On the concircular curvature tensor of a (κ, μ)-manifold

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Dedicated to the memory of Grigorios TSAGAS (1935-2003)

Abstract

We give a classification of (κ, μ)-manifolds, whose concircular curvature tensor Z and Ricci tensor S satisfy $Z(\xi, X) \cdot S = 0$.

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1 Introduction

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a *concircular transformation* ([9], [16]). A concircular transformation is always a conformal transformation ([9]). Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus, the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [3]). An interesting invariant of a concircular transformation is the *concircular curvature tensor* Z. It is defined by ([16], [17])

$$Z = R - \frac{r}{n(n-1)} R_0,$$

where R is the curvature tensor, r is the scalar curvature and

$$R_0(X,Y)W = g(Y,W)X - g(X,W)Y, \qquad X, Y, W \in TM.$$

Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

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It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$ [2]. On the other hand, on a manifold M equipped with a Sasakian structure (η, ξ, φ, g), it follows that (see equation (2.6))

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y = R_0(X,Y)\xi, \quad X, Y \in TM.$$

As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case; D. Blair, T. Koufogiorgos and B. J. Papantoniou [5] considered the (κ, μ)-*nullity condition* (see Section 2) on a contact metric manifold and gave several reasons for studying it. Thus, they introduced the class of contact metric manifolds M with contact metric structures (η, ξ, φ, g), which satisfies

$$R(X,Y)\xi = (\kappa I + \mu h) R_0(X,Y)\xi, \qquad X, Y \in TM,$$

where $(\kappa, \mu) \in \mathbf{R}^2$ and $2h$ is the Lie derivative of φ in the direction ξ. A contact metric manifold belonging to this class is called a (κ, μ)-*manifold*. Characteristic examples of non-Sasakian (κ, μ)-manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one and certain Lie groups [8].

In a previous paper [6], D. E. Blair and the authors started a study of concircular curvature tensor of contact metric manifolds. Main result of this paper [6] states that a $(2n+1)$-dimensional $N(\kappa)$-contact metric manifold M satisfies $Z(\xi, X) \cdot Z = 0$ if and only if M is locally isometric to the sphere $S^{2n+1}(1)$, M is locally isometric to the Example 2.1 (Example 3.1 of [6]) or M is 3-dimensional and flat. An $N(\kappa)$-contact metric manifold is a (κ, μ)-manifold with $\mu = 0$. Example 2.1 is an $N(\kappa)$-contact metric manifold with $\kappa = 1 - \frac{1}{n}$, $n > 1$. In this example it is $Z(\xi, .)$ that vanishes while Z itself is not zero. B. J. Papantoniou [12] and D. Perrone [13] included the studies of contact metric manifolds satisfying $R(X, \xi) \cdot S = 0$, where S is the Ricci tensor. Motivated by these studies, we continue the study of the paper [6] and classify (κ, μ)-manifolds with concircular curvature tensor Z satisfying $Z(\xi, X) \cdot S = 0$. In fact, we prove the following theorems.

Theorem 1.1 *A Ricci flat (κ, μ)-manifold must be flat and 3-dimensional.*

Theorem 1.2 *A non-Sasakian Einstein (κ, μ)-manifold is flat and 3-dimensional.*

Theorem 1.3 *Let M^{2n+1} be a non-Sasakian η-Einstein (κ, μ)-manifold. Then the concircular curvature tensor Z satisfies $Z(\xi, X) \cdot S = 0$ if and only if M^{2n+1} is flat and 3-dimensional.*

Theorem 1.4 *Let M^{2n+1} be a (κ, μ)-manifold. The concircular curvature tensor Z satisfies $Z(\xi, X) \cdot S = 0$ if and only if we have one of the following:*
(a) *M^{2n+1} is flat and 3-dimensional.*
(b) *M^{2n+1} is locally isometric to the* Example 2.1.
(c) *M^{2n+1} is an Einstein-Sasakian manifold.*

The section 2 contains a brief introduction to contact metric manifolds and $\mathcal{D}$-homothetic deformation. In this section we also recall Example 3.1 of [6] as Example 2.1. Section 3 contains some basic results. In the section 4, we prove the above theorems.

2 Contact metric manifolds

A differentiable 1-form η on a $(2n+1)$-dimensional differentiable manifold M is called a *contact form* if $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, and M equipped with a contact form is a *contact manifold*. Since rank of $d\eta$ is $2n$ on the Grassmann algebra $\bigwedge T_p^* M$ at each point $p \in M$, therefore there exists a unique global vector field ξ, called the *characteristic vector field*, such that

$$\eta(\xi) = 1, \qquad \text{and} \qquad d\eta(\xi, \cdot) = 0. \tag{2.1}$$

Moreover, it is well-known that there exist a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(X) = g(X, \xi), \tag{2.2}$$

$$\varphi^2 = -I + \eta \otimes \xi, \quad d\eta(X, Y) = g(X, \varphi Y), \tag{2.3}$$

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y) \tag{2.4}$$

for $X, Y \in TM$. The structure (η, ξ, φ, g) is called a *contact metric structure* and the manifold M endowed with such a structure is said to be a *contact metric manifold*.

The contact metric structure (η, ξ, φ, g) on M gives rise to a natural almost Hermitian structure on the product manifold $M \times \mathbf{R}$. If this structure is integrable, then M is said to be a *Sasakian manifold*. A Sasakian manifold is characterized by the condition

$$\nabla_X \varphi = R_0(\xi, X), \qquad X \in TM, \tag{2.5}$$

where ∇ is Levi-Civita connection. Also, a contact metric manifold M is Sasakian if and only if the curvature tensor R satisfies

$$R(X, Y)\xi = R_0(X, Y)\xi, \quad X, Y \in TM. \tag{2.6}$$

In a contact metric manifold M, the $(1,1)$-tensor field h is symmetric and satisfies

$$h\xi = 0, \ \ h\varphi + \varphi h = 0, \ \ \nabla\xi = -\varphi - \varphi h, \ \ \mathrm{trace}(h) = \mathrm{trace}(\varphi h) = 0. \tag{2.7}$$

The (κ, μ)-*nullity distribution* $N(\kappa, \mu)$ ([5],[12]) of a contact metric manifold M is defined by

$$N(\kappa, \mu) : p \to N_p(\kappa, \mu) = \{W \in T_pM \mid R(X, Y)W = (\kappa I + \mu h)R_0(X, Y)W\}$$

for all $X, Y \in TM$, where $(\kappa, \mu) \in \mathbf{R}^2$. A contact metric manifold M with $\xi \in N(\kappa, \mu)$ is called a (κ, μ)-*manifold*. In this case, we have $h^2 = (\kappa - 1)\varphi^2$. In fact, (κ, μ)-manifolds exist for all values of $\kappa \leq 1$ and all μ. The class of (κ, μ)-manifolds contains Sasakian manifolds for $\kappa = 1$ and $h = 0$. If $\mu = 0$, the (κ, μ)-nullity distribution $N(\kappa, \mu)$ is reduced to the κ-*nullity distribution* $N(\kappa)$ [15]. If $\xi \in N(\kappa)$, then we call a contact metric manifold M an $N(\kappa)$-*contact metric manifold* [15]. For more details we refer to [1] and [4].

We also recall the notion of a $\mathcal{D}$-*homothetic deformation*. For a given contact metric structure (φ, ξ, η, g), this is the structure defined by

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\varphi} = \varphi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is a positive constant. While such a change preserves the state of being contact metric, K-contact, Sasakian or strongly pseudo-convex CR, it destroys a condition like $R(X,Y)\xi = 0$ or $R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y)$. However the form of the (κ,μ)-nullity condition is preserved under a $\mathcal{D}$-homothetic deformation with

$$\bar{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}.$$

Given a non-Sasakian (κ,μ)-manifold M, E. Boeckx [8] introduced an invariant

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-\kappa}}$$

and showed that for two non-Sasakian (κ,μ)-manifolds $(M_i,\varphi_i,\xi_i,\eta_i,g_i)$, $i = 1,2$, we have $I_{M_1} = I_{M_2}$ if and only if up to a $\mathcal{D}$-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds. Thus we know all non-Sasakian (κ,μ)-manifolds locally as soon as we have for every odd dimension $2n+1$ and for every possible value of the invariant I, one (κ,μ)-manifold (M,φ,ξ,η,g) with $I_M = I$. For $I > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where we have $I = \frac{1+c}{|1-c|}$. E. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I \le -1$.

In the following, we recall Example 3.1 of [6].

Example 2.1 [6] For $n > 1$, the Boeckx invariant for a $(2n+1)$-dimensional $\left(1-\frac{1}{n},0\right)$-manifold is $\sqrt{n} > -1$. Therefore, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c so chosen that the resulting $\mathcal{D}$-homothetic deformation will be a $\left(1-\frac{1}{n},0\right)$-manifold. That is for $\kappa = c(2-c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{\kappa + a^2 - 1}{a^2}, \quad 0 = \frac{\mu + 2a - 2}{a}$$

for a and c. The result is

$$c = \frac{\left(\sqrt{n} \pm 1\right)^2}{n-1}, \quad a = 1 + c$$

and taking c and a to be these values we obtain a $N\left(1-\frac{1}{n}\right)$-contact metric manifold.

The above example is used in Theorem 1.4.

3 Some basic results

From the definition of the concircular curvature tensor Z, in an almost contact metric manifold M^{2n+1} we have

$$Z = R - \frac{r}{2n\,(2n+1)} R_0. \tag{3.8}$$

For a (κ,μ)-manifold, we have

$$R(X,Y)\xi = (\kappa I + \mu h) R_0(X,Y)\xi, \tag{3.9}$$

which is equivalent to

$$R(\xi, X) = R_0(\xi, (\kappa I + \mu h) X). \tag{3.10}$$

From (3.9), we get

$$R(\xi, X)\xi = \kappa(\eta(X)\xi - X) - \mu hX. \tag{3.11}$$

Now, we prove the following

Proposition 3.1 *In a (κ, μ)-manifold M^{2n+1}, the concircular curvature tensor Z satisfies*

$$Z(X,Y)\xi = \left(\left(\kappa - \frac{r}{2n(2n+1)}\right) I + \mu h\right) R_0(X,Y)\xi, \tag{3.12}$$

$$Z(\xi, X) = \left(\kappa - \frac{r}{2n(2n+1)}\right) R_0(\xi, X) + \mu R_0(\xi, hX). \tag{3.13}$$

Consequently, we have

$$Z(\xi, X)\xi = \left(\kappa - \frac{r}{2n(2n+1)}\right)(\eta(X)\xi - X) - \mu hX. \tag{3.14}$$

$$\eta(Z(X,Y)\xi) = 0, \tag{3.15}$$

$$\begin{aligned} \eta(Z(\xi, X)Y) &= \left(\kappa - \frac{r}{2n(2n+1)}\right)(g(X,Y) - \eta(X)\eta(Y)) \\ &+ \mu g(hX, Y). \end{aligned} \tag{3.16}$$

Proof. From (3.8), (3.9) and (3.10) the equations (3.12) and (3.13) follow easily. □

Next, we have the following

Proposition 3.2 *In a (κ, μ)-manifold M^{2n+1}, we have*

$$\begin{aligned} S(Z(\xi, X)Y, \xi) &= 2n\kappa\mu g(hX, Y) \\ &+ 2n\kappa\left(\kappa - \frac{r}{2n(2n+1)}\right)(g(X,Y) - \eta(X)\eta(Y)), \end{aligned} \tag{3.17}$$

$$\begin{aligned} S(Z(\xi, X)\xi, Y) &= 2n\kappa\left(\kappa - \frac{r}{2n(2n+1)}\right)\eta(X)\eta(Y) \\ &- \left(\kappa - \frac{r}{2n(2n+1)}\right) S(X,Y) - \mu S(hX, Y). \end{aligned} \tag{3.18}$$

Proof. For a (κ, μ)-manifold M^{2n+1}, it is well known that

$$S(X, \xi) = 2n\kappa\eta(X). \tag{3.19}$$

From (3.19) and (3.16) we get (3.17), while (3.18) follows from (3.14) and (3.19). □

Now, we prove a key Lemma for later use.

Lemma 3.3 *Let M^{2n+1} be a (κ,μ)-manifold satisfying $Z(\xi,X)\cdot S=0$. Then*

$$\begin{aligned}(3.20)\qquad 0 &= \left(\kappa-\frac{r}{2n\,(2n+1)}\right)(S\,(X,Y)-2n\kappa g\,(X,Y))\\ &+ \mu\,(S\,(hX,Y)-2n\kappa g\,(hX,Y))\,.\end{aligned}$$

Proof. In an almost contact metric manifold, the condition $Z\,(\xi,X)\cdot S=0$ implies that

$$(3.21)\qquad S\,(Z\,(\xi,X)\,Y,\xi)+S\,(Y,Z\,(\xi,X)\,\xi)=0,$$

which in view of (3.17) and (3.18) gives (3.20). □

It is well known that in a non-Sasakian (κ,μ)-manifold M^{2n+1} the Ricci operator Q is given by [5]

$$\begin{aligned}(3.22)\qquad Q &= (2(n-1)-n\mu)I+(2(n-1)+\mu)h\\ &+ (2(1-n)+n(2\kappa+\mu))\eta\otimes\xi.\end{aligned}$$

Consequently, the Ricci tensor S and the scalar curvature r are given by

$$\begin{aligned}(3.23)\qquad S\,(X,Y) &= (2\,(n-1)-n\mu)\,g\,(X,Y)+(2\,(n-1)+\mu)\,g\,(hX,Y)\\ &+ (2\,(1-n)+n\,(2\kappa+\mu))\,\eta\,(X)\,\eta\,(Y)\,,\end{aligned}$$

$$(3.24)\qquad r=2n\,(2n-2+\kappa-n\mu)\,.$$

From (3.23), we also have

$$\begin{aligned}(3.25)\qquad S\,(hX,Y) &= (2\,(n-1)-n\mu)\,g\,(hX,Y)\\ &- (\kappa-1)\,(2\,(n-1)+\mu)\,g\,(X,Y)\\ &+ (\kappa-1)\,(2\,(n-1)+\mu)\,\eta\,(X)\,\eta\,(Y)\,,\end{aligned}$$

where $\eta\circ h=0$, $h^2=(\kappa-1)\,\varphi^2$ and (2.4) are used.

We also recall the following theorems for later use.

Theorem 3.4 (Olszak [11] or see [4] pp. 98-99) *A contact metric manifold of constant curvature is necessarily a Sasakian manifold of constant curvature $+1$ or is 3-dimensional and flat.*

Theorem 3.5 (Blair [2] or see [4] p. 101) *Let M^{2n+1} be a contact metric manifold satisfying $R(X,Y)\xi=0$. Then, M^{2n+1} is locally isometric to $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$.*

4 Proof of Theorems

In this section, we prove Theorems 1.1, 1.2, 1.3 and 1.4.

Proof of Theorem 1.1. Let M^{2n+1} be a Ricci flat (κ,μ)-manifold. Then from (3.19), we get

$$0=S\,(\xi,\xi)=2n\kappa,$$

which implies that $\kappa = 0$. Using $\kappa = 0$ in (3.24) and (3.25), we get

$$n\mu = 2\,(n-1) \tag{4.26}$$

and

$$0 = S\,(hX,Y) = (2\,(n-1)+\mu)\,(g\,(X,Y) - \eta\,(X)\,\eta\,(Y)) + (2\,(n-1) - n\mu)\,g\,(hX,Y) \tag{4.27}$$

respectively. The above equation implies that

$$\mu = -2\,(n-1)\,. \tag{4.28}$$

Since n is positive, from (4.26) and (4.28) we get $n = 1$ and consequently $\mu = 0$. Thus, in view of Theorem 3.5 the proof is complete. □

Now we give a proof of Theorem 1.2.

Proof of Theorem 1.2. To prove that a non-Sasakian Einstein (κ,μ)-manifold is 3-dimensional and flat, we proceed as follows. If $QX = aX$ and since we know Q, we have

$$\begin{aligned} aX &= (2\,(n-1) - n\mu)\,X + (2\,(n-1)+\mu)\,hX \\ &+ (2\,(1-n) + n\,(2\kappa+\mu))\,\eta\,(X)\,\xi. \end{aligned} \tag{4.29}$$

Setting $X = \xi$, we get $a = 2n\kappa$. Applying to eigenvectors of h, say $hX = \lambda X$, $h\varphi X = -\lambda\varphi X$, and comparing we see that the coefficient of hX must vanish. Thus, we get $\mu = -2(n-1)$ and then

$$2n\kappa = 2(n-1) + 2n(n-1) = 2(n^2-1). \tag{4.30}$$

Therefore $\kappa = \frac{n^2-1}{n} < 1$, so $n = 1$ is the only case. This gives $\mu = 0$ which with $n = 1$ gives $\kappa = 0$. □

Theorem 1.2 is a generalization of Theorem 5.2 of [15], which states that an Einstein $N(\kappa)$-contact metric manifold of dimension ≥ 5 is necessarily Sasakian.

Before proving Theorem 1.3, we give a brief introduction to η-Einstein (κ,μ)-manifold. A contact metric manifold M is said to be *η-Einstein* ([10] or see [4] p. 105) if the Ricci tensor S satisfies

$$S = ag + b\eta\otimes\eta, \tag{4.31}$$

where a and b are some smooth functions on the manifold. In particular if $b = 0$, then M becomes an *Einstein manifold*. In dimensions ≥ 5 it is known that for any η-Einstein K-contact manifold, a and b are constnts [14].

Example 4.1 A contact metric manifold, obtained by a $\mathcal{D}$-homothetic deformation of the contact metric structure on the tangent sphere bundle of a Riemannian manifold M^{n+1} of constant curvature $\frac{n^2\pm 2n+1}{n^2-1}$, is a non-Sasakian η-Einstein (κ,μ)-manifold.

From (3.23) and (4.31), we see that a non-Sasakian (κ,μ)-manifold M^{2n+1} is η-Einstein if and only if $\mu=-2(n-1)$. In this case Ricci tensor is given by

$$S=2\left(n^2-1\right)g-2\left(n^2-n\kappa-1\right)\eta\otimes\eta. \tag{4.32}$$

Putting $\mu=-2(n-1)$ in (3.24), we get

$$r=2n\left(\kappa+2(n-1)(n+1)\right). \tag{4.33}$$

A 3-dimensional contact metric manifold is η-Einstein if and only if it is an $N(\kappa)$-contact metric manifold [7]. More precisely, in a 3-dimensional $N(\kappa)$-contact metric manifold, it follows that

$$S=\left(\frac{r}{2}-\kappa\right)g+\left(3\kappa-\frac{r}{2}\right)\eta\otimes\eta. \tag{4.34}$$

Now, we provide a proof of Theorem 1.3 as follows:

Proof of Theorem 1.3. From (3.17), we get

$$\begin{aligned} S(Z(\xi,X)Y,\xi) &= 4n(1-n)\kappa g(hX,Y) \\ &+ 2n\kappa\left(\kappa-\frac{r}{2n(2n+1)}\right)\left(g(X,Y)-\eta(X)\eta(Y)\right). \end{aligned} \tag{4.35}$$

In view of (4.32) and (3.18), we get

$$\begin{aligned} S(Z(\xi,X)\xi,Y) &= 4(n-1)\left(n^2-1\right)g(hX,Y) \\ &- 2\left(n^2-1\right)\left(\kappa-\frac{r}{2n(2n+1)}\right)\left(g(X,Y)-\eta(X)\eta(Y)\right). \end{aligned} \tag{4.36}$$

If M satisfies $Z(\xi,X)\cdot S=0$, from (4.35), (4.36) and (3.21), we get

$$\begin{aligned} 0 &= S(Z(\xi,X)Y,\xi)+S(Z(\xi,X)\xi,Y) \\ &= 2\left(1+n\kappa-n^2\right)\left(\kappa-\frac{r}{2n(2n+1)}\right)\left(g(X,Y)-\eta(X)\eta(Y)\right) \\ &- 4(n-1)\left(1+n\kappa-n^2\right)g(hX,Y). \end{aligned}$$

Contracting the above equation and using $\operatorname{trace}(h)=0$, we get

$$4n\left(1+n\kappa-n^2\right)\left(\kappa-\frac{r}{2n(2n+1)}\right)=0.$$

In view of (4.33), $\kappa-\frac{r}{2n(2n+1)}=0$ is equivalent to $\kappa=\frac{n^2-1}{n}$, which is equivalent to $1+n\kappa-n^2=0$. In this case M^{2n+1} reduces to an Einstein manifold. Therefore in view of Theorem 1.2, M^{2n+1} is flat and 3-dimensional. The converse is straightforward. □

Finally, we prove Theorem 1.4.

Proof of Theorem 1.4. Let M be a $(2n+1)$-dimensional (κ,μ)-manifold satisfying $Z(\xi,X)\cdot S=0$. We have the following four possible cases.

Case I. $\kappa = 0 = \mu$. From (3.9) we have $R(X,Y)\xi = 0$. Thus, in view of Theorem 3.5, M satisfies the statement **(a)**.

Case II. $\kappa \neq 0 = \mu$. Using $\mu = 0$ in (3.20), we have

$$\left(\kappa - \frac{r}{2n\left(2n+1\right)}\right)\left(S\left(X,Y\right) - 2n\kappa g\left(X,Y\right)\right) = 0. \tag{4.37}$$

Therefore, either $r = 2n\left(2n+1\right)\kappa$ or $S = 2n\kappa g$. In the second case M^{2n+1} reduces to an Einstein manifold. Therefore in view of Theorem 1.2, we have either the statement **(a)** or the statement **(c)**.

If $r = 2n\left(2n+1\right)\kappa$, we note from (3.24) that the scalar curvature of an $N(\kappa)$-contact metric manifold is $r = 2n\left(2n-2+\kappa\right)$. Comparing gives $\kappa = 1 - \frac{1}{n}$ and hence M is locally isometric to the Example 2.1 for $n > 1$ and to the flat case if $n = 1$. This is the statement **(b)**. Conversely it is straightforward to check that when $\kappa = 1 - \frac{1}{n}$, $QX = 2(n-1)(X + hX)$ and in turn $Z(\xi, X)\cdot S = 0$.

Case III. $\kappa = 0 \neq \mu$.

Case IIIa. $\kappa = 0 \neq \mu$ and $n = 1$. Using $\kappa = 0$ and $n = 1$ in (3.23), (3.20), (3.25) we get

$$S\left(X,Y\right) = -\mu\left(g\left(X,Y\right) - \eta\left(X\right)\eta\left(Y\right)\right) + \mu g\left(hX,Y\right),$$

$$rS\left(X,Y\right) = 6\mu S\left(hX,Y\right),$$

$$S\left(hX,Y\right) = -\mu g\left(hX,Y\right) + \mu\left(g\left(X,Y\right) - \eta\left(X\right)\eta\left(Y\right)\right)$$

respectively. From the above three relations, we get $\left(\frac{r}{6\mu} + 1\right)S\left(X,Y\right) = 0$. Either $\frac{r}{6\mu} + 1 = 0$ or $S = 0$. If $\frac{r}{6\mu} + 1 = 0$, then $r = -6\mu$. Putting $\kappa = 0$ and $n = 1$ in (3.24), we get $r = -2\mu$. Thus $\frac{r}{6\mu} + 1 = 0$ is not possible. If $S = 0$, then in view of Theorem 1.1, we get $\mu = 0$, which is a contradiction. Thus, the Case IIIa is not possible.

Case IIIb. $\kappa = 0 \neq \mu$ and $n > 1$. Using $\kappa = 0$ in (3.23), (3.20), (3.25) we get

$$\begin{aligned} S\left(X,Y\right) &= \left(2\left(n-1\right) - n\mu\right)\left(g\left(X,Y\right) - \eta\left(X\right)\eta\left(Y\right)\right) \\ &+ \left(2\left(n-1\right) + \mu\right)g\left(hX,Y\right), \end{aligned}$$

$$rS\left(X,Y\right) = 2n\left(2n+1\right)\mu S\left(hX,Y\right),$$

$$\begin{aligned} S\left(hX,Y\right) &= \left(2\left(n-1\right) - n\mu\right)g\left(hX,Y\right) \\ &+ \left(2\left(n-1\right) + \mu\right)\left(g\left(X,Y\right) - \eta\left(X\right)\eta\left(Y\right)\right) \end{aligned}$$

respectively. From the above three equations, we get

$$S\left(X,Y\right) = a(g(X,Y) - \eta(X)\eta(Y))$$

for some suitable a. Now, in view of Theorem 1.3, we see that the Case IIIb is also not possible.

Case IV. $\kappa \neq 0 \neq \mu$.

Case IVa. $\kappa \neq 0 \neq \mu$ and $n = 1$. Putting $n = 1$ in (3.23), (3.20), (3.25), we get

$$S\left(X,Y\right) = -\mu g\left(X,Y\right) + \mu g\left(hX,Y\right) + \left(2\kappa + \mu\right)\eta\left(X\right)\eta\left(Y\right),$$

$$\left(\kappa - \frac{r}{6}\right) S(X,Y) = 2\kappa \left(\kappa - \frac{r}{6}\right) g(X,Y) + 2\kappa\mu g(hX,Y) - \mu S(hX,Y),$$

$$S(hX,Y) = -\mu g(hX,Y) - (\kappa - 1)\mu g(X,Y) + (\kappa - 1)\mu\eta(X)\eta(Y)$$

respectively. Eliminating $g(hX,Y)$ and $S(hX,Y)$ from the above three equations, we have

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

for some suitable a and b. Thus, M is an η-Einstein manifold. Since in the η-Einstein case $\mu = -2(n-1)$, therefore for $n = 1$, we get $\mu = 0$, which is a contradiction. Thus the Case IVa is not possible.

Case IVb. $\kappa \neq 0 \neq \mu$ and $n > 1$. After eliminating $g(hX,Y)$ and $S(hX,Y)$ from (3.23), (3.20) and (3.25); we get $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, for some suitable a and b. Hence, in view of Theorem 1.3, the Case IVb also does not exist. Thus the proof is complete. □

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