Examples of parallel spin-tensors

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Abstract

Let (M, g) be a Riemannian, oriented, spin manifold. The existence of the parallel spinors (that is the spinors with vanishing covariant derivative) imposes the conditions over the base space M. In the *space of spin-tensors,* we prove that there are parallel spin-tensors for every connection ∇.

Key words: spin manifold, parallel spin-tensors.
Mathematics Subject Classification: 55R10, 53C27.

1 Introduction

Recall that there exists a linear irreducible complex representation of the Clifford algebra $Cl_n, n = 2m$, over the complex vector space $\Sigma = \mathbf{C}^{2^m}$, which makes the group $Spin_n$ as a subgroup of $SU(2^m)$. For $m \geq 2$, let $p : Spin_n \to SO(n)$ be the universal covering homomorphism with kernel $\mathbf{Z}_2$.

Let (M, g) be a spin manifold of dimension $n = 2m, m \geq 2$, and $f : SpinM \to SO(M)$ a spin structure over M. Recall that $SO(M)$ is the principal fibre bundle of oriented orthogonal frames of (M, g) and $SpinM$ is a principal fibre bundle over M with the structure group $Spin_n$, f being a fibre bundle homomorphism which restricted to the fibres corresponds to the covering p [4].

Let $S(M)$ be the space of spinors of M, which is the vector fibre bundle over M associated with the principal fibre bundle $SpinM$ with the fibre type Σ. Each section of this fibre bundle $S(M)$ is by definition *a spinor.*

Every connection ∇ of the principal fibre bundle $SO(M)$ (in the particular case, the Levi-Civita connection of the Riemannian manifold (M, g)) induces a connection on the principal fibre bundle $SpinM$ [4], [2] and, consequently, a covariant derivative in the *space of spinors.* In the general case, the existence of the parallel spinors (that is, the spinors with vanishing covariant derivative) imposes the conditions over the base space M [1]

In the *space of spin-tensors* [3] we prove that there are parallel spin-tensors for every connection ∇.

Balkan Journal of Geometry and Its Applications, Vol.9, No.2, 2004, pp. 83-86.

2 The space of spin-tensors

Let $\{U_\alpha\}_{\alpha\in A}$ be a trivialization covering of the principal fibre bundles $SpinM$ and $SO(M)$. Denote respectively by $\{s_{\alpha\beta}\}_{\alpha,\beta\in A}$ and $\{t_{\alpha\beta}\}_{\alpha,\beta\in A}$ the corresponding transition functions of $SpinM$ and $SO(M)$, respectively. Let W be a vector space and W^* its dual space. For every $A\in GL(W)$ we denote by $A^*\in GL(W^*)$ the dual of A^{-1}.

Let $su(2^m)$ be the Lie algebra of the special unitary group $SU(2^m)$. The special orthogonal group $SO(n), n=2m$, acts on the vector space $su(2^m)$ as follows. For each $B\in SO(n)$, there exists $\pm A\in Spin_n\subseteq SU(2^m)$ such that $p(\pm A)=B$. By definition, for every $X\in su(2^m)$,

$$B(X)=AXA^{-1}. \tag{2.1}$$

This linear representation of the group $SO(n)$ will be denoted by ρ.

Let P_{ijhk} be the principal fibre bundle with the base space M, and the transition functions ${s_{\alpha\beta}}^{[i]}\otimes {s_{\alpha\beta}}^{*[j]}\otimes {t_{\alpha\beta}}^{[h]}\otimes {t_{\alpha\beta}}^{*[k]}$ corresponding to the open covering $\{U_\alpha\}_{\alpha\in A}$. We consider $E(P_{ijhk})$ its associated fibre bundle with standard fibre $\mathbf{C}^{2^m\,[i]}\otimes\mathbf{C}^{2^m*[j]}\otimes su(2^m)^{[h]}\otimes su(2^m)^{*[k]}$.

By definition, a *spin-tensor of kind* (i,j,h,k) *of* M is a section of $E(P_{ijhk})$.

Let $\{e_i\}_i$ be the canonical basis of $\Sigma=\mathbf{C}^{2^m}$ and let $\{E_a\}_a$ be a basis of $su(2^m)$. If B is the Killing form on $SU(2^m)$,

$$B(X,Y)=tr(XY),(\forall)X,Y\in su(2^m),$$

then we consider for every a,b the real number $B_{ab}=B(E_a,E_b)$. Let $(B^{ab})_{a,b}$ be the inverse matrix of $(B_{ab})_{ab}$. We have for all a,i,j,k,l :

$$E^i_{aj}E^{ak}_l=\delta^i_l\delta^k_j, \tag{2.2}$$

where E^i_{aj} and E^{ai}_j is the element of line i and column j of the matrix E_a and E^a respectively.

The formula (2.1) shows that for every $g\in Spin_n$, we have:

$$p(g)X=gXg^{-1},(\forall)X\in su(2^m), \tag{2.3}$$

and therefore

$$p_*(A)X=[A,X],(\forall)A\in spin_n. \tag{2.4}$$

3 Some examples of parallel spin-tensors

We preserve the previous notations.

Proposition 3.1. *For every $\alpha\in A$, we consider the map*

$$\varphi_\alpha:U_\alpha\to\mathbf{C}^{2^m}\otimes\mathbf{C}^{2^m*}\otimes su(2^m),\varphi_\alpha(x)=E^{ai}_je_i\otimes e^j\otimes E_a.$$

Then: i) The family $\{\varphi_\alpha\}_{\alpha\in A}$ defines a constant spin-tensor φ of kind $(1,1,1,0)$; ii) The spin-tensor φ is parallel relative to every connection ∇ of the principal fibre bundle $SO(M)$ (in particular, relative to the Levi-Civita connection).

Proof. i) We recall (see e.g., [2]) that the family of applications $\{\varphi_\alpha\}_{\alpha\in A}$ defines a spin-tensor of kind $(1,1,1,0)$ if and only if

$$\varphi_\beta(x) = S_{\beta\alpha}(x)\varphi_\alpha(x), (\forall)\alpha,\beta \in A, x \in U_\alpha \cap U_\beta,$$

where

$$S_{\beta\alpha}(x) = s_{\alpha\beta}(x) \otimes {s_{\alpha\beta}}^*(x) \otimes t_{\alpha\beta}(x).$$

Therefore, we need to prove that

$$(s_{\alpha\beta}(x) \otimes {s_{\alpha\beta}}^*(x) \otimes t_{\alpha\beta}(x))(E^{ai}_j e_i \otimes e^j \otimes E_a) = E^{ai}_j e_i \otimes e^j \otimes E_a. \tag{3.5}$$

Indeed

$$(s_{\alpha\beta}(x) \otimes {s_{\alpha\beta}}^*(x) \otimes t_{\alpha\beta}(x))(E^{ai}_j e_i \otimes e^j \otimes E_a) =$$
$$= E^{ai}_j (s_{\alpha\beta}(x))^h_i (s_{\beta\alpha}(x))^j_k (t_{\alpha\beta}(x))^b_a (e_h \otimes e^k \otimes E_b).$$

But, on the other side, since $t_{\alpha\beta}(x) = p(s_{\alpha\beta}(x))$, we may write using (2.1)

$$(t_{\alpha\beta}(x))^b_a E_b = s_{\beta\alpha}(x) E_a s_{\alpha\beta}(x)$$

and this means that for all c

$$tr((t_{\alpha\beta}(x))^b_a E_b E^c) = tr(s_{\beta\alpha}(x) E_a s_{\alpha\beta}(x) E^c).$$

We have equivalently

$$t_{\alpha\beta}(x)^b_a = tr(s_{\beta\alpha}(x) E_a s_{\alpha\beta}(x) E^b).$$

Using this formula and (2.2), we obtain that (3.5) is true.

ii) Let $\{\omega_\alpha\}_{\alpha\in A}$ be the family of 1-forms of the connection ∇ and let $\{\theta_1, ..., \theta_n\}$ be a local field of frames on U_α. We have $\varphi_{\alpha *}(\theta_r) = 0$ for all $r = 1, ..., n$, since the spin-tensor φ is constant. Then, by definition, for all $x \in U_\alpha$ [2]

$$(\nabla_{\theta_r}\varphi)_\alpha(x) =$$
$$E^{ai}_j \omega_\alpha(\theta_r)^s_i e_s \otimes e^j \otimes E_a - E^{ai}_j e_i \otimes \omega_\alpha(\theta_r)^j_k e^k \otimes E_a + E^{ai}_j e_i \otimes e^j \otimes p_*(\omega_\alpha(\theta_r)) E_a =$$
$$\{[\omega_\alpha(\theta_r), E_a]^i_j + E^{bi}_j \omega_\alpha(\theta_r)^q_p [E_b, E^a]^p_q\} e_i \otimes e^j \otimes E_a.$$

Using (2.2), we obtain $(\nabla_{\theta_r}\varphi)_\alpha(x) = 0$. □

Similar considerations prove that the family $\{\psi_\alpha\}_{\alpha\in A}$ given by

$$\psi_\alpha : U_\alpha \to \mathbf{C}^{2^m} \otimes \mathbf{C}^{2^m *} \otimes su(2^m)^*, \psi_\alpha(x) = E^i_{aj} e_i \otimes e^j \otimes E^a,$$

or the family $\{\chi_\alpha\}_{\alpha\in A}$

$$\chi_\alpha : U_\alpha \to \mathbf{C}^{2^m} \otimes \mathbf{C}^{2^m *}, \chi_\alpha(x) = e_i \otimes e^i,$$

or the family $\{\sigma_\alpha\}_{\alpha\in A}$

$$\sigma_\alpha : U_\alpha \to su(2^m) \otimes su(2^m)^*, \sigma_\alpha = E_a \otimes E^a$$

define a constant spin-tensor (of kind $(1,1,0,1), (1,1,0,0), (0,0,1,1)$ respectively) which is parallel relative to every connection ∇ of the principal fibre bundle $SO(M)$.

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