A locally symmetric Kähler Einstein structure on the cotangent bundle of a space form

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Abstract

We obtain a locally symmetric Kähler Einstein structure on the cotangent bundle of a Riemannian manifold of negative constant sectional curvature. Similar results are obtained on a tube around zero section in the cotangent bundle, in the case of a Riemannian manifold of positive constant sectional curvature. The obtained Kähler Einstein structures cannot have constant holomorphic sectional curvature.

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1 Introduction

The differential geometry of the cotangent bundle T^*M of a Riemannian manifold (M, g) is almost similar to that of the tangent bundle TM. However there are some differences, because the lifts (vertical, complete, horizontal etc.) to T^*M cannot be defined just like in the case of TM.

In [8] V. Oproiu and the present author have obtained a natural Kähler Einstein structure (G, J) of diagonal type induced on T^*M from the Riemannian metric g. The obtained Kähler structure on T^*M, depends on one essential parameter u, which is a smooth function depending on the energy density t on T^*M. If the Kähler structure is Einstein they get a second order differential equation fulfilled by the parameter u. In the case of the general solution, they have obtained that (T^*M, G, J) has constant holomorphic sectional curvature.

In this paper we study the singular case where the parameter $u = A, A \in \mathbf{R}$. The considered natural Riemannian metric G of diagonal type on T^*M is defined by using one parameter v which is a smooth function depending on the energy density t on T^*M. The vertical distribution VT^*M and the horizontal distribution HT^*M are orthogonal to each other but the dot products induced on them from G are not isomorphic (isometric).

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Next, the natural almost complex structures J on T^*M that interchange the vertical and horizontal distributions depends of one essential parameter v.

After that, we obtain that G is Hermitian with respect to J and it follows that the fundamental 2-form ϕ, associated to the almost Hermitian structure (G, J) is the fundamental form defining the usual symplectic structure on T^*M, hence it is closed.

From the integrability condition for J it follows that the base manifold M must have constant sectional curvature c and the parameter v must be constant.

If the constant sectional curvature c is negative then we obtain a locally symmetric Kähler Einstein structure defined on the whole T^*M. If the constant sectional curvature c of M is positive then we get a similar structure defined on a tube around zero section in T^*M. The Kähler Einstein manifolds obtained cannot have constant holomorphic sectional curvature.

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates but many results from this paper may be expressed in an invariant form. The well known summation convention is used throughout this paper, the range for the indices h, i, j, k, l, r, s being always$\{1, ..., n\}$ (see [7], [11]). We shall denote by $\Gamma(T^*M)$ the module of smooth vector fields on T^*M.

2 Some geometric properties of T^*M

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its cotangent bundle by $\pi : T^*M \longrightarrow M$. Recall that there is a structure of a $2n$-dimensional smooth manifold on T^*M, induced from the structure of smooth n-dimensional manifold of M. From every local chart $(U, \varphi) = (U, x^1, \dots, x^n)$ on M, it is induced a local chart $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \dots, q^n, p_1, \dots, p_n)$, on T^*M, as follows. For a cotangent vector $p \in \pi^{-1}(U) \subset T^*M$, the first n local coordinates $q^1, \dots, q^n$ are the local coordinates $x^1, \dots, x^n$ of its base point $x = \pi(p)$ in the local chart (U, φ) (in fact we have $q^i = \pi^* x^i = x^i \circ \pi,\ i = 1, \dots n)$. The last n local coordinates $p_1, \dots, p_n$ of $p \in \pi^{-1}(U)$ are the vector space coordinates of p with respect to the natural basis $(dx^1_{\pi(p)}, \dots, dx^n_{\pi(p)})$, defined by the local chart (U, φ), i.e. $p = p_i dx^i_{\pi(p)}$.

An M-tensor field of type (r, s) on T^*M is defined by sets of n^{r+s} components (functions depending on q^i and p_i), with r upper indices and s lower indices, assigned to induced local charts $(\pi^{-1}(U), \Phi)$ on T^*M, such that the local coordinate change rule is that of the local coordinate components of a tensor field of type (r, s) on the base manifold M (see [2] for further details in the case of the tangent bundle). An usual tensor field of type (r, s) on M may be thought of as an M-tensor field of type (r, s) on T^*M. If the considered tensor field on M is covariant only, the corresponding M-tensor field on T^*M may be identified with the induced (pullback by π) tensor field on T^*M.

Some useful M-tensor fields on T^*M may be obtained as follows. Let v, w : $[0, \infty) \longrightarrow \mathbf{R}$ be a smooth functions and let $\|p\|^2 = g^{-1}_{\pi(p)}(p, p)$ be the square of the norm of the cotangent vector $p \in \pi^{-1}(U)$ (g^{-1} is the tensor field of type (2,0) having the components $(g^{kl}(x))$ which are the entries of the inverse of the matrix $(g_{ij}(x))$ defined by the components of g in the local chart (U, φ)). The components $g_{ij}(\pi(p))$, p_i, $v(\|p\|^2)p_i p_j$ define M-tensor fields of types $(0, 2)$, $(0, 1)$, $(0, 2)$ on T^*M, respectively. Similarly, the components $g^{kl}(\pi(p))$, $g^{0i} = p_h g^{hi}$, $w(\|p\|^2)g^{0k}g^{0l}$ define M-tensor fields of type $(2,0)$, $(1,0)$, $(2,0)$ on T^*M, respectively. Of course, all the components considered above are in the induced local chart $(\pi^{-1}(U), \Phi)$.

The Levi Civita connection $\dot\nabla$ of g defines a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M. \tag{2.1}$$

of the tangent bundle to T^*M into vertical distributions $VT^*M = \text{Ker } \pi_*$ and the horizontal distribution HT^*M.

If $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$ is a local chart on T^*M, induced from the local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$, the local vector fields $\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}$ on $\pi^{-1}(U)$ define a local frame for VT^*M over $\pi^{-1}(U)$ and the local vector fields $\frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n}$ define a local frame for HT^*M over $\pi^{-1}(U)$, where

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma^0_{ih}\frac{\partial}{\partial p_h}, \quad \Gamma^0_{ih} = p_k \Gamma^k_{ih}$$

and $\Gamma^k_{ih}(\pi(p))$ are the Christoffel symbols of g.

The set of vector fields $(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n})$ defines a local frame on T^*M, adapted to the direct sum decomposition (1).

We consider

$$t = \frac{1}{2}\|p\|^2 = \frac{1}{2}g^{-1}_{\pi(p)}(p,p) = \frac{1}{2}g^{ik}(x)p_i p_k, \quad p \in \pi^{-1}(U) \tag{2.2}$$

the energy density defined by g in the cotangent vector p. We have $t \in [0, \infty)$ for all $p \in T^*M$.

From now on we shall work in a fixed local chart (U, φ) on M and in the induced local chart $(\pi^{-1}(U), \Phi)$ on T^*M.

3 An almost Kähler structure on T^*M

Consider a real valued smooth function v defined on $[0, \infty) \subset \mathbf{R}$ and a real constant A. We define the following symmetric M-tensor field of type $(0,2)$ on T^*M having the components

$$G_{ij}(p) = Ag_{ij}(\pi(p)) + v(t)p_i p_j. \tag{3.3}$$

It follows easily that the matrix (G_{ij}) is positive definite if and only if $A > 0$, $A + 2tv > 0$. The inverse of this matrix has the entries

$$H^{kl}(p) = \frac{1}{A}g^{kl}(\pi(p)) + w(t)g^{0k}g^{0l}, \tag{3.4}$$

where

$$w = -\frac{v}{A(A+2tv)}. \tag{3.5}$$

The components H^{kl} define a symmetric M-tensor field of type $(2,0)$ on T^*M.

Remark. If the matrix (G_{ij}) is positive definite, then its inverse (H^{kl}) is positive definite too.

Using the M-tensor fields defined by $G_{ij},\ H^{kl}$, the following Riemannian metric may be considered on T^*M

$$G = G_{ij}dq^i dq^j + H^{ij}Dp_i Dp_j, \tag{3.6}$$

where $Dp_i = dp_i - \Gamma^0_{ij}dq^j$ is the absolute (covariant) differential of p_i with respect to the Levi Civita connection $\dot{\nabla}$ of g . Equivalently, we have

$$G(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j}) = G_{ij},\quad G(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}) = H^{ij},\quad G(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j}) = \ G(\frac{\delta}{\delta q^j},\frac{\partial}{\partial p_i}) = 0.$$

Remark that $HT^*M,\ VT^*M$ are orthogonal to each other with respect to G, but the Riemannian metrics induced from G on $HT^*M,\ VT^*M$ are not the same, so the considered metric G on T^*M is not a metric of Sasaki type. Remark also that the system of 1-forms $(dq^1,...,dq^n,Dp_1,...,Dp_n)$ defines a local frame on T^*T^*M, dual to the local frame $(\frac{\delta}{\delta q^1},\dots,\frac{\delta}{\delta q^n},\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_n})$ adapted to the direct sum decomposition (1).

Next, an almost complex structure J is defined on T^*M by the same M-tensor fields $G_{ij},\ H^{kl}$, expressed in the adapted local frame by

$$J\frac{\delta}{\delta q^i} = G_{ik}\frac{\partial}{\partial p_k},\quad J\frac{\partial}{\partial p_i} = -H^{ik}\frac{\delta}{\delta q^k} \tag{3.7}$$

From the property of the M-tensor field H^{kl} to be defined by the inverse of the matrix defined by the components of the M-tensor field G_{ij}, it follows easily that J is an almost complex structure on T^*M.

Theorem 1. *(T^*M,G,J) is an almost Kähler manifold.*

Proof. Since the matrix (H^{kl}) is the inverse of the matrix (G_{ij}), it follows easily that

$$G(J\frac{\delta}{\delta q^i},J\frac{\delta}{\delta q^j}) = G(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j}),\quad G(J\frac{\partial}{\partial p_i},J\frac{\partial}{\partial p_j}) = G(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}),$$

$$G(J\frac{\partial}{\partial p_i},J\frac{\delta}{\delta q^j}) = G(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j}) = 0.$$

Hence

$$G(JX,JY) = G(X,Y),\quad \forall\ X,Y\in\Gamma(T^*M).$$

Thus (T^*M,G,J) is an almost Hermitian manifold.

The fundamental 2-form associated with this almost Hermitian structure is ϕ, defined by

$$\phi(X,Y) = G(X,JY),\quad \forall\ X,Y\in\Gamma(T^*M).$$

By a straightforward computation we get

$$\phi(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j})=0,\quad \phi(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j})=0,\quad \phi(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j})=\delta^i_j.$$

Hence

$$\phi = Dp_i\wedge dq^i = dp_i\wedge dq^i, \tag{3.8}$$

due to the symmetry of $\Gamma^0_{ij}=p_h\Gamma^h_{ij}$. It follows that ϕ does coincide with the fundamental 2-form defining the usual symplectic structure on T^*M. Of course, we have $d\phi=0$, i.e. ϕ is closed. Therefore (T^*M,G,J) is an almost Kähler manifold.

4 A Kähler structure on T^*M

We shall study the integrability of the almost complex structure defined by J on T^*M. To do this we need the following well known formulas for the brackets of the vector fields $\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^i},\ i=1,...,n$

$$[\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}]=0,\quad [\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j}]=\Gamma^i_{jk}\frac{\partial}{\partial p_k},\quad [\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j}]=R^0_{kij}\frac{\partial}{\partial p_k}, \tag{4.9}$$

where $R^h_{kij}(\pi(p))$ are the local coordinate components of the curvature tensor field of $\dot\nabla$ on M and $R^0_{kij}(p)=p_hR^h_{kij}$. Of course, the components R^0_{kij}, R^h_{kij} define M-tensor fields of types (0,3), (1,3) on T^*M, respectively.

Theorem 2. *The Nijenhuis tensor field of the almost complex structure J on T^*M is given by*

$$\begin{cases} N(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j})=-\{Av(\delta^h_i g_{jk}-\delta^h_j g_{ik})+R^h_{kij}\}p_h\frac{\partial}{\partial p_k},\\[2mm] N(\frac{\delta}{\delta q^i},\frac{\partial}{\partial p_j})=-H^{kl}H^{jr}\{Av(\delta^h_i g_{rl}-\delta^h_r g_{il})+R^h_{lir}\}p_h\frac{\delta}{\delta q^k},\\[2mm] N(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j})=-H^{ir}H^{jl}\{Av(\delta^h_l g_{rk}-\delta^h_r g_{lk})+R^h_{klr}\}p_h\frac{\partial}{\partial p_k}.\end{cases} \tag{4.10}$$

Proof. Recall that the Nijenhuis tensor field N defined by J is given by

$$N(X,Y)=[JX,JY]-J[JX,Y]-J[X,JY]-[X,Y],\ \ \forall\ \ X,Y\in\Gamma(T^*M).$$

Then, we have $\frac{\delta}{\delta q^k}t=0,\ \frac{\partial}{\partial p_k}t=g^{0k}$ and $\dot\nabla_iG_{jk}=0,\ \dot\nabla_iH^{jk}=0$, where

$$\dot\nabla_iG_{jk}=\frac{\delta}{\delta q^i}G_{jk}-\Gamma^l_{ij}G_{lk}-\Gamma^l_{ik}G_{lj}$$

$$\dot\nabla_iH^{jk}=\frac{\delta}{\delta q^i}H^{jk}+\Gamma^j_{il}H^{lk}+\Gamma^k_{il}H^{lj}$$

The above expressions for the components of N can be obtained by a quite long, straightforward computation.

Theorem 3. *The almost complex structure J on T^*M is integrable if and only if the base manifold M has constant sectional curvature c and the function v is given by*

$$v = -\frac{c}{A}. \tag{4.11}$$

Proof. From the condition $N = 0$, one obtains

$$\{Av(\delta_i^h g_{jk} - \delta_j^h g_{ik}) + R_{kij}^h\}p_h = 0.$$

Differentiating with respect to p_l, taking $p_h = 0\ \forall h \in \{1, ..., n\}$, it follows that the curvature tensor field of $\dot{\nabla}$ has the expression

$$R_{kij}^l = -Av(0)(\delta_i^l g_{jk} - \delta_j^l g_{ik}).$$

Using by the Schur theorem(in the case where M is connected and $dimM \geq 3$) it follows that (M, g) has the constant sectional curvature $c = -Av(0)$. Then we obtain the expression (11) of v.

Conversely, if (M, g) has constant curvature c and v is given by (11), it follows in a straightforward way that $N = 0$.

Remark. The function v must fulfill the condition

$$A + 2tv = \frac{A^2 - 2ct}{A} > 0, \quad A > 0. \tag{4.12}$$

If $c < 0$ then (T^*M, G, J) is a Kähler manifold.

If $c > 0$ then (T_A^*M, G, J) is a Kähler manifold, where T_A^*M is the tube around zero section in T^*M defined by the condition $0 \leq \|p\|^2 < \frac{A^2}{c}$.

If $c = 0$ then (M, g) is a flat manifold and G becomes a flat metric on T^*M.

The components of the Kähler metric G on T^*M are

$$\begin{cases} G_{ij} = Ag_{ij} - \frac{c}{A}p_i p_j, \\ H^{ij} = \frac{1}{A}g^{ij} + \frac{c}{A(A^2-2ct)}g^{0i}g^{0j}. \end{cases} \tag{4.13}$$

5 A Kähler Einstein structure on T^*M

In this section we shall study the property of the Kähler manifold (T^*M, G, J) to be Einstein.

The Levi Civita connection ∇ of the Riemannian manifold (T^*M, G) is determined by the conditions

$$\nabla G = 0, \qquad T = 0,$$

where T is its torsion tensor field. The explicit expression of this connection is obtained from the formula

$$2G(\nabla_X Y, Z) = X(G(Y, Z)) + Y(G(X, Z)) - Z(G(X, Y)) +$$

$$+G([X, Y], Z) - G([X, Z], Y) - G([Y, Z], X); \qquad \forall\ X, Y, Z \in \Gamma(T^*M).$$

The final result can be stated as follows.

Theorem 4. *The Levi Civita connection ∇ of G has the following expression in the local adapted frame* $(\frac{\delta}{\delta q^1},\dots,\frac{\delta}{\delta q^n},\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_n})$:

$$(5.14)\qquad \begin{cases} \nabla_{\frac{\partial}{\partial p_i}}\frac{\partial}{\partial p_j} = Q^{ij}_h\frac{\partial}{\partial p_h}, & \nabla_{\frac{\delta}{\delta q^i}}\frac{\partial}{\partial p_j} = -\Gamma^j_{ih}\frac{\partial}{\partial p_h} + P^{hj}_i\frac{\delta}{\delta q^h},\\ \nabla_{\frac{\partial}{\partial p_i}}\frac{\delta}{\delta q^j} = P^{hi}_j\frac{\delta}{\delta q^h}, & \nabla_{\frac{\delta}{\delta q^i}}\frac{\delta}{\delta q^j} = \Gamma^h_{ij}\frac{\delta}{\delta q^h} + S_{hij}\frac{\partial}{\partial p_h},\end{cases}$$

where $Q^{ij}_h, P^{hi}_j, S_{hij}$ are M-tensor fields on T^*M, defined by

$$(5.15)\qquad \begin{cases} Q^{ij}_h = \frac{1}{2}G_{hk}(\frac{\partial}{\partial p_i}H^{jk} + \frac{\partial}{\partial p_j}H^{ik} - \frac{\partial}{\partial p_k}H^{ij}),\\ P^{hi}_j = \frac{1}{2}H^{hk}(\frac{\partial}{\partial p_i}G_{jk} - H^{il}R^0_{ljk}),\\ S_{hij} = -\frac{1}{2}G_{hk}\frac{\partial}{\partial p_k}G_{ij} + \frac{1}{2}R^0_{hij}.\end{cases}$$

After replacing of the expressions of the involved M-tensor fields , we obtain

$$(5.16)\qquad Q^{ij}_h = \frac{c}{A}H^{ij}p_h, \qquad P^{hi}_j = -\frac{c}{A}H^{hi}p_j, \qquad S_{hij} = \frac{c}{A}G_{hj}p_i.$$

The curvature tensor field K of the connection ∇ is obtained from the well known formula

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad \forall\ X,Y,Z \in \Gamma(T^*M).$$

The components of curvature tensor field K with respect to the adapted local frame $(\frac{\delta}{\delta q^1},\dots,\frac{\delta}{\delta q^n},\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_n})$ are obtained easily:

$$(5.17)\qquad \begin{cases} K(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j})\frac{\delta}{\delta q^k} = \frac{c}{A}(\delta^h_i G_{jk} - \delta^h_j G_{ik})\frac{\delta}{\delta q^h},\\ K(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j})\frac{\partial}{\partial p_k} = \frac{c}{A}(\delta^k_j G_{hi} - \delta^k_i G_{hj})\frac{\partial}{\partial p_h},\\ K(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j})\frac{\delta}{\delta q^k} = \frac{c}{A}(\delta^j_k H^{hi} - \delta^i_k H^{hj})\frac{\delta}{\delta q^h},\\ K(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j})\frac{\partial}{\partial p_k} = \frac{c}{A}(\delta^i_h H^{jk} - \delta^j_h H^{ik})\frac{\partial}{\partial p_h},\\ K(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j})\frac{\delta}{\delta q^k} = \frac{c}{A}\delta^i_j G_{hk}\frac{\partial}{\partial p_h},\\ K(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j})\frac{\partial}{\partial p_k} = -\frac{c}{A}\delta^i_j H^{hk}\frac{\delta}{\delta q^h}.\end{cases}$$

Remark. From the local coordinates expressions of the curvature tensor field K we obtain that the Kähler manifold (T^*M,G,J) cannot have constant holomorphic sectional curvature.

The Ricci tensor field Ric of ∇ is defined by the formula:

$$Ric(Y,Z) = trace(X \longrightarrow K(X,Y)Z), \quad \forall\, X,Y,Z \in \Gamma(T^*M).$$

It follows

$$\begin{cases} Ric(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) = \frac{cn}{A} G_{ij}, \\ Ric(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = \frac{cn}{A} H^{ij}, \\ Ric(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}) = Ric(\frac{\delta}{\delta q^j}, \frac{\partial}{\partial p_i}) = 0. \end{cases}$$

Thus

$$Ric = \frac{cn}{A} G. \tag{5.18}$$

By using the expressions (17), we have computed the covariant derivatives of curvature tensor field K in the local adapted frame $(\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_i})$ with respect to the connection ∇ and we obtained in all twelve cases the result is zero.

Hence we may state our main result.

Theorem 5. *Assume that the Riemannian manifold (M,g) has constant sectional curvature c, the conditions (12) are fulfilled and the components of the metric G are given by (13).*

1. If $c < 0$ then (T^*M, G, J) is a locally symmetric Kähler Einstein manifold.

2. If $c > 0$ then $(T^*_A M, G, J)$ is a locally symmetric Kähler Einstein manifold.

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