Linearized geometric dynamics of Tobin-Benhabib-Miyao economic flow

Constantin Udrişte and Armando Ciancio

Abstract

The aim of this paper is to study the influence of the Euclidean-Lagrangian structure of the state space on the generalized Tobin flow in economics, confirming the existence of optimal economic fluctuations (small oscillations).

Section 1 reviews the generalized Tobin economic flow as formulated by Benhabib and Miyao. Section 2 recalls some tools of single-time geometric dynamics which describe a geodesic motion under a gyroscopic field of forces. Section 3 studies the linearized geometric dynamics produced by the Tobin-Benhabib-Miyao flow and by the Euclidean-Lagrangian structure of the economic state space.

In this way we estimate the time economic evolution for spotlighting the expectations of the agents.

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1 Tobin-Benhabib-Miyao economic flow

The Tobin model [4] regarding the role of money on economic growth has been extended by Benhabib and Miyao [1] to incorporate the role of expectation parameters, and to show that the variation of this parameter produces a Hopf Bifurcation in a three sector economy (see also [5],[6]).

We introduce the state space by the following variables:

- k = the capital labour ratio;
- m = the money stock per head;
- q = the expected rate of inflation.

Then, the model is an ODEs system

$$\dot{k} = sf(k) - (1-s)(\theta - q)m - nk, \dot{m} = m(\theta - \bar{p} - n), \dot{q} = \mu(\bar{p} - q), \tag{1.1}$$

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where

$$\bar{p} = \varepsilon\Big(m - L(k,q)\Big) + q, \tag{1.2}$$

is the actual rate of inflation. The real functions $f(k)$ and $L(k,q)$ are differentiable, and $s, \theta, n, \mu, \varepsilon$ are parameters (s = saving ratio; θ = rate of money expansion, n = population growth rate, μ = speed of adjustment of expectations, ε = speed of adjustment of price level). Keeping the parameters $s, \theta, n, \varepsilon$ like constants, we obtain an ODEs system with one parameter μ.

An equilibrium point $(k^*(\mu), m^*(\mu), q^*(\mu))$, at which $\dot{k} = 0 = \dot{m} = \dot{q}$, is the solution of the algebraic system

$$\theta = q + n, L(k,q) = m, s\, f(k) \; - (1-s)m\, n - k\, n = 0. \tag{1.3}$$

(suppose we have isolated solutions). Denoting $x = (x_1, x_2, x_3) = (k - k^*, m - m^*, q - q^*)$, the linearization about the equilibrium point $(0,0,0)$ is

$$\dot{x} = A(\mu)\, x, \tag{1.4}$$

where $A(\mu)$ is the Jacobian matrix of the function

$$\Big(sf(k) - (1-s)(\theta - q)m - nk \;, m(\theta - \bar{p} - n) \;, \mu(\bar{p} - q) \Big), \tag{1.5}$$

computed at $(k = k^*, m = m^*, q = q^*)$, i.e.,

$$\left(\begin{array}{ccc} s\, f' \; - n & -(1-s)n & (1-s)m \\ \varepsilon m L_1 & -2\varepsilon m & m(\varepsilon L_2 - 1) \\ -\mu\varepsilon L_1 & \mu\varepsilon & -\mu\varepsilon L_2 \end{array}\right)_{(k^*, m^*, q^*)} \tag{1.6}$$

The characteristic equation is

$$\det\Big(A(\mu) - \lambda(\mu) I\Big) = -\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3 = 0\,, \tag{1.7}$$

where $c_1 = \mathrm{tr}A(\mu)$, c_2 = sum of principal minors of order two, $c_3 = \det A(\mu)$. If $(-1)^i c_i > 0$ $(i = 1,2,3)$ and $c_1 c_2 < c_3$, then we have solutions of type $\lambda_1(\mu) < 0$, $\lambda_{2,3}(\mu) = \alpha(\mu) \pm i\,\beta(\mu)$, $\alpha(\mu) < 0$, and consequently the equilibrium point is asymptotically stable. In the hypothesis $c_1 c_2 = c_3$, we can find μ_0 such that $\lambda_1(\mu_0) < 0$, and $\alpha(\mu_0) = 0$, $\frac{d\alpha}{d\mu}(\mu_0) \neq 0$. By the Hopf Bifurcation Theorem, there exist periodic solutions

$$\Big(k(t,\mu), m(t,\mu), q(t,\mu)\Big),\; t \in R$$

around the equilibrium point (k^*, m^*, q^*).

Medio [2],[3] generalized the previous model and studied the birth of limit cycles given by Hopf Bifurcation, in the framework of λ - matrices and gyroscopic models.

2 Geometric dynamics produced by a flow and a Riemannian metric

Our theory [7]-[10] can be applied equally to any kind of flow on a manifold endowed with a geometric structure capable to produce *"square of the length"* (density of energy) and *"derivatives"*. This geometric structure transforms a flow into a geodesic motion in a gyroscopic field of forces. As an example, we can use a Euclidean-Lagrangian structure associated to the metric δ_{ij} or a Riemannian-Lagrangian structure associated to the metric g_{ij}.

Let us start with an arbitrary flow

$$\dot{x}_i = X_i(x)\,, \quad i = 1, 2, 3 \tag{2.8}$$

on the Euclidean space $(R^3,\, \delta_{ij})$. Modifying the prolongation by derivation in a suitable way, we obtain a *gyroscopic prolongation*

$$\frac{d^2 x_i}{dt^2} = \delta^{jk}\Big(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\Big)\frac{dx_k}{dt} + \frac{\partial f}{\partial x_i}\,, \tag{2.9}$$

where

$$f = 0.5\; \delta^{ij}\, X_i\, X_j \tag{2.10}$$

is the density of economic energy. This new prolongation determines a geometric dynamics, i.e., a geodesic motion in a gyroscopic field of forces. Another way to realize a geometric dynamics is to consider the least squares Lagrangian

$$L(x, \dot{x}) = 0.5\, \delta^{ij}\, \Big(\dot{x}_i - X_i(x)\Big)\Big(\dot{x}_j - X_j(x)\Big) \tag{2.11}$$

and to write the Euler-Lagrange equations which are just (9). Automatically, the geometric dynamics conserves the Hamiltonian

$$H(x, \dot{x}) = 0.5\, \delta^{ij}\, \Big(\dot{x}_i - X_i(x)\Big)\Big(\dot{x}_j + X_j(x)\Big). \tag{2.12}$$

The linearization of the first order ODEs system (8) around an equilibrium point $(0, 0, 0)$ is of the form

$$\dot{x} = Ax. \tag{2.13}$$

Then the linearization of the second order ODEs system (9), around the same equilibrium point, is

$$\frac{d^2 x}{dt^2} = (A - A^T)\,\frac{dx}{dt} + A^T\, A\, x\ . \tag{2.14}$$

For this second order system, the equilibrium point is

$$x(t) = 0, \frac{dx}{dt}(t) = 0.$$

Of course, the second order ODEs system (9) can be linearized also around a nonzero critical point of the density of energy f, but this study will be made in a further paper.

3 Linearized geometric dynamics around Tobin-Benhabib-Miyao flow

The Tobin-Benhabib-Miyao flow and the Euclidean-Lagrangian structure of the state space determine a geometric dynamics. We shall analyse its linearization around the equilibrium point which is described by a second order ODEs system of type (14). The solution of this system is of the form $x = ue^{\lambda t}, t \in R$, where u is a nonzero solution of the linear system

$$\Big(-A^T A - \lambda(A - A^T) + \lambda^2 I\Big)u = 0. \tag{3.15}$$

In other words, λ is a latent value of the λ-matrix in paranthesis, i.e., solution of the equation

$$\det\Big(-A^T A - \lambda(A - A^T) + \lambda^2 I\Big) = 0\,, \tag{3.16}$$

and u is a latent vector (nonzero solution of the equation 15). On the other hand, the following proposition is true.

Theorem. *If λ is a proper value of a real matrix A, then λ and $-\lambda$ are latent values of the previuos λ-matrix. Consequently the latent values satisfy $\sum \lambda = 0$.*

Proof. We use the decomposition

$$0 = det\Big(-A^T A - \lambda(A - A^T) + \lambda^2 I\Big) = det(\lambda I - A)det(\lambda I + A^T). \tag{3.17}$$

In order to obtain informations about the nature of the latent values λ, associated to the latent vector u, we build an equation of degree two satisfied by λ.

Let λ be a real latent value and u be the corresponding real latent vector. Premultiplying by u^T we find

$$\lambda^2 = \frac{u^T A^T A u}{u^T u}. \tag{3.18}$$

Let λ be a complex latent value and u be the associated complex latent vector. Premultiplying by $\bar{u}^T$ (conjugate transpose of u) gives

$$m\,\lambda^2 + i\,g\,\lambda + n = 0\,, \tag{3.19}$$

where

$$m = \bar{u}^T\,u > 0\,,\; i\,g = -\bar{u}^T\,(A - A^T)\,u\,,\; n = -\bar{u}^T\,A^T\,A\,u < 0\,.$$

The discriminant of the equation (19) is

$$\Delta\,(\mu) = -g^2\, - \,4\,m\,n\,.$$

Theorem. *1) The ODEs system (14) has saddle point properties around the equilibrium point iff $\Delta(\mu) > 0$.*

2) If $\Delta(\mu) < 0$, no saddle point properties exist.

Let μ_0 be such that $\Delta(\mu_0) = 0$ and $\Delta(\mu) > 0$ or < 0 if $\mu < \mu_0$ respectively $\mu > \mu_0$ with $\frac{d\,\Delta}{d\,\mu}(\mu_0) < 0$. These are just the conditions in the "*Hopf Bifurcation Theorem* *for second order systems*". Consequently, when μ passes μ_0, the system will undergo a bifurcation and lose its saddle point properties. Assuming simple latent values around μ_0, we find:

Corollary. 1) *if $g(\mu) = 0$ for $\mu \in N_\varepsilon(\mu_0)$, then certain latent value lying on the real axis crosses imaginary axis from left to right, causing "total instability";*

2) if $g(\mu) \neq 0$, then the loss of stability is of the "flutter type", i.e., a pair of complex conjugate latent values crosses the imaginary axis from the left, causing Hopf Bifurcation and giving birth to closed orbits around the equilibrium point.

Remark. This limit cycle is optimal because it fulfils all optimality requirements, including the transversality condition

$$lim_{t\to\infty} k(t)\, q(t)\, e^{-\mu\, t} \;=\; 0. \tag{3.20}$$

Thus an economy satisfying all standard neo-classical competitive conditions such as perfect foresight, zero profit, market clearing, can exhibit permanent small oscillations in prices and capital stocks. Therefore we recovered the concept of optimal economic fluctuations.

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Constantin Udrişte,
University Politehnica of Bucharest, Department of Mathematics I,
Splaiul Independenţei No. 313, RO-060042 Bucharest, Romania.
e-mail address: udriste@mathem.pub.ro

Armando Ciancio,
Department of Mathematics, Faculty of Economics,
University of Messina, Via dei Verdi 75, 98122 Messina, Italy.