of Euclidian metrics Δ, provides an abstract gyroscopic field F, which, in our opinion, must be intimately related to biologic phenomenas taken in study. In order to observe the informations that may be suggested by this field F, we introduce a natural geometrical energy of F, like the Yang-Mills energies studied by Bourgouignon and Lawson [7] or Teleman [29]. In this direction, recall that the $L(G)$-valued 2-forms F, where G is a Lie subgroup of the linear group of matrices with real entries $GL_n(\mathbb{R})$ and $L(G)$ is its Lie algebra, models from a geometrical point of view important *physical fields*. As examples, for specific several subgroups $G \subset GL_n(\mathbb{R})$, a $L(G)$-valued 2-form F can model one from the *gravitational*, *electromagnetic*, *strong nuclear* or *weak nuclear physical fields*. Moreover, we recall that a G-valued 1-form of connection $\bigtriangledown$, verifying the equality $d\bigtriangledown = F$, is called a *potential* of the physical field F.

In such geometrical and physical approach, we assert that our 2-forms F produced by biologic DEs systems and pairs of Euclidian metrics, may be regarded as $o(n)$-valued 1-forms on the time manifold T, setting

$$\mathbf{F} = \mathbf{F}_{(1)}dt \in \Gamma(\Lambda^1(\mathbf{T}^*T) \otimes o(n)),$$

where

$$\mathbf{F}_{(1)} = \begin{pmatrix} 0 & F^{(1)}_{(1)2} & F^{(1)}_{(1)3} & ... & ... & F^{(1)}_{(1)n} \\ -F^{(1)}_{(1)2} & 0 & F^{(1)}_{(2)3} & ... & ... & F^{(1)}_{(2)n} \\ -F^{(1)}_{(1)3} & F^{(1)}_{(2)3} & 0 & ... & ... & F^{(1)}_{(3)n} \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & 0 & F^{(1)}_{(n-1)n} \\ -F^{(1)}_{(1)n} & -F^{(1)}_{(2)n} & -F^{(1)}_{(3)n} & ... & -F^{(1)}_{(n-1)n} & 0 \end{pmatrix} \in o(n),$$

$o(n)$ beeing the set of skew-matrices as the Lie algebra $L(O(n))$ of the subgroup of orthogonal matrices $O(n) \subset GL_n(\mathbb{R})$. As a conclusion, let us introduce the following geometrical, physical and abstract biological concept of *energy.*

Definition 4.1. *The Lagrangian function of Yang-Mills type, which is given by the formula*

$$\mathcal{EYM}_{DEs(\Delta)}(\mathbf{F}) = ||\mathbf{F}_{(1)}||^2 = \frac{1}{2}Trace\left(\mathbf{F}_{(1)} \cdot^T \mathbf{F}_{(1)}\right) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[F^{(1)}_{(i)j}\right]^2,$$

is called the **geometric gyroscopic energy of the biologic phenomenas governed by the DEs system taken in study.**

In the next sections, we study the expressions of the abstract biological Yang-Mills energies produced by the calcium oscillations in a large variety of cell types.

4.1 Intracellular calcium oscillations induced by self-modulation of the inositol $1, 4, 5$- triphosphate signal

The mathematical model [13] that describes calcium oscillations which can arise in a model based on the mechanism of calcium-induced calcium release, takes into account the calcium-stimulated degradation of inositol triphosphate ($InsP_3$).

In some cell types, particularly in hepatocytes, calcium oscillations have been observed in response to stimulation by specific agonists. As these cells are not electrically excitable, it is likely that this calcium oscillations rely on the interplay between two intracellular mechanisms capable of destabilizing the steady state: an increase in $InsP_3$ is expected to lead to an increase in the frequency of calcium spikes, but at the same time the $InsP_3$-induced rise will also lead to increased $InsP_3$ hydrolysis due to the calcium activation of the $InsP_3$ 3-kinase.

The classical mathematical model for the study of cytosolic calcium oscillations and their associated degradations of $InsP_3$ in endoplasmic reticulum contains three variables $Z(t)$, $Y(t)$ and $A(t)$, where

- Z is the concentration of free calcium in the cytosol;

- Y is the concentration of free calcium in the internal pool;
- A is the $InsP_3$ concentration.

The time evolution of these variables is governed by the following first order differential equations of cytosolic calcium oscillations (for more details, please see [5]), denoted by us with $[Ca^{2+} - InsP_3]$:

$$\begin{cases} \dfrac{dZ}{dt} = V_{M_3}\dfrac{Z^m}{K_Z^m + Z^m}\cdot\dfrac{Y^2}{K_Y^2 + Y^2}\cdot\dfrac{A^4}{K_A^4 + A^4} - V_{M_2}\dfrac{Z^2}{K_2^2 + Z^2}+ \\ \qquad + k_f Y - kZ + V_0 + \beta V_1 \\ \dfrac{dY}{dt} = V_{M_2}\dfrac{Z^2}{K_2^2 + Z^2} - k_f Y - V_{M_3}\dfrac{Z^m}{K_Z^m + Z^m}\cdot\dfrac{Y^2}{K_Y^2 + Y^2}\cdot\dfrac{A^4}{K_A^4 + A^4} \\ \dfrac{dA}{dt} = \beta V_{M_4} - V_{M_5}\dfrac{A^p}{K_5^p + A^p}\cdot\dfrac{Z^n}{K_d^n + Z^n} - \varepsilon A, \end{cases}$$

where

- V_0 refers to a constant input of calcium from the extracellular medium;
- V_1 is the maximum rate of stimulus-induced influx of calcium from the extracellular medium;
- β is a constant parameter reflecting the degree of stimulation of the cell by an agonist and thus only varies between 0 and 1;
- the rates $V_2 = V_{M_2}\dfrac{Z^2}{K_2^2 + Z^2}$ and $V_3 = V_{M_3}\dfrac{Z^m}{K_Z^m + Z^m}\cdot\dfrac{Y^2}{K_Y^2 + Y^2}\cdot\dfrac{A^4}{K_A^4 + A^4}$ refer to the pumping of cytosolic calcium into the internal stores and to the release of calcium from these stores into the cytosol in a process activated by cytosolic calcium, respectively. The constants V_{M_2} and V_{M_3} represent the maximum values of the preceding rates;
- the parametres K_2, K_Y, K_Z and K_A are treshold constants for pumping, release, and activation of release by calcium and by $InsP_3$;
- k_f is a rate constant measuring the passive, linear leak of Y into Z;
- k relates on the assumed linear transport of cytosolic calcium into the extracellular medium;
- V_{M_4} is the maximum rate of stimulus-induced synthesis of $InsP_3$;
- $V_5 = V_{M_5}\dfrac{A^p}{K_5^p + A^p}\cdot\dfrac{Z^n}{K_d^n + Z^n}$ is the rate of phosphorylation of the $InsP_3$ by the 3-kinase, which is caracterized by a maximum value V_{M_5} and a half-saturation constant K_5;
- m, n and p are the Hill's coefficients related to the cooperative processes;

- ε is the rate of phosphorylation of the $InsP_3$ by the 5-phosphatase.

From a biologic point of view, we recall that the preceding DEs system is based on the mechanism of Ca^{2+}- induced Ca^{2+} release ($CICR$), that takes into account the Ca^{2+} stimulates the degradations of the inositol $1, 4, 5$ triphosphate ($InsP3$) by a 3-kinase.

From a geometrical point of view, the DEs system of calcium oscillations and the degradation of the inositol trisphosphate $InsP_3$ may be regarded as a DEs system on the particular 1-jet space $J^1(T, M) \subset J^1(\mathbb{R}, \mathbb{R}^n)$, where $\dim T = 1$ and $\dim M = n = 3$. Let us denote the coordinates of the manifold M by $x^1 = Z$, $x^2 = Y$ and $x^3 = A$. In this geometrical context, the DEs system $[Ca^{2+} - InsP_3]$, as a particular DEs system of the form (2.2.2.10), is determined by the following tensorial components $X^{(i)}_{(1)}(x^1, x^2, x^3)$:

$$\begin{aligned} X^{(1)}_{(1)}(Z, Y, A) = {} & V_{M_3}\frac{Z^m}{K_Z^m + Z^m} \cdot \frac{Y^2}{K_Y^2 + Y^2} \cdot \frac{A^4}{K_A^4 + A^4} - V_{M_2}\frac{Z^2}{K_2^2 + Z^2} + \\ & + V_0 + \beta V_1 + k_f Y - kZ, \\ X^{(2)}_{(1)}(Z, Y, A) = {} & V_{M_2}\frac{Z^2}{K_2^2 + Z^2} - k_f Y - V_{M_3}\frac{Z^m}{K_Z^m + Z^m} \cdot \frac{Y^2}{K_Y^2 + Y^2} \cdot \frac{A^4}{K_A^4 + A^4}, \\ X^{(3)}_{(1)}(Z, Y, A) = {} & \beta V_{M_4} - V_{M_5}\frac{A^p}{K_5^p + A^p} \cdot \frac{Z^n}{K_d^n + Z^n} - \varepsilon A. \end{aligned}$$

Consequently, using the general results from the preceding sections, some partial derivatives and computations, we find the following important geometrical result that characterizes the cytosolic calcium oscillations in hepatocytes and the degradation of $InsP_3$ through endoplasmic reticulum.

Theorem 4.1. *The biologic gyroscopic field* $\mathbf{F}$ *produced by the DEs system* $[Ca^{2+} - InsP_3]$ *and the pair of Euclidian metrics* Δ *has the components*

$$\begin{aligned} F^{(1)}_{(1)2} = {} & \frac{1}{2}\left\{k_f - 2V_{M_2}K_2^2\frac{Z}{(K_2^2 + Z^2)^2} + V_{M_3}\frac{Z^{m-1}}{K_Z^m + Z^m} \cdot \frac{Y}{K_Y^2 + Y^2} \cdot \frac{A^4}{K_A^4 + A^4} \cdot \right. \\ & \left. \cdot \left[2\frac{K_Y^2 Z}{K_Y^2 + Y^2} + m\frac{K_Z^m Y}{K_Z^m + Z^m}\right]\right\}, \\ F^{(1)}_{(1)3} = {} & \frac{2V_{M_3}K_A^4 Z^m}{K_Z^m + Z^m} \cdot \frac{Y^2}{K_Y^2 + Y^2} \cdot \frac{A^3}{(K_A^4 + A^4)^2} + \frac{n}{2} \cdot \frac{A^p}{K_5^p + A^p} \cdot \frac{V_{M_5}K_d^n Z^{n-1}}{(K_d^n + Z^n)^2}, \\ F^{(1)}_{(2)3} = {} & -2V_{M_3}K_A^4 \cdot \frac{Z^m}{K_Z^m + Z^m} \cdot \frac{Y^2}{K_Y^2 + Y^2} \cdot \frac{A^3}{(K_A^4 + A^4)^2}. \end{aligned}$$

Proof. Particularizing the formulas that define the gyroscopic components of a DEs system of order one and a pair of Riemannian metrics, for the particular DEs system $[Ca^{2+} - InsP_3]$ and the pair of Euclidian metrics $\Delta = (1, \delta_{ij})$, we find

$$F^{(1)}_{(1)2} = \frac{1}{2}\left[\frac{\partial X^{(1)}_{(1)}}{\partial Y} - \frac{\partial X^{(2)}_{(1)}}{\partial Z}\right], \quad F^{(1)}_{(1)3} = \frac{1}{2}\left[\frac{\partial X^{(1)}_{(1)}}{\partial A} - \frac{\partial X^{(3)}_{(1)}}{\partial Z}\right],$$

$$F^{(1)}_{(2)3} = \frac{1}{2}\left[\frac{\partial X^{(2)}_{(1)}}{\partial A} - \frac{\partial X^{(3)}_{(1)}}{\partial Y}\right].$$

So, the computations imply the required result. □

We recall that the formulas from Theorem 3.1 show that the spatial components $N^{(i)}_{(1)j}$ of the nonlinear connection $\Gamma_{[Ca^{2+}-InsP_3]} = (0, N^{(i)}_{(1)j})$ produced by the DEs system $[Ca^{2+} - InsP_3]$ and the pair of Euclidian metrics $\Delta = (1, \delta_{ij})$ are exactly the same with the gyroscopic components $N^{(i)}_{(1)j} = F^{(1)}_{(i)j}$. Consequently, the Theorem 2.1 and his Corollary imply the following interesting qualitative geoemetrical result with biological energetical connotations.

Theorem 4.2. *The C^2 solutions of the DEs system $[Ca^{2+} - InsP_3]$ may be regarded as harmonic curves on the 1-jet space $J^1(T, M)$ of the nonlinear connection $\Gamma_{[Ca^{2+}-InsP_3]} = (0, F^{(1)}_{(i)j})$. In other words, the C^2 solutions $(Z(t), Y(t), A(t))$ of the first order DEs system $[Ca^{2+} - InsP_3]$ verify also the following second order biological DEs system:*

$$(4.4.4.1) \qquad \begin{cases} \dfrac{d^2Z}{dt^2} + F^{(1)}_{(1)2}\dfrac{dY}{dt} + F^{(1)}_{(1)3}\dfrac{dA}{dt} + \displaystyle\sum_{k=1}^{3} \partial_z X^{(1)}_{(1)} \cdot X^{(k)}_{(1)} = 0 \\[2ex] \dfrac{d^2Y}{dt^2} - F^{(1)}_{(1)2}\dfrac{dZ}{dt} + F^{(1)}_{(2)3}\dfrac{dA}{dt} \displaystyle\sum_{k=1}^{3} \partial_z X^{(1)}_{(1)} \cdot X^{(k)}_{(2)} = 0 \\[2ex] \dfrac{d^2A}{dt^2} - F^{(1)}_{(1)3}\dfrac{dZ}{dt} - F^{(1)}_{(2)3}\dfrac{dY}{dt} \displaystyle\sum_{k=1}^{3} \partial_z X^{(1)}_{(1)} \cdot X^{(k)}_{(3)} = 0. \end{cases}$$

Remark 4.1. *The importance of the second order system (4.4.4.1) is that its equations are equivalent with the Euler-Lagrange equations of the least squares Lagrangian on the 1-jet space $J^1(T, M)$, given by*

$$\mathcal{LSQ} = \left(\dot{Z} - X^{(1)}_{(1)}(Z, Y, A)\right)^2 + \left(\dot{Y} - X^{(2)}_{(1)}(Z, Y, A)\right)^2 + \left(\dot{A} - X^{(3)}_{(1)}(Z, Y, A)\right)^2,$$

where $(t, Z, Y, A, \dot{Z}, \dot{Y}, \dot{A})$ are the coordinates on $J^1(T, M) \equiv T \times \mathbf{T}M$. Therefore, the C^2 solutions $(Z(t), Y(t), A(t))$ of the first order DEs system $[Ca^{2+} - InsP_3]$, which characterizes the intracellular calcium oscillations in non-excitable cells involving Ca^{2+}-activated $InsP_3$ degradation, are minimizing the least squares biological Lagrangian $\mathcal{LSQ}$.

Particularizing the general definition of the geometric gyroscopic energy of a general DEs system coming from Theoretical Biology to our present biological phenomenas, we deduce that the geometric gyroscopic energy produced by the intracellular calcium oscillations in some non-excitable cell types has the form

$$\mathcal{EYM}_{[Ca^{2+}-InsP_3](\Delta)}(Z,Y,A) = \left[F^{(1)}_{(1)2}\right]^2 + \left[F^{(1)}_{(1)3}\right]^2 + \left[F^{(1)}_{(2)3}\right]^2.$$

Note that we have three important sets of parameter values corresponding to three types of complex Ca^{2+} oscillations in this model that involves Ca^{2+}-activated $InsP_3$ degradation including bursting, chaos and quasiperiodicity are listed in the following table. For more details, please see [13].

Parameters	**Bursting**	**Chaos**	**Quasiperiodicity**
β	0.46	0.65	0.51
n	2	4	4
m	4	2	2
p	1	1	2
K_2 (μM)	0.1	0.1	0.1
K_5 (μM)	1	0.3194	0.3
K_A (μM)	0.1	0.1	0.2
K_d (μM)	0.6	1	0.5
K_Y (μM)	0.2	0.3	0.2
K_Z (μM)	0.3	0.6	0.5
k (s^{-1})	0.1667	0.1667	0.1667
k_f (s^{-1})	0.0167	0.0167	0.0167
ε (s^{-1})	0.0167	0.2167	0.0017
V_0 $(\mu M\ s^{-1})$	0.0333	0.0333	0.0333
V_1 $(\mu M\ s^{-1})$	0.0333	0.0333	0.0333
V_{M_2} $(\mu M\ s^{-1})$	0.1	0.1	0.1
V_{M_3} $(\mu M\ s^{-1})$	0.3333	0.5	0.3333
V_{M_4} $(\mu M\ s^{-1})$	0.0417	0.05	0.0833
V_{M_5} $(\mu M\ s^{-1})$	0.5	0.8333	0.5

These parameter values are corresponding to the various types of complex oscillatory behaviour observed in the model defined by equations $[Ca^{2+} - InsP_3]$ and obviously produce particular geometric biological Yang-Mills energies.

Theorem 4.3. *The following formulas for the geometric biological energies of Yang-Mills type are true:*

(i) Biological Yang-Mills energy of bursting cytosolic calcium oscillations in the model involving Ca^{2+} activated $InsP_3$ degradation.

$$\mathcal{EYM}^{\text{bursting}}_{[Ca^{2+}-InsP_3](\Delta)} = \left\{0.00835 - \frac{0.001Z}{(0.01+Z^2)^2} + \frac{0.3333Z^3}{0.0081+Z^4}\cdot\frac{Y}{0.04+Y^2}\right.$$

$$\cdot\frac{A^4}{0.0001+A^4}\cdot\left[\frac{0.04Z}{0.04+Y^2}+\frac{0.0162Y}{0.0081+Z^4}\right]\Bigg\}^2+\Bigg\{\frac{0.6666\cdot 10^{-4}Z^4}{0.0081+Z^4}\cdot\frac{Y^2}{0.04+Y^2}\cdot$$

$$\frac{A^3}{(0.0001+A^4)^2}+\frac{A}{(1+A)^2}\cdot\frac{0.18Z}{(0.36+Z^2)^2}\Bigg\}^2+\frac{0.44435556\cdot 10^{-8}Z^8}{(0.0081+Z^4)^2}\cdot$$

$$\cdot\frac{Y^4}{(0.04+Y^2)^2}\cdot\frac{A^6}{(0.0001+A^4)^4};$$

(ii) Biological Yang-Mills energy of chaos cytosolic calcium oscillations in the model involving Ca^{2+} activated $InsP_3$ degradation.

$$\mathcal{EYM}^{\text{chaos}}_{[Ca^{2+}-InsP_3](\Delta)}=\Bigg\{0.00835-\frac{0.001Z}{(0.01+Z^2)^2}+\frac{0.5Z}{0.36+Z^2}\cdot\frac{Y}{0.09+Y^2}\cdot$$

$$\cdot\frac{A^4}{0.0001+A^4}\cdot\left[\frac{0.09Z}{0.09+Y^2}+\frac{0.36Y}{0.36+Z^2}\right]\Bigg\}^2+\Bigg\{\frac{0.0001Z^2}{0.36+Z^2}\cdot\frac{Y^2}{0.09+Y^2}\cdot$$

$$\cdot\frac{A^3}{(0.0001+A^4)^2}+\frac{2A}{0.3194+A}\cdot\frac{0.8333Z^3}{(1+Z^4)^2}\Bigg\}^2+\frac{10^{-8}Z^4}{(0.36+Z^2)^2}\cdot\frac{Y^4}{(0.09+Y^2)^2}\cdot$$

$$\cdot\frac{A^6}{(0.0001+A^4)^4};$$

(iii) Biological Yang-Mills energy of quasiperiodicity cytosolic calcium oscillations in the model involving Ca^{2+} activated $InsP_3$ degradation.

$$\mathcal{EYM}^{\text{quasiperiodicity}}_{[Ca^{2+}-InsP_3](\Delta)}=\Bigg\{0.00835-\frac{0.001Z}{(0.01+Z^2)^2}+\frac{0.3333Z}{0.25+Z^2}\cdot\frac{Y}{0.04+Y^2}\cdot$$

$$\cdot\frac{A^4}{0.0016+A^4}\cdot\left[\frac{0.04Z}{0.09+Y^2}+\frac{0.25Y}{0.25+Z^2}\right]\Bigg\}^2+\Bigg\{\frac{0.00106656Z^2}{0.25+Z^2}\cdot\frac{Y^2}{0.04+Y^2}\cdot$$

$$\cdot\frac{A^3}{(0.0016+A^4)^2}+\frac{2A^2}{0.09+A^2}\cdot\frac{0.03125Z^3}{(0.0625+Z^4)^2}\Bigg\}^2+\frac{0.00001137550233Z^4}{(0.25+Z^2)^2}\cdot$$

$$\cdot\frac{Y^4}{(0.04+Y^2)^2}\cdot\frac{A^6}{(0.0016+A^4)^4}.$$

4.2 Calcium oscillations in a model involving endoplasmic reticulum, mitochondria and cytosolic proteins

The next mathematical model represents a possible mechanism for complex calcium oscillations based on the interplay between three calcium stores in the biological living cells: the endoplasmic reticulum (ER), mitochondria and cytosolic proteins. The majority of calcium released from the ER is first very quickly sequestred by mitochondria. Afterwards, a much slower release of calcium from the mitochondria servs as the calcium suply for the intermediate calcium exchanges between the ER and the cytosolic proteins. We would like to point out that the oscillations of cytosolic calcium concentration play a vital role in providing the intracellular signalling. Moreover, a lot of cellular processes, like cell secretion or egg fertilisation for instance, are controlled by the oscillatory regime of the cytosolic calcium concentration.

In this second mathematical model, we have three variables $Ca_{cyt}(t)$, $Ca_{ER}(t)$ and $Ca_m(t)$, where

- Ca_{cyt} means the free cytosolic calcium concentration;
- Ca_{ER} means the free calcium concentration in the ER;
- Ca_m means the free calcium concentration in the mitochondria.

The preceding variables of calcium are governed by the following first order DEs system of the calcium oscillations through endoplasmic reticulum, mitochondria and the cytosolic proteins (for more details, please see [14]), denoted by us with $[Ca^{2+} - ER - cyt.pr - m]$:

$$\begin{cases} \dfrac{dCa_{cyt}}{dt} = k_{ch}\dfrac{Ca_{cyt}^2}{K_1^2 + Ca_{cyt}^2}(Ca_{ER} - Ca_{cyt}) + k_{leak}(Ca_{ER} - Ca_{cyt}) - \\ \qquad - k_{pump}Ca_{cyt} + \left(k_{out}\dfrac{Ca_{cyt}^2}{K_1^2 + Ca_{cyt}^2} + k_m\right)Ca_m - \\ \qquad - k_{in}\dfrac{Ca_{cyt}^8}{K_2^8 + Ca_{cyt}^8} + k_- \left(Ca_{tot} - Ca_{cyt} - \dfrac{\rho_{ER}}{\beta_{ER}}Ca_{ER} - \right. \\ \qquad \left. - \dfrac{\rho_m}{\beta_m}Ca_m\right) - k_+ Ca_{cyt}\left(\mathrm{Pr}_{tot} - Ca_{tot} + Ca_{cyt} + \right. \\ \qquad \left. + \dfrac{\rho_{ER}}{\beta_{ER}}Ca_{ER} + \dfrac{\rho_m}{\beta_m}Ca_m\right), \\ \dfrac{dCa_{ER}}{dt} = \dfrac{\beta_{ER}}{\rho_{ER}}\left[k_{pump}Ca_{cyt} - k_{ch}\dfrac{Ca_{cyt}^2}{K_1^2 + Ca_{cyt}^2}(Ca_{ER} - Ca_{cyt}) - \right. \\ \qquad \left. - k_{leak}(Ca_{ER} - Ca_{cyt})\right] \\ \dfrac{dCa_m}{dt} = \dfrac{\beta_m}{\rho_m}\left[k_{in}\dfrac{Ca_{cyt}^8}{K_2^8 + Ca_{cyt}^8} - \left(k_{out}\dfrac{Ca_{cyt}^2}{K_1^2 + Ca_{cyt}^2} + k_m\right)Ca_m\right], \end{cases}$$

where

- $\Pr_{tot}$ is the total concentration of cytosolic proteins;
- Ca_{tot} represents the total cellular Ca^{2+} concentration;
- K_1 represents the half-saturation for Ca^{2+};
- K_2 represents the half-saturation for Ca^{2+} of uniporters in the mitochondrial membrane;
- $V_{pump} = k_{pump}Ca_{cyt}$ is the adenosine triphosphate (ATP)- dependent calcium uptake from the cytosol into the ER;
- $V_{ch} = k_{ch}\dfrac{Ca_{cyt}^2}{K_1^2 + Ca_{cyt}^2}(Ca_{ER} - Ca_{cyt})$ is the calcium efflux from the ER through channels following the calcium-induced calcium release mechanism;
- $V_{leak} = k_{leak}(Ca_{ER} - Ca_{cyt})$ represents the calcium leak flux from the ER into the cytosol;
- $V_{in} = k_{in}\dfrac{Ca_{cyt}^8}{K_2^8 + Ca_{cyt}^8}$ is the active calcium uptake by mitochondrial uniporters;
- $V_{out} = \left(k_{out}\dfrac{Ca_{cyt}^2}{K_1^2 + Ca_{cyt}^2} + k_m\right)Ca_m$ is a very small non-specific leak flux;
- k_- and k_+ denote the off and the on rate constants of the calcium binding;
- ρ_{ER} and ρ_m represent the volume ratio between the ER and the cytosol or between the mitochondria and the cytosol, respectively;
- β_{ER} and β_m are constant factors for relating the concentrations of free calcium in the ER and the mitochondria to the respective total concentrations;
- k_{pump} is the rate constant of the ATP-ases;
- k_{ch} represents the maximal permeability of the calcium channels in the ER membrane;
- k_{leak} is the rate constant for calcium leak flux through the ER membrane;
- k_{in} represents the maximal permeability of the uniporters in the mitochondrial membrane;
- k_{out} represents the maximal rate for calcium flux through pores;
- k_m stands for the non-specific leak flux;

Remark 4.2. *From a biological point of view, note that, in addition to the endoplasmic reticulum as the main intracellular calcium store used in the first mathematical model, in this second model, the mitochondrial and cytosolic Ca^{2+}- binding proteins are also taken into account. We recall that this model was proposed in [14] especially for the study of the physiological role of mitochondria and the cytosolic proteins in generating complex Ca^{2+} oscillations.*

From a differential geometric point of view, we underline that the first order DEs system of calcium oscillations through endoplasmic reticulum, mitochondria and cytosolic proteins may be regarded as a DEs system on the particular 1-jet space $J^1(T,M) \subset J^1(\mathbb{R},\mathbb{R}^n)$, where $\dim T = 1$ and $\dim M = n = 3$. Denoting the coordinates of the manifold M by $x^1 = Ca_{cyt}$, $x^2 = Ca_{ER}$ and $x^3 = Ca_m$, we remark that the DEs system of order one $[Ca^{2+} - ER - cyt.pr - m]$ is determined by the following tensorial components $X^{(i)}_{(1)}(x^1,x^2,x^3)$:

$$\begin{aligned}
X^{(1)}_{(1)}(Ca_{cyt},Ca_{ER},Ca_m) = {} & \frac{k_{ch}Ca^2_{cyt}}{K_1^2+Ca^2_{cyt}}(Ca_{ER}-Ca_{cyt}) + k_{leak}(Ca_{ER}-Ca_{cyt}) - \\
& - k_{pump}Ca_{cyt} + \left(k_{out}\frac{Ca^2_{cyt}}{K_1^2+Ca^2_{cyt}} + k_m\right)Ca_m - \\
& - k_{in}\frac{Ca^8_{cyt}}{K_2^8+Ca^8_{cyt}} + k_-\left(Ca_{tot} - Ca_{cyt} - \frac{\rho_{ER}}{\beta_{ER}}Ca_{ER} - \right. \\
& \left. - \frac{\rho_m}{\beta_m}Ca_m\right) - k_+ Ca_{cyt}\left(\Pr_{tot} - Ca_{tot} + Ca_{cyt} + \right. \\
& \left. + \frac{\rho_{ER}}{\beta_{ER}}Ca_{ER} + \frac{\rho_m}{\beta_m}Ca_m\right), \\
X^{(2)}_{(1)}(Ca_{cyt},Ca_{ER},Ca_m) = {} & \frac{\beta_{ER}}{\rho_{ER}}\left[k_{pump}Ca_{cyt} - k_{leak}(Ca_{ER}-Ca_{cyt}) - \right. \\
& \left. - k_{ch}\frac{Ca^2_{cyt}}{K_1^2+Ca^2_{cyt}}(Ca_{ER}-Ca_{cyt})\right], \\
X^{(3)}_{(1)}(Ca_{cyt},Ca_{ER},Ca_m) = {} & \frac{\beta_m}{\rho_m}\left[\frac{k_{in}Ca^8_{cyt}}{K_2^8+Ca^8_{cyt}} - \left(k_{out}\frac{Ca^2_{cyt}}{K_1^2+Ca^2_{cyt}} + k_m\right)Ca_m\right].
\end{aligned}$$

As in the preceding biological case, it is obvious that again some partial derivatives and computations imply geometrical results which characterize the microscopic changes produced by the calcium oscillations in the model involving endoplasmic reticulum, mitochondria and cytosolic proteins.

Theorem 4.4. *The adapted components of the gyroscopic field* $\mathbf{F}$ *produced by the DEs system* $[Ca^{2+} - ER - cyt.pr - m]$ *and the pair of Euclidian metrics* Δ *are given by the following expressions:*

$$F^{(1)}_{(1)2} = \frac{1}{2}\left[\frac{\partial X^{(1)}_{(1)}}{\partial Ca_{ER}} - \frac{\partial X^{(2)}_{(1)}}{\partial Ca_{cyt}}\right] = \frac{1}{2}\left\{k_{ch}\frac{Ca^2_{cyt}}{K_1^2 + Ca^2_{cyt}} + k_{leak} - \frac{\beta_{ER}}{\rho_{ER}}\left[k_{pump} - \right.\right.$$

$$\left. - \frac{2k_{ch}K_1^2 Ca_{cyt}}{\left(K_1^2 + Ca^2_{cyt}\right)^2}(Ca_{ER} - Ca_{cyt}) + k_{ch}\frac{Ca^2_{cyt}}{K_1^2 + Ca^2_{cyt}} + k_{leak}\right] -$$

$$\left. - \frac{\rho_{ER}}{\beta_{ER}}(k_- + k_+ Ca_{cyt})\right\},$$

$$F^{(1)}_{(1)3} = \frac{1}{2}\left[\frac{\partial X^{(1)}_{(1)}}{\partial Ca_m} - \frac{\partial X^{(3)}_{(1)}}{\partial Ca_{cyt}}\right] = \frac{1}{2}\left\{k_{out}\frac{Ca^2_{cyt}}{K_1^2 + Ca^2_{cyt}} + k_m - \frac{\beta_m}{\rho_m}\cdot\right.$$

$$\cdot\left[8k_{in}K_2^8\frac{Ca^7_{cyt}}{\left(K_2^8 + Ca^8_{cyt}\right)^2} - 2k_{out}K_1^2\frac{Ca_m Ca_{cyt}}{\left(K_1^2 + Ca^2_{cyt}\right)^2}\right] -$$

$$\left. - \frac{\rho_m}{\beta_m}(k_- + k_+ Ca_{cyt})\right\},$$

$$F^{(1)}_{(2)3} = \frac{1}{2}\left[\frac{\partial X^{(2)}_{(1)}}{\partial Ca_m} - \frac{\partial X^{(3)}_{(1)}}{\partial Ca_{ER}}\right] = 0.$$

Now, taking again into account that the spatial components $N^{(i)}_{(1)j}$ of the nonlinear connection $\Gamma_{[Ca^{2+}-ER-cyt.pr-m]} = (0, N^{(i)}_{(1)j})$ produced by the DEs system $[Ca^{2+} - ER - cyt.pr - m]$ and the pair of Euclidian metrics $\Delta = (1, \delta_{ij})$ coincid with the components $N^{(i)}_{(1)j} = F^{(1)}_{(i)j}$, we naturally establish the following qualitative geometrical result with biological energetical meaning.

Theorem 4.5. *The C^2 solutions of the DEs system $[Ca^{2+} - ER - cyt.pr - m]$ may be viewed as harmonic curves on the 1-jet space $J^1(T, M)$ of the nonlinear connection $\Gamma_{[Ca^{2+}-ER-cyt.pr-m]} = (0, F^{(1)}_{(i)j})$. In other words, the solutions $(Ca_{cyt}(t), Ca_{ER}(t), Ca_m(t))$ of the first order DEs system $[Ca^{2+}-ER-cyt.pr-m]$ verify the second order biological DEs system*

$$(4.4.4.2)\quad \begin{cases} \dfrac{d^2 Ca_{cyt}}{dt^2} + F^{(1)}_{(1)2}\dfrac{dCa_{ER}}{dt} + F^{(1)}_{(1)3}\dfrac{dCa_m}{dt}\displaystyle\sum_{k=1}^{3} X^{(k)}_{(1)}\partial_{Ca_{cyt}} X^{(k)}_{(1)} = 0 \\[2ex] \dfrac{d^2 Ca_{ER}}{dt^2} - F^{(1)}_{(1)2}\dfrac{dCa_{cyt}}{dt}\displaystyle\sum_{k=1}^{3} X^{(k)}_{(1)}\partial_{Ca_{ER}} X^{(k)}_{(1)} = 0 \\[2ex] \dfrac{d^2 Ca_m}{dt^2} - F^{(1)}_{(1)3}\dfrac{dCa_{cyt}}{dt}\displaystyle\sum_{k=1}^{3} X^{(k)}_{(1)}\partial_{Ca_m} X^{(k)}_{(1)} = 0. \end{cases}$$

Remark 4.3. *The second order differential equations (4.4.4.2) are obviously equivalent with the Euler-Lagrange equations system of the least squares Lagrangian*

$$\mathcal{LSQ} = \left(\dot{C}a_{cyt} - X^{(1)}_{(1)}\right)^2 + \left(\dot{C}a_{ER} - X^{(2)}_{(1)}\right)^2 + \left(\dot{C}a_m - X^{(3)}_{(1)}\right)^2,$$

where $(t, Ca_{cyt}, Ca_{ER}, Ca_m, \dot{C}a_{cyt}, \dot{C}a_{ER}, \dot{C}a_m)$ *are the coordinates on the 1-jet space* $J^1(T, M) \equiv T \times \mathbf{T}M$. *In other words, the* C^2 *solutions* $(Ca_{cyt}(t),\ Ca_{ER}(t),\ Ca_m(t))$ *of the first order DEs system* $[Ca^{2+} - ER - cyt.pr - m]$, *that characterizes the calcium oscillations in this model involving endoplasmic reticulum, mitochondria and cytosolic proteins, are minimizing the least squares Lagrangian* $\mathcal{LSQ}$.

In the sequel, particularizing the general definition of the geometric energy of a general DEs system coming from Theoretical Biology to our present biological phenomenas, we deduce that the geometric Yang-Mills energy produced by the calcium oscillations in this model that takes into account endoplasmic reticulum, mitochondria and cytosolic proteins has the form

$$\mathcal{EYM}_{[Ca^{2+}-ER-cyt.pr-m](\Delta)}(Ca_{cyt}, Ca_{ER}, Ca_m) = \left[F^{(1)}_{(1)2}\right]^2 + \left[F^{(1)}_{(1)3}\right]^2.$$

Also. it is important to note that three sets of parameter values corresponding to three types of complex Ca^{2+} oscillations, including bursting, chaos and birhythmicity are listed in the following table, in which the parameter values correspond to the various types of complex oscillatory whose behaviour was observed in the model $[Ca^{2+} - ER - cyt.pr - m]$. For more details, please see [14].

Parameters	**Bursting**	**Chaos**	**Birhythmicity**
Ca_{tot} (μM)	90	90	90
Pr_{tot} (μM)	120	120	120
ρ_{ER}	0.01	0.01	0.01
ρ_m	0.01	0.01	0.01
β_{ER}	0.0025	0.0025	0.0025
β_m	0.0025	0.0025	0.0025
K_1 (μM)	5	5	5
K_2 (μM)	0.8	0.8	0.8
k_{ch} (s^{-1})	4100	2780-2980, 3598-3636	1968-2456
k_{pump} (s^{-1})	20	20	20
k_{leak} (s^{-1})	0.05	0.05	0.05
k_{in} $(\mu M\ s^{-1})$	300	300	300
k_{out} (s^{-1})	125	125	125
k_m (s^{-1})	0.00625	0.00625	0.00625
k_+ $(\mu M\ s^{-1})$	0.1	0.1	0.1
k_- (s^{-1})	0.01	0.01	0.01

Obviously the preceding parameter values produce particular geometric biological Yang-Mills energies for calcium oscillations phenomenas through endoplasmic reticulum, mitochondria and cytosolic proteins.

Theorem 4.6. *The following formulas for the geometric biological energies of Yang-Mills type are true:*

(i) Biological Yang-Mills energy of bursting calcium oscillations in the model involving endoplasmic reticulum, mitochondria and cytosolic proteins.

$$\mathcal{EYM}^{\text{bursting}}_{[Ca^{2+}-ER-cyt.pr-m](\Delta)} = \frac{1}{4}\left\{\frac{3075Ca^2_{cyt}}{25+Ca^2_{cyt}} + (Ca_{ER}-Ca_{cyt})\cdot\right.$$

$$\left.\cdot\frac{51250Ca_{cyt}}{(25+Ca^2_{cyt})^2} - 0.4Ca_{cyt} - 5.0025\right\}^2 + \frac{1}{4}\left\{\frac{1562.5Ca_mCa_{cyt}}{(25+Ca^2_{cyt})^2} - \right.$$

$$\left.-\frac{100.663296Ca^7_{cyt}}{(0.16777216+Ca^8_{cyt})^2} + \frac{125Ca^2_{cyt}}{25+Ca^2_{cyt}} - 0.4Ca_{cyt} - 0.03375\right\}^2,$$

(ii) Biological Yang-Mills energy of chaos calcium oscillations in the model involving endoplasmic reticulum, mitochondria and cytosolic proteins.

$$\mathcal{EYM}^{\text{chaos}}_{[Ca^{2+}-ER-cyt.pr-m](\Delta)} = \frac{1}{4}\left\{\frac{0.75k_{ch}Ca^2_{cyt}}{25+Ca^2_{cyt}} + (Ca_{ER}-Ca_{cyt})\cdot\right.$$
$$\left.\cdot\frac{12.5k_{ch}Ca_{cyt}}{(25+Ca^2_{cyt})^2} - 0.4Ca_{cyt} - 5.0025\right\}^2 + \frac{1}{4}\left\{\frac{1562.5Ca_mCa_{cyt}}{(25+Ca^2_{cyt})^2} - \right.$$
$$\left.-\frac{100.663296Ca^7_{cyt}}{(0.16777216+Ca^8_{cyt})^2} + \frac{125Ca^2_{cyt}}{25+Ca^2_{cyt}} - 0.4Ca_{cyt} - 0.03375\right\}^2,$$

where $k_{ch} \in [2780, 2980] \cup [3598, 3636]$;

(iii) Biological Yang-Mills energy of birhythmicity calcium oscillations in the model involving endoplasmic reticulum, mitochondria and cytosolic proteins.

$$\mathcal{EYM}^{\text{birhythmicity}}_{[Ca^{2+}-ER-cyt.pr-m](\Delta)} = \frac{1}{4}\left\{\frac{0.75k_{ch}Ca^2_{cyt}}{25+Ca^2_{cyt}} + (Ca_{ER}-Ca_{cyt})\cdot\right.$$

$$\left.\cdot\frac{12.5k_{ch}Ca_{cyt}}{(25+Ca^2_{cyt})^2} - 0.4Ca_{cyt} - 5.0025\right\}^2 + \frac{1}{4}\left\{\frac{1562.5Ca_mCa_{cyt}}{(25+Ca^2_{cyt})^2} - \right.$$
$$\left.-\frac{100.663296Ca^7_{cyt}}{(0.16777216+Ca^8_{cyt})^2} + \frac{125Ca^2_{cyt}}{25+Ca^2_{cyt}} - 0.4Ca_{cyt} - 0.03375\right\}^2,$$

where $k_{ch} \in [1968, 2456]$.

5 Theoretical biological interpretations

In authors's opinion, from a biological point of view, the appeareance in our geometrical studies of an unknown gyroscopic field **F**, directly and naturally provided by a DEs system of form (2.2.2.10) that governs some biological phenomenas, and the pair of Euclidian metrics Δ, may probably have interesting connections with the intrinsic biological phenomenas studied. Moreover, from a point of view of classification of field theories exposed by Gotay, Isenberg and Marsden in [11], we appreciate that our study may be included in the category of *parametrized metrical field theories.*

As a conclusion, taking into account our preceding discussions and some possible theoretical biological interpretations, we believe that our geometric biological field **F** may be regarded as follows:

- Or this biological field **F** must vanish in order to realize a stability of the biological phenomenas studied. This should be probably because a such electromagnetic field must not exist in the biological phenomenas.
- Or this biological field **F** must not vanish, having a natural and microscopic character in the biological phenomenas studied. In other words, this microscopic biological field **F** may be probably regarded as beeing provided not necessarily by the DEs systems involved in studies but by the pair of metrics Δ that have the well known physical meaning of gravitational potentials produced intrinsically by the biological matter.

Open problems.

1. There is a biological meaning of our preceding geometrical Yang-Mills energies ? In the affirmative case, what are their real biological meanings?
2. We think that the study of classical geometries (i. e. the fundamental forms and the main curvatures) of the surfaces of constant level, provided by the geometric Yang-Mills energies of DEs systems studied in this paper, together with some eventually computer drawn graphics, may represent an interesting research topic for the Theoretical Biology.
3. Developments of analogous multi-time geometric electromagnetisms on 1-jet spaces produced by PDEs systems of order one coming also from Theoretical Biology, together with new eventually theoretical biologic interpretations, are parts of the work in progress of authors of this paper.

Acknowledgements. The authors are grateful to Professors M. Anastasiei, Gh. Atanasiu, V. Balan, M. Lupu, J. E. Marsden, P. J. Olver, V. Prepeliţă, C. Radu, E. Stoica, K. Teleman and C. Udrişte for stimulative discussions on the geometrical methods used in the preceding biological applicative research.

The first author of this paper emphasizes that G.S. Asanov's paper [2] provides the foundations for a similar framework. Special thanks go to Professor R. G. Beil, the reviewer of the paper [17], which has kindly provided essential papers for the study of differential geometry of I-jet spaces.

References

[1] P. L. Antonelli, R. Miron, *Lagrange and Finsler Geometry. Applications to Physics and Biology*, Kluwer Academic Publishers, 1996.

[2] G.S. Asanov, *Jet extension of Finslerian Gauge approach*, Fortscritte der Physik, 38 (1990), 8, 571-610.

[3] V. Balan, *Notable curves in geometrized $J^1(T, M)$ framework,* Balkan Journal of Geometry and Its Applications, 8 (2003), 2, 1-10.

[4] M. Berridge, P. Lipp, M. Bootman, *Calcium signalling*, Curr. Biol., 9 (1999), R157-R159.

[5] J. A. M. Borghans, G. Dupont, A. Goldbeter, *Complex intracellular calcium oscillations: A theoretical exploration of possible mechanisms*, Biophys. Chem., 66 (1997), 25-41.

[6] K. S. R. Cuthbertson, P. H. Cobbold, *Phorbol ester and sperm activate mouse oocytes by inducing sustained oscillations in cell Ca^{2+}*, Nature, 316 (1985), 541-542.

[7] J. P. Bourgouignon, H. B. Lawson Jr., *Yang-Mills Theory: Its Physical Origins and Differential Geometric Aspects,* Seminar on Differential Geometry, Princeton University Press, New Jersey, 1982, 395-421.

[8] J. Eells, L. Lemaire, *A Report on Harmonic Maps,* Bulletin of London Mathematical Society, 20 (1988), 385-524.

[9] G. Giachetta, L. Mangiarotti, G. Sardanashvily, *Covariant Hamiltonian Field Theory*, http://xxx.lanl.gov/hep-th/9904062, 1999.

[10] A. Goldbeter, *Biochemical oscillations and cellular rhythms*, Cambridge University Press, Cambridge, 1996.

[11] M. Gotay, J. Isenberg, J. E. Marsden, *Momentum Maps and the Hamiltonian Structure of Classical Relativistic Field Theories I*, http://xxx.lanl.gov/hep-th/9801019.

[12] T. Haberichter, M. Marhl, R. Heinrich, *Birhythmicity , trirhythmicity and chaos in bursting calcium oscillations*, Biophysical Chemistry, 90 (2001), 17-30.

[13] G. Houart, G. Dupont, A. Goldbeter, *Bursting, chaos and birhythmicity originating from self-modulation of the inositol* $1,4,5$*-trisphosphate signal in model for intracellular Ca^{2+} oscillations*, Bulletin of Mathematical Biology, 61 (1999), 507-530.

[14] M. Marhl, T. Haberichter, M. Brumen, R. Heinrich, *Complex calcium oscillations and the role of mitochondria and cytosolic proteins*, Biosystems, 57 (2000), 75-86.

[15] J. E. Marsden, S. Pekarsky, S. Shkoller and M. West, *Variational methods, multisymplectic geometry and continuum mechanics,* J. Geom. Phys., 38 (2001), 253-284.

[16] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications,* Kluwer Academic Publishers, Dordrecht 1994.

[17] M. Neagu, *Generalized metrical multi-time Lagrange geometry of physical fields,* Journals from de Gruyter, Forum Mathematicum, 15 (2003), 63-92.

[18] M. Neagu, *Ricci and Bianchi identities for h-normal Γ-linear connections on* $J^1(T, M)$, Hindawi Publishing Corporation, International Journal of Mathematics and Mathematical Sciences, 34 (2003), 2177-2191.

[19] M. Neagu, *Riemann-Lagrange Geometry of 1-Jet Spaces* (Romanian), University "Politehnica" of Bucharest, Ph.D. Thesis 2001.

[20] M. Neagu, *The Geometry of autonomous metrical multi-time Lagrange space of electrodynamics*, Hindawi Publishing Corporation, International Journal of Mathematics and Mathematical Sciences, 29 (2002), 1, 7-15.

[21] M. Neagu, *The geometry of relativistic rheonomic Lagrange spaces*, Workshop on Differential Geometry, Global Analysis and Lie Algebras, University "Aristotle" of Thessaloniki, Greece; Editor: Prof. Dr. Gr. Tsagas, Proceedings 5, 2001, 142-168.

[22] M. Neagu, C. Udrişte, *From PDEs systems and metrics to geometric multi-time field theories*, Seminar of Mechanics, Differential Dynamical Systems, 79 (2001), Faculty of Mathematics, West University of Timişoara, Romania.

[23] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics 107, Springer-Verlag, New-York 1986.

[24] H. Poincaré, *Sur une forme nouvelle des équations de la m'echanique,* C. R. Acad Sci. 132 (1901), 369-371.

[25] J. A. Rottingen, J.G. Iversen, *Rulled by waves? Intracellular and intercellular calcium signalling*, Acta Physiol. Scand., 169 (2000), 203-219.

[26] D. Saunders, *The Geometry of Jet Bundles,* Cambridge University Press, New York 1989.

[27] T. J. Shuttleworth, J. L. Thompson, Ca^{2+} *entry modulated oscillations frequency by triggering* Ca^{2+} *release*, Biochem. J., 313 (1996) , 815-819.

[28] E. Sóos, *Géométrie et Electromagnetisme,* Seminar of Mechanics, Differential Dynamical Systems 35 (1992), Faculty of Mathematics, West University of Timişoara, Romania.

[29] K. Teleman, *Donaldson-Witten Theory,* Cours for students presented in the years 1994-1995 at Faculty of Mathematics, University of Bucharest, Romania.

[30] C. Udrişte, *Geometric Dynamics,* Springer-Verlag, Southeast Asian Bulletin of Mathematics, 24 (2000), 313-322.

[31] C. Udrişte, M. Neagu, *Geometrical interpretation of solutions of certain PDEs*, BJGA 4 (1999),1, 145-152.

[32] C. Udrişte, A. Udrişte, *Electromagnetic dynamical systems*, BJGA 2 (1997), 1, 129-140.

[33] C. Udrişte, *Geometric Dynamics*, KAP, Dordrecht, 2000.

[34] A. Vondra, *Symmetries of connections on fibered manifolds,* Archivum Mathematicum, 30 (1994), 97-115.

[35] N. M. Woods, K. S. R. Cuthbertson, P. H. Cobbold, *Repetitive transient rises in cytoplasmic free calcium in hormone-stimulated hepatocytes*, Nature, 319 (1986), 600-602.

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