Submanifold Dirac operators with torsion

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Abstract

The submanifold Dirac operator has been studied for this decade, which is closely related to Frenet-Serret and generalized Weierstrass relations. In this article, we will give a submanifold Dirac operator defined over a surface immersed in $\mathbb{E}^4$ with U(1)-gauge field as torsion in the sense of the Frenet-Serret relation, which also has data of immersion of the surface in $\mathbb{E}^4$.

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1 Introduction

The submanifold quantum mechanics was opened by Jensen and Koppe [8] and de Casta [4]. The submanifold Dirac equations were studied in [3, 11, 12, 13, 14, 15, 20, 21], which are closely related to recent movements in differential geometry. The same Dirac operator as one in [15] appeared in a lecture by Pinkall in ICM of 1998 [22], in which conformal surfaces in the Euclidean space were studied using the Dirac operator. The related Dirac equation is known as the generalized Weierstrass relation in the category of differential geometry [10, 22, 24].

Recently, this author gave an algebraic construction of the submanifold quantum mechanics, which exhibits nature of submanifold [19, 20].

On the other hand, for a space curve in three dimensional Euclidean space $\mathbb{E}^3$, Takagi and Tanzawa found a submanifold Schrödinger operator with a gauge field [25],

$$\mathcal{S} := -(\partial_s - \sqrt{-1}a)^2 - \frac{1}{4}k^2, \tag{1.1}$$

whereas the original one in [4, 8] is given by

$$\mathcal{S} := -\partial_s^2 - \frac{1}{4}k^2,$$

where k is a curvature of the curve. The existence of the gauge field a is due to the fact that the codimension is two.

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In this article, we will generalize the Takagi-Tanzawa Schrödinger operator (1.1) to the submanifold Dirac operator over a surface in $\mathbb{E}^4$ using the algebraic construction.

After giving a geometrical setting of our system in §2, and explaining our conventions of the system of the Dirac equations in §3, we will define the submanifold Dirac equation following the algebraic construction in §4. As the submanifold Dirac equation is on a surface in $\mathbb{E}^4$ closely related to the generalized Weierstrass relation in [10, 15, 22], we will show Theorem 4.1 as the generalized Weierstrass relation with another proof based upon [20]. Further in §5, we will introduce another submanifold Dirac operator which has a gauge field associated with the torsion in the sense of the Frenet-Serret relation. We will provide Theorem 5.1 which is also connected with the generalized Weierstrass relation.

Before finishing the Introduction, we will comment on our theory from viewpoints of mathematics.

Theorem 4.1 can be easily extended to more general $k-$spin submanifold in n dimensional Euclidean space $\mathbb{E}^n$ as shown in [20]. However since [20] is somewhat complicate due to general dimensionality, in this article, we will restrict ourselves to four dimensional theory in order to make theory simple and study the effects from the torsion in the case of the codimention $n-k>1$ or $(n=4, k=2)$.

However due to the potentiality, our theory might be even complicate for differential geometers by comparing with the theories of [10, 22]. Thus we will give an answer of a question why we persist the submanifold Dirac operators.

One of our motivations on this study is to construct a continuous variant of Frobenius reciprocity for linear representation of subgroup in a linear representation theory of finite group [23]. This article is a first step from [20]. For class functions φ and ψ over a finite group G and over its subgroup H respectively, the Frobenius reciprocity is given by,

$$\langle \mathrm{Res}\varphi, \psi\rangle_H = \langle \varphi, \mathrm{Ind}\psi\rangle_G \tag{1.2}$$

for a certain restriction map Res, a map of an induced representation Ind and pairings $\langle\rangle_G$ and $\langle\rangle_H$ defined over G and H respectively [23]. On the other hand, our Theorem 4.1 and 5.1 can be expressed by following: For a point pt in a surface S in $\mathbb{E}^4$ and for a spinor field ψ over S satisfying a certain Dirac equation $\not{D}\psi = 0$, we find a spinor field φ in $S \subset \mathbb{E}^4$ using data of $\mathbb{E}^4$ and the relation,

$$\langle \varphi, \psi\rangle_S = \langle \varphi, \psi\rangle_{\mathbb{E}^4} \quad \text{at} \quad pt, \tag{1.3}$$

for pointwise bilinear forms at pt, $\langle \varphi, \psi\rangle_S$ and $\langle \varphi, \psi\rangle_{\mathbb{E}^4}$, properly defined over S and $\mathbb{E}^4$ respectively. Here using a natural spin representation of SO(4), $\langle \varphi, \psi\rangle_{\mathbb{E}^4}$ gives a germ of sheaf of $T^*\mathbb{E}^4$ which express the tangential component $T^*S \subset T^*\mathbb{E}^4|_S$ at the point pt. In other words, the solution ψ of the differential equation and the pairing $\langle,\rangle_S$ at pt gives the local data of embedding or immersion of S into $\mathbb{E}^4$. These theorems give the generalized Weierstrass relation and its essentials. Every bilinear representation of SO(4) is uniquely identified with a spin representation and the spinor fields can be characterized by kernel of a Dirac operator. Thus we have searched such a Dirac operator whose kernel has the data of immersions for this decade [20]. Simultaneously the fact gives answers of questions why the Dirac operator appears in generalized Weirestrass relation and quaternion expresses the case of $n = 4$. It implies that our theory is natural and should contrast with others. Further we note that $\not{D}$ is constructed by restriction manipulations, which might be related to Res in the group theory [23]. This author recognizers our theory as as a continuous variant of the Frobenius reciprocity and wishes to extend these schemes to more general situations than [20]. This article is its first attempt.

Further we have studied more simple submanifold system, loops in $\mathbb{E}^2$, for this decade using the submanifold Dirac operator, spinor representation of operator in Frenet-Serret relation for the curvature k and the arclength s,

$$\not{D} = \begin{pmatrix} \partial_s & k/2 \\ -k/2 & \partial_s \end{pmatrix}. \tag{1.4}$$

This Dirac operator connects with the various mathematics, such as hyperelliptic function theory, D-module theory, automorphic function theory and so on [17, 18]. The Dirac operator plays contributes classification of loop space of $\mathbb{E}^2$ in the category of the differential geometry, [17, 18]. For $\not{D}\psi = 0$, we rewrite it by

$$\begin{pmatrix} \partial_s & \\ & \partial_s \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} & -k/2 \\ -k/2 & \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{1.5}$$

the left hand side is in the category of differential ring whereas the right hand is in a category of function space. If we restrict the right hand side by a holomorphic function space, we encounter the hyperelliptic functions, via the Korteweg-de Vries or the modified Korteweg-de Vries equation. The half of the curvature is connected with the Weierstrass hyperelliptic al functions and might be related to modular function theory as mentioned in [17]. The genus of hyperelliptic functions which the Dirac operator brings us are connected to an infinite dimensional manifold [18]. Further as mentioned in [11, 12], it leads us to index theorems.

Thus it is expected that higher dimensional variant is related to classification of immersions of submanifolds in $\mathbb{E}^n$ as partially mentioned in [16] for the case of $(k, n) = (2, 3)$. From the viewpoint, it is natural to have a question how the internal group appears for the case of codimension $n - k > 1$. Thus we focus on the case of a surface in $\mathbb{E}^4$, $(n - k = 2)$.

2 Geometrical preliminaries

In this section, we will give a geometrical preliminary. Though it is not difficult to extend our theory more general, we will concentrate our attention on a case of a smooth surface S embedded in four dimensional Euclidean space $\mathbb{E}^4$, $i : S \hookrightarrow \mathbb{E}^4$. We identify $i(S)$ with S. Since our theory is local, we construct our theory over $S \cap U$ instead of S itself for an appropriate open set $U \subset \mathbb{E}^4$ so that $S \cap U$ is topologically trivial and its closure is compact in $\mathbb{E}^4$. For simplicity by replacing S with its piece $S \cap U$, we assume that S is homeomorphic to $\mathbb{R}^2$ and is in a compact subspace in $\mathbb{E}^4$ hereafter.

We fix the notations of the Cartesian coordinates in $\mathbb{E}^4$ by $x := (x^1, x^2, x^3, x^4)$. Let S be locally expressed by real parameters (s^1, s^2). Let a tubular neighborhood of S be denoted by T_S, $\pi_{T_S} : T_S \to S$. Due to the above assumptions, T_S is homeomorphic to $\mathbb{R}^4$. Let $q := (q^3, q^4)$ be a normal coordinate of T_S whose absolute value $\sqrt{(q^3)^2 + (q^4)^2}$

is the distance from the surface S; $dq^{\dot\alpha}$ belongs to kernel of $\pi_{T_S *}$ and $dq^{\dot\alpha}(\partial_{s^\alpha}) = 0$, $(\alpha = 1,2, \dot\alpha = 3,4)$: $\partial_\alpha := \partial_{s^\alpha} \equiv \partial/\partial s^\alpha$. The normal bundle $T_{pt}^\perp S$ is given by $T_{pt}\mathbb{E}^4/T_{pt}S$ at a point $pt \in S$. Further let $\partial_{\dot\alpha} := \partial_{q^{\dot\alpha}} \equiv \partial/\partial q^{\dot\alpha}$, $(\dot\alpha = 3,4)$. We use the notation, $u = (u^\mu) = (u^1, u^2, u^3, u^4) := (s^1, s^2, q^3, q^4)$. Hereafter we will assume that the indices "α, β, $\cdots$" are for (s^1, s^2), "$\dot\alpha$, $\dot\beta$, $\cdots$" for (q^3, q^4), "μ, ν, $\cdots$" for $u = (s, q)$ and "i, j, $\cdots$" for the Cartesian coordinates x.

For a point of T_S expressed by the Cartesian coordinates x in $\mathbb{E}^4$ can be uniquely written by

$$x = \pi_{T_S} x + q^3 \mathbf{e}_3 + q^4 \mathbf{e}_4,$$

where $\mathbf{e}_3$ and $\mathbf{e}_4$ are normal unit vectors $T^\perp_{\pi_{T_S} x} S$ which satisfy

$$\partial_\alpha \mathbf{e}_{\dot\beta} = \Gamma^{\beta}{}_{\dot\beta\alpha} \mathbf{e}_\beta,$$

for $e^i{}_\beta := \partial_\beta(\pi_{T_S} x^i)$. A moving frame $E^i{}_\mu := \partial_\mu x^i$, $(\mu, i = 1,2,3,4)$, in T_S is expressed by

$$E^i{}_\alpha = e^i{}_\alpha + q^{\dot\alpha} \Gamma^\beta{}_{\dot\alpha\alpha} e^i{}_\beta, \quad E^i{}_{\dot\alpha} = e^i{}_{\dot\alpha}.$$

In general, more general normal unit vectors $\tilde{\mathbf{e}}_{\dot\alpha} \in T_{pt}^\perp S$ at $pt \in S$ obey a relation,

$$\partial_\alpha \tilde{\mathbf{e}}_{\dot\beta} = \tilde\Gamma^{\beta}{}_{\dot\beta\alpha} \tilde{\mathbf{e}}_\beta + \tilde\Gamma^{\dot\alpha}{}_{\alpha\dot\beta} \tilde{\mathbf{e}}_{\dot\alpha}. \tag{2.6}$$

These bases can be connected with

$$\begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{e}}_3 \\ \tilde{\mathbf{e}}_4 \end{pmatrix}, \quad \partial_\alpha \theta = \Gamma^3{}_{\alpha 4}, \tag{2.7}$$

due to the relation $\Gamma^{\dot\alpha}{}_{\alpha\dot\beta} = -\Gamma^{\dot\beta}{}_{\alpha\dot\alpha}$.

Thus the induced metric, $g_{T_S \mu\nu} := \delta_{ij} E^i_\mu E^j_\nu$, in T_S from that in the Euclidean space $\mathbb{E}^4$ is given as

$$g_{T_S} = \begin{pmatrix} g_{S_q} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.8}$$

$$g_{S_q \alpha\beta} = g_{S\alpha\beta} + [\Gamma^\gamma{}_{\dot\alpha\alpha} g_{S\gamma\beta} + g_{S\alpha\gamma}\Gamma^\gamma{}_{\dot\alpha\beta}] q^{\dot\alpha} + [\Gamma^\delta{}_{\dot\alpha\alpha} g_{S\delta\gamma} \Gamma^\gamma{}_{\dot\beta\beta}] q^{\dot\alpha} q^{\dot\beta}, \tag{2.9}$$

where $g_{S\alpha\beta} := \delta_{ij} e^i_\alpha e^j_\beta$. The determinant of the metric is expressed as

$$\det g_{T_S} = \rho \det g_S, \quad \rho = (1 + \Gamma^\alpha_{\dot\alpha\alpha} q^{\dot\alpha} + \mathcal{O}((q^{\dot\alpha})^2, q^3 q^4))^2. \tag{2.10}$$

As we use primitive facts in sheaf theory [6], we will give our conventions. For a fiber bundle A over a paracompact differential manifold M and an open set $U \subset M$, let A_M denote a sheaf given by a set of smooth local sections of the fiber bundle A and $A_M(U) \equiv \Gamma(U, A_M)$ sections of A_M over U. For example, $\mathbb{C}_M$ is a sheaf given by smooth local sections of complex line bundle over M.

Further for open sets $U \subset V \subset M$, we will denote the restriction map of a sheaf A_M by ρ_{UV}. Using the direct limit for $\{U \mid pt \in U \subset M\}$, we have a *stalk* A_{pt} of A_M

by setting $A_{pt} \equiv \Gamma(pt, A_M) := \lim_{pt \leftarrow U} A_M(U)$. An element of A_{pt} is called *germ*. Similarly for a compact subset K in M, $i_K : K \hookrightarrow M$ and for $\{U \mid K \subset U \subset M\}$, we have $\Gamma(K, A_M) := \lim_{K \leftarrow U} A_M(U)$. On the other hand, for a topological subset N of M, $i_N : N \hookrightarrow M$, there is an inverse sheaf, $A_M|_N := i_N^* A_M$ given by the sections $A_M|_N(U) = \Gamma(i_N(U), A_M)$ for $U \subset N$. When N is a compact set, *i.e.*, K, we have an equality $\Gamma(K, A_M) = \Gamma(K, i_K^* A_M)$ (Theorem 2.2 in [6]) and we identify them in this article.

We say that a sheaf A_M over M is ***soft*** if and only if for every compact subset $K \subset M$, a sheaf morphism $\Gamma(M, A_M) \to \Gamma(K, A_M)$ is surjective. For example, $\mathbb{C}_{\mathbb{R}^n}$ is soft (Theorem 3.2 [6]) and thus for a point *pt* and an open set U, $pt \in U \subset \mathbb{R}^n$, $\mathbb{C}_{\mathbb{R}^n}(U) \to \Gamma(pt, \mathbb{C}_{\mathbb{R}^n})$ is surjective. Our theory is of germs and based upon these facts.

3 The Dirac system in $\mathbb{E}^4$

For the above geometrical setting, we will consider a Dirac equation over T_S as an equation over a preHilbert space $\mathcal{H} = (\Gamma_c(T_S, \mathrm{Cliff}^*_{T_S}) \times \Gamma_c(T_S, \mathrm{Cliff}_{T_S}), \langle , \rangle, \varphi)$. Here 1) $\Gamma_c(T_S, \mathrm{Cliff}_{T_S})$ is a set of global sections of the compact support Clifford module Cliff_{T_S} over T_S, $\mathrm{Cliff}^*_{T_S}$ is a natural Hermite conjugate of Cliff_{T_S}, 2) $\langle , \rangle$ is the L^2-type pairing, for $(\overline{\Psi}_1, \Psi_2) \in \Gamma_c(T_S, \mathrm{Cliff}^*_{T_S}) \times \Gamma_c(T_S, \mathrm{Cliff}_{T_S})$,

$$\langle \overline{\Psi}_1, \Psi_2 \rangle = \int_{T_S} d^4x \overline{\Psi}_1 \Psi_2, \tag{3.11}$$

and 3) φ is the isomorphism from Cliff_{T_S} to $\mathrm{Cliff}^*_{T_S}$. Further in this article, we will express the preHilbert space using the triplet with the inner product $(\circ, \cdot) := \langle \varphi \circ, \cdot \rangle$. Here in (3.11), we have implicitly used another paring given by the pointwise product for the germs at $pt \in T_S$, *i.e.*, $\overline{\Psi}_1 \Psi_2|_{pt} \in \Gamma(pt, \mathbb{C}_{T_S})$, which also gives us a preHilbert space $\mathcal{H}^{pt} = (\Gamma(pt, \mathrm{Cliff}^*_{T_S}) \times \Gamma(pt, \mathrm{Cliff}_{T_S}), \cdot, \varphi_{pt})$. Here φ_{pt} is the Hermite conjugate operation.

Let the sheaf of the Clifford ring over $\mathbb{E}^4$ and T_S be denoted by $\mathrm{CLIFF}_{\mathbb{E}^4}$ and CLIFF_{T_S}, $\mathrm{CLIFF}_{T_S} \equiv \mathrm{CLIFF}_{\mathbb{E}^4}|_{T_S}$. Since the gamma-matrix, the generator of $\mathrm{CLIFF}_{\mathbb{E}^4}$, depends upon the orthonormal system $\{e\}$, we will sometimes refer it by $\gamma_{\{e\}}(a_i e^i) := a_i \gamma_{\{e\}}(e^i)$. In the same way, we use a representation of the Clifford module $\mathrm{Cliff}_{\mathbb{E}^4}$, $\mathrm{Cliff}_{T_S} = \mathrm{Cliff}_{\mathbb{E}^4}|_{T_S}$, using the orthonormal system $\{e\}$ as $\Psi_{\{e\}}$. Using the Pauli matrices,

$$\tau_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 := \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \tau_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_4 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we will use the convention,

$$\gamma_{\{dx\}}(dx^i) := \tau_1 \otimes \tau_i, \ (i = 1, 2, 3), \quad \gamma_{\{dx\}}(dx^4) := \tau_2 \otimes \tau_4.$$

However for abbreviation, let $\gamma^i := \gamma_{\{dx\}}(dx^i)$.

Here the Dirac operator is given by $\not{D}_{\{dx\},x} := \gamma^i \partial_i$, and the Dirac equation is given by

$$\sqrt{-1}\, \not{D}_{\{dx\},x} \Psi_{\{dx\}} = 0 \quad \text{over} \quad T_S. \tag{3.12}$$

Immediately we have a proposition for the solution space of the Dirac equation.

Proposition 3.1. *Let us define a set of constant section in the Clifford module* $\mathrm{Cliff}_{\mathbb{E}^4}$ *in* $\mathbb{E}^4$*:*

$$\Psi^{[1]} := \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \Psi^{[2]} := \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \Psi^{[3]} := \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \Psi^{[4]} := \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

$\overline{\Psi}^{[a]} := \varphi(\Psi^{[a]})$ *is given by the Hermite conjugate of each* $\Psi^{[a]}$.

1. *They satisfy the relation,*

$$\overline{\Psi}^{[a]}\Psi^{[b]} = \delta^{a,b}.$$

   We call this relation orthonormality relation in this article.

2. *A germ of solutions of Dirac equation (3.12) are expressed by* $\sum_a b^a(pt)\Psi^{[a]}$ *for* $b^a \in \Gamma(pt, \mathbb{C}_{\mathbb{E}^4})$ *at a point* $pt \in \mathbb{E}^4$.

Due to properties of the gamma-matrices, $\delta_{ij}\overline{\Psi}_1\gamma^i\Psi_2 dx^j$ is a one-form over T_S. Direct computations lead us the following Proposition which gives the properties of Clifford module.

Proposition 3.2. *Let us define a set of constant sections in Clifford module* $\mathrm{Cliff}_{\mathbb{E}^4}$ *in* $\mathbb{E}^4$*:*

$$\Psi^{(1)} := \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \Psi^{(2)} := \frac{1}{2}\begin{pmatrix}1\\\sqrt{-1}\\1\\\sqrt{-1}\end{pmatrix},$$

$$\Psi^{(3)} := \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad \Psi^{(4)} := \frac{1}{2}\begin{pmatrix}1\\1\\\sqrt{-1}\\\sqrt{-1}\end{pmatrix}.$$

$\overline{\Psi}^{(k)} := \varphi(\Psi^{(k)})$ *is given by the Hermite conjugate of each* $\Psi^{(k)}$. *They satisfy the relation,*

$$\delta_{ij}\overline{\Psi}^{(k)}\gamma^i\Psi^{(k)}dx^j = dx^k \quad \textit{(not summed over k)}.$$

We call this relation $\mathrm{SO}(4)$*-representation in this article.*

Remark 3.1. Using a $\mathbb{C}$-valued smooth compact function $b \in \Gamma_c(\mathbb{E}^4, \mathbb{C}_{\mathbb{E}^4})$ over $\mathbb{E}^4$ such that $b \equiv 1$ at $U \subset T_S$ and its support is in T_S, $b\Psi^{[a]}$, $b\Psi^{(k)}$ and their partners belong to $\Gamma_c(T_S, \mathrm{Cliff}_{T_S})$ and $\Gamma_c(T_S, \mathrm{Cliff}^*_{T_S})$. Hereafter we assume $\Psi^{[a]}$, $\Psi^{(k)}$ and their partners are sections of $\Gamma_c(T_S, \mathrm{Cliff}_{T_S})$ and $\Gamma_c(T_S, \mathrm{Cliff}^*_{T_S})$ in the sense.

Next let us give expressions of these players of the Dirac system in terms of the coordinate system u in T_S. An orthonormal bases of T^*T_S will be denoted as $d\xi = (d\zeta^1, d\zeta^2, dq^3, dq^4)$. Then the expressions are given by the transformations,

$$\Psi_{\{d\xi\}}(u) := \mathrm{e}^{-\Omega}\Psi_{\{dx\}}(x), \quad \overline{\Psi}_{\{d\xi\}}(u) := \overline{\Psi}_{\{dx\}}(x)\mathrm{e}^{\Omega},$$

$$\mathrm{e}^{-\Omega}\gamma_{\{dx\}}(dx^i)\mathrm{e}^{\Omega} = E^i{}_\mu\gamma_{\{d\xi\}}(du^\mu) =: E^i{}_\mu\gamma^\mu.$$

Here e^{Ω} and $\mathrm{e}^{-\Omega}$ are sections of the spin group sheaf $\mathrm{SPIN}_{T_S} \equiv \mathrm{SPIN}_{\mathbb{E}^4}|_{T_S}$. For $\Psi \in \Gamma_c(T_S, \mathrm{Cliff}_{T_S})$, the pairing (3.11) is expressed by

$$\langle \overline{\Psi}_1, \Psi_2 \rangle = \int (\det g_S)^{1/2} \rho^{1/2} d^2 s d^2 q \ \overline{\Psi}_{1,\{d\zeta\}} \Psi_{2,\{d\zeta\}}, \tag{3.13}$$

and the Dirac equation (3.12) is expressed by

$$\sqrt{-1}\, \not{D}_{\{d\xi\},u} \Psi_{\{d\xi\}} = 0, \quad \not{D}_{\{d\xi\},u} = \gamma^{\mu}(\partial_{\mu} + \partial_{\mu}\Omega). \tag{3.14}$$

Using Proposition 3.1, we have the following

Corollary 3.1. *For an open set $U \subset T_S$ and $\mathrm{e}^{-\Omega} \in \Gamma(U, \mathrm{SPIN}_{\mathbb{E}^4})$, by letting $\Psi^{[a]}_{\{d\xi\}} := \mathrm{e}^{-\Omega}\Psi^{[a]} \in \Gamma(U, \mathrm{Cliff}_{T_S})$ and $\overline{\Psi}^{[a]}_{\{d\xi\}} := \overline{\Psi}^{[a]}\mathrm{e}^{\Omega} \in \Gamma(U, \mathrm{Cliff}^*_{T_S})$ over U, the orthonormal relation holds*

$$\overline{\Psi}^{[a]}_{\{d\xi\}} \Psi^{[b]}_{\{d\xi\}} = \delta^{a,b} \quad \text{at} \quad U \subset T_S. \tag{3.15}$$

Inversely for given such orthonormal bases $\Psi^{[b]}_{\{d\xi\}} \in \Gamma(U, \mathrm{Cliff}_{T_S})$ and $\varphi(\Psi^{[b]}_{\{d\xi\}})$ satisfying (3.15), the relation,

$$\Psi^{[a]}_{\{d\xi\}} = \mathrm{e}^{-\Omega}\Psi^{[a]} \quad \text{for} \quad a = 1, 2, 3, 4,$$

completely characterizes the spin matrix $\mathrm{e}^{-\Omega} \in \Gamma(U, \mathrm{SPIN}_{T_S})$.

Proposition 3.2 gives the following

Corollary 3.2. *For an open set $U \subset T_S$ and $\mathrm{e}^{-\Omega} \in \Gamma(U, \mathrm{SPIN}_{\mathbb{E}^4})$, by letting $\Psi^{(i)}_{\{d\xi\}} := \mathrm{e}^{-\Omega}\Psi^{(i)} \in \Gamma(U, \mathrm{Cliff}_{T_S})$ and $\overline{\Psi}^{(i)}_{\{d\xi\}} := \overline{\Psi}^{(i)}\mathrm{e}^{\Omega} \in \Gamma(U, \mathrm{Cliff}^*_{T_S})$ over U, the $\mathrm{SO}(4)$-representation holds*

$$g_{T_S\,\mu\nu} \overline{\Psi}^{(i)}_{\{d\xi\}} \gamma^{\mu} \Psi^{(i)}_{\{d\xi\}} du^{\nu} = dx^i \quad \text{at} \quad U \subset T_S \quad \text{(not summed over } i).$$

4 The submanifold Dirac operator over S in $\mathbb{E}^4$

In this section, we will define the submanifold Dirac operator over S in $\mathbb{E}^4$ and investigate its properties. In the papers in [11, 12, 13, 14, 15, 21], we add a mass type potential in (3.14), which confines a particle in the tubular neighborhood T_S; the mass potential makes the support of the Clifford module in T_S. After taking a squeezing limit of the mass potential, we decompose the normal and the tangential modes, suppress the normal mode, and obtain the submanifold Dirac equation as an effective equation for low energy states. Instead of the scheme, we will choose another construction and give a novel definition of the Dirac operator as in Definition 4.1 [20].

Let us consider such a Dirac particle algebraically. Confinement of the particle into a surface requires that the momentum and position of the particle vanish. In order to realize the vanishing momentum, we wish to consider kernel of $\partial_{\dot{\alpha}}$. However $p_{\dot{\alpha}} := \sqrt{-1}\partial_{\dot{\alpha}}$ is not self-adjoint in general due to the existence of ρ in (3.13).

For an operator P over Cliff_{T_S}, let $Ad(P)$ be defined by the relation $\langle\overline{\Psi}_1, P\Psi_2\rangle{=}\langle\overline{\Psi}_1 Ad(P), \Psi_2\rangle$ if exists. Further for $\Psi \in \Gamma_c(T_S, \mathrm{Cliff}_{T_S})$, P^* is defined by $P^*\Psi = \varphi^{-1}(\varphi(\Psi)Ad(P))$. If $p_{\dot\alpha}$ is not self-adjoint, the kernel of $p_{\dot\alpha}$ is not isomorphic to the kernel of $Ad(p_{\dot\alpha})$. Thus the kernel of $p_{\dot\alpha}$ cannot become a preHilbert space and φ_{pt} or φ_{pt}^{-1} is not well-defined there. It means that SO(4)-representation, Corollary 3.2, which should be regarded as a fundamental properties of the Clifford module, will neither be well-defined.

Accordingly we introduce another preHilbert space $\mathcal{H}' \equiv (\Gamma_c(T_S, \tilde{\mathrm{Cliff}}^*_{T_S}) \times \Gamma_c(T_S, \tilde{\mathrm{Cliff}}_{T_S}), \langle,\rangle_{\mathrm{sa}}, \tilde\varphi)$ so that $p_{\dot\alpha}$'s become self-adjoint operators there. Using the half-density (Theorem 18.1.34 in [5]), we construct *s*elf-adjointization: $\eta_{\mathrm{sa}} : \mathcal{H} \to \mathcal{H}'$ by,

$$\eta_{\mathrm{sa}}(\overline{\Psi}) := \rho^{1/4}\overline{\Psi}, \quad \eta_{\mathrm{sa}}(\Psi) := \rho^{1/4}\Psi, \quad \eta_{\mathrm{sa}}(P) := \rho^{1/4}P\rho^{-1/4}.$$

Here since ρ does not vanish in T_S, η_{sa} gives an isomorphism $\eta_{\mathrm{sa}} : \mathrm{Cliff}^*_{T_S} \times \mathrm{Cliff}_{T_S} \to \tilde{\mathrm{Cliff}}^*_{T_S} \times \tilde{\mathrm{Cliff}}_{T_S}$. For $(\overline{\Psi}_1, \Psi_2) \in \Gamma_c(T_S, \tilde{\mathrm{Cliff}}^*_{T_S}) \times \Gamma_c(T_S, \tilde{\mathrm{Cliff}}_{T_S})$, by letting $\tilde\varphi := \eta_{\mathrm{sa}}\varphi\eta_{\mathrm{sa}}^{-1}$, the pairing is defined by

$$\langle\overline{\Psi}_1, \Psi_2\rangle_{\mathrm{sa}} := \int_{T_S} (\det g_S)^{1/2} d^2s d^2q\ \overline{\Psi}_1\Psi_2. \tag{4.16}$$

Here we have the properties of η_{sa} that 1) $\langle\circ, \cdot\rangle_{\mathrm{sa}} = \langle\eta_{\mathrm{sa}}^{-1}\circ, \eta_{\mathrm{sa}}^{-1}\cdot\rangle$, 2) for an operator P of Cliff_{T_S}, $\eta_{\mathrm{sa}}(P) = \eta_{\mathrm{sa}}P\eta_{\mathrm{sa}}^{-1}$, and 3) $p_{\dot\alpha}$'s themselves become self-adjoint in $\mathcal{H}'$, *i.e.*, $p_{\dot\alpha} = p^*_{\dot\alpha}$. The self-adjointization is not a unitary operation in some sense because due to the operation, the inner product changes from $\langle\varphi\circ, \cdot\rangle$ to $\langle\tilde\varphi\circ, \cdot\rangle_{\mathrm{sa}}$ if we regard them as inner products for $\Gamma_c(T_S, \tilde{\mathrm{Cliff}}_{T_S}) \times \Gamma_c(T_S, \tilde{\mathrm{Cliff}}_{T_S})$. Due to this trick, $p_{\dot\alpha}$'s become self-adjoint.

Noting $\rho = 1$ at a point in S, Corollaries 3.1 and 3.2 lead the following lemma.

Lemma 4.1. *1. For* $(\overline{\Psi}, \Psi) \in \Gamma(S, \mathrm{Cliff}^*_{T_S}) \times \Gamma(S, \mathrm{Cliff}_{T_S})$, $\eta_{\mathrm{sa}}(\Psi) = \Psi$ *and* $\eta_{\mathrm{sa}}(\overline{\Psi}) = \overline{\Psi}$ *at* S.

2. For the quantities defined in Corollary 3.1, the orthonormality relation holds:

$$\eta_{\mathrm{sa}}(\overline{\Psi}^{[a]}_{\{d\xi\}})\eta_{\mathrm{sa}}(\Psi^{[b]}_{\{d\xi\}}) = \delta^{a,b} \quad at \quad S.$$

3. For the quantities defined in Corollary 3.2, by letting $\Phi^{(i)} := \eta_{\mathrm{sa}}(\Psi^{(i)}_{\{d\xi\}})$ *and* $\overline{\Phi}^{(i)} := \eta_{\mathrm{sa}}(\overline{\Psi}^{(i)}_{\{d\xi\}})$, *the* SO(4)*-representation holds*

$$g_{T_S\mu\nu}\overline{\Phi}^{(i)}\gamma^\mu\Phi^{(i)}du^\nu = dx^i \quad at \quad S \quad (not\ summed\ over\ i).$$

We have the following proposition.

Proposition 4.1. *By letting* $p_q := a_3p_3 + a_4p_4$ *for real generic numbers* a_3 *and* a_4, *the projection,*

$$\pi_{p_q} : \tilde{\mathrm{Cliff}}^*_{T_S} \times \tilde{\mathrm{Cliff}}_{T_S} \to \mathrm{Ker}(Ad(p_q)) \times \mathrm{Ker}(p_q),$$

induces the projection in the preHilbert space [1], i.e.*,*

1. *$\tilde\varphi|_{\mathrm{Ker}(p_q)}$: $\mathrm{Ker}(p_q) \to \mathrm{Ker}(Ad(p_q))$ is isomorphic. We simply express $\tilde\varphi|_{\mathrm{Ker}(p_q)}$ by $\tilde\varphi$ hereafter.*

2. *$\mathcal{H}_{p_q} := (\Gamma_c(T_S, \mathrm{Ker}(Ad(p_q))) \times \Gamma_c(T_S, \mathrm{Ker}(p_q)), \langle,\rangle_{\mathrm{sa}}, \tilde\varphi)$ is a preHilbert space.*

3. *$\varpi_{p_q} := \pi_{p_q}|_{\tilde{\mathrm{Cliff}}_{T_S}}$, $\varpi_{p_q} = \varpi_{p_q}^2 = \varpi_{p_q}^*$ in $\mathcal{H}_{p_q}$.*

4. *ϖ_{p_q} induces a natural restriction of pointwise multiplication for a point in T_s, $\mathcal{H}_{p_q}^{pt} := (\Gamma(pt, \mathrm{Ker}(Ad(p_q))) \times \Gamma(pt, \mathrm{Ker}(p_q)), \cdot, \tilde\varphi_{pt})$ becomes a preHilbert space. The Hermite conjugate map $\tilde\varphi_{pt}$ is still an isomorphism.*

Proof. Since $p_{\dot\alpha}$ is self-adjoint, $\mathrm{Ker}(p_q) = \mathrm{Ker}(p_q^*)$ and $\mathrm{Ker}(p_q)$ is isomorphic to $\mathrm{Ker}(Ad(p_q))$, *i.e.*, $\varphi(\varpi_{p_q}\Psi) = \varphi(\Psi)Ad(\varpi_{p_q})$. $\varpi_{p_q}^*\Psi = \varphi^{-1}(\varphi(\Psi)Ad(\varpi_{p_q}))$ gives $\varpi_{p_q} = \varpi_{p_q}^*$. □

Since T_S is homeomorphic to $\mathbb{R}^4$, $\mathbb{C}_{T_S}$ is soft (Theorem 3.1 in [6]). Hence we have the following proposition.

Proposition 4.2. Cliff_{T_S} *is soft.*

Proof. Due to Proposition 3.1 (2), Cliff_{T_S} is a sheaf of $\mathbb{C}_{T_S}$ vector bundle. From the proof of Theorem 3.2 in [6], it is justified. □

Due to the Proposition 4.2, for a point pt in S, $pt \in U \subset T_S$ and for a germ $\Psi_{pt} \in \Gamma(pt, \mathrm{Cliff}_{T_S})$, there exists $\Psi_c \in \Gamma_c(T_S, \mathrm{Cliff}_{T_S})$ $\Psi_o \in \Gamma(U, \mathrm{Cliff}_{\mathbb{E}^4})$ such that

$$\Psi_{pt} = \Psi_c, \quad \Psi_{pt} = \Psi_o, \quad \text{at} \quad pt.$$

Thus when we deal with an element of $\Gamma(pt, \mathrm{Cliff}_{T_S})$, we need not distinguish which it comes from $\Gamma_c(T_S, \mathrm{Cliff}_{T_S})$ or $\Gamma(U, \mathrm{Cliff}_{\mathbb{E}^4})$.

Remark 4.1. At a point pt in S, we can find $(\overline{\Phi}^{(i)}, \Phi^{(i)})$ in $\mathcal{H}_{p_q}^{pt}$ obeying the SO(4)-representation,

$$g_{T_S\mu\nu}\overline{\Phi}^{(i)}\gamma^\mu\Phi^{(i)}du^\nu = dx^i \quad \text{at} \quad pt \quad (\text{not summed over } i).$$

because 1) we can easily find such an element in $\mathcal{H}'$ and 2) extend its domain to a vicinity of S so that its value preserves for the normal direction, , i.e., $\partial_{\dot\alpha}\Phi^{(i)} = 0$ and $\overline{\Phi}^{(i)}Ad(\partial_{\dot\alpha}) = 0$; $(\overline{\Phi}^{(i)}, \Phi^{(i)})$ belongs to $\mathrm{Ker}(p_q)^* \times \mathrm{Ker}(p_q)$.

Since we kill a normal translation freedom in $\mathcal{H}_{p_q}$, we can choose a position q and make q vanish. Thus we will give our definition of the submanifold Dirac operator.

Definition 4.1. *We define the submanifold Dirac operator for the surface S in $\mathbb{E}^4$ by,*

$$\not{D}_{S\hookrightarrow\mathbb{E}^4} := \eta_{\mathrm{sa}}(\not{D})|_{\mathrm{Ker}(p_q)}|_{q=0},$$

as an endomorphism of Clifford submodule $\mathrm{Ker}(p_q)|_S \subset \tilde{\mathrm{Cliff}}_{T_S}|_S$, i.e., $\not{D}_{S\hookrightarrow\mathbb{E}^4}$: $\mathrm{Ker}(p_q)|_S \to \mathrm{Ker}(p_q)|_S$.

Here we note that the first restriction $|_{\mathrm{Ker}(p_q)}$ is as an operator but the second one $|_{q=0}$ is given by ρ_{S,T_S}, which is given by a direct limit $\rho_{S,T_S} := \lim_{S\to U}\rho_{U,T_S}$.

Here we will connect the $\mathrm{CLIFF}_{T_S}|_S$ with the sheaf of proper Clifford ring CLIFF_S over S. We will fix the orthonormal base $\{d\zeta\} \equiv (d\zeta^\alpha)$ associated with the local coordinate (ds^α) and the gamma-matrix as a generator in CLIFF_S by $\gamma_{S,\{d\zeta\}}(d\zeta^\alpha)$. For abbreviation, $\tilde\sigma^\alpha := \gamma_{S,\{d\zeta\}}(d\zeta^\alpha)$ and $\sigma^\alpha := \gamma_{S,\{d\zeta\}}(ds^\alpha)$.

We have an inclusion as vector space for generators,

$$\iota_g : \mathrm{CLIFF}_S \ni \tilde\sigma^\alpha \mapsto \tau_1 \otimes \tilde\sigma^\alpha \in \mathrm{CLIFF}_{T_S}|_S.$$

Let $\iota_g(\tilde\sigma^\alpha\tilde\sigma^\beta) := \iota_g(\tilde\sigma^\alpha)\iota_g(\tilde\sigma^\beta)$, $\iota_g(\tilde\sigma^\alpha\tilde\sigma^\beta\tilde\sigma^\gamma) := \iota_g(\tilde\sigma^\alpha)\iota_g(\tilde\sigma^\beta)\iota_g(\tilde\sigma^\gamma)$ and so on. This does not become the homomorphism between the Clifford rings whereas the natural ring homeomorphism is given by

$$\iota_r : \mathrm{CLIFF}_S \ni c \mapsto 1 \otimes c \in \mathrm{CLIFF}_{T_S}|_S.$$

However the inclusion ι_g generates the homomorphism of the spin groups because $\iota_g(\tilde\sigma^\alpha\tilde\sigma^\beta) = \iota_r(\tilde\sigma^\alpha\tilde\sigma^\beta)$. A spin matrix $\exp(\Omega_S) \in \Gamma(pt, \mathrm{SPIN}_S)$, of the spin group sheaf SPIN_S properly defined over S, is given by $\exp(\Omega_S) = \exp(a_{\alpha\beta}\tilde\sigma^\alpha\tilde\sigma^\beta)$. On the other hand, a germ $\exp(\Omega)$ of SPIN_{T_S} at a point pt in S is given by $\exp(\Omega) = \exp(a_{\mu\nu}\gamma^\mu\gamma^\nu)$. Thus ι_g and ι_r induce the natural inclusion of SPIN_S into SPIN_{T_S} as a sheaf morphism by $\exp(\Omega) = \exp(a_{\alpha\beta}(1 \otimes \tilde\sigma^\alpha\tilde\sigma^\beta))$.

Using these facts, we will give explicit form of the submanifold Dirac operator, which was obtained in [15] using a mass potential.

Proposition 4.3. *The submanifold Dirac operator of the surface S in $\mathbb{E}^4$ can be expressed by*

$$\not{D}_{S\hookrightarrow\mathbb{E}^4} = \iota_g(\sigma^\alpha\nabla_\alpha) + \frac{1}{2}\gamma^3\Gamma^\alpha{}_{3\alpha} + \frac{1}{2}\gamma^4\Gamma^\alpha{}_{4\alpha}, \tag{4.17}$$

where ∇_α is the proper spin connection over S and $\gamma^{\dot\alpha} := \gamma_{\{d\xi\}}(dq^{\dot\alpha})$.

Proof. First we note that $\eta_{\mathrm{sa}}(\not{D}_{\{d\xi\},u})$ has a decomposition,

$$\eta_{\mathrm{sa}}(\not{D}_{\{d\xi\},u}) = \not{\mathbb{D}}^{\|}_{\{d\xi\},u} + \not{\mathbb{D}}^{\perp}_{\{d\xi\},u},$$

where $\not{\mathbb{D}}^{\perp}_{\{d\xi\},u} := \gamma^{\dot\alpha}\partial_{\dot\alpha}$ and $\not{\mathbb{D}}^{\|}_{\{d\xi\},u}$ does not include the normal derivative $p_{\dot\alpha}$. $\not{D}^{\perp}_{\{d\xi\},u}$ vanishes at $\mathrm{Ker}(p_q)$. Due to the constructions, $\iota_g(\sigma^\alpha)$ and $\gamma^{\dot\alpha}$ become generator of the CLIFF_{T_S} at sufficiently vicinity of S. The geometrically independency due to (2.8) and direct computations give above the result. □

Now we will give the first theorem:

Theorem 4.1. *Let a point pt in S be expressed by the Cartesian coordinates (x^i) and $\mathbb{C}^4_S$ sheaf of complex vector bundle over S with rank four. A set of germs of $\Gamma(pt, \mathbb{C}^4_S)$ satisfying the submanifold Dirac equation,*

$$\sqrt{-1}\,\not{D}_{S\hookrightarrow\mathbb{E}^4}\psi = 0 \quad at \quad pt,$$

is given by $\{b_a\psi^{[a]} \mid a = 1,2,3,4,\ b_a \in \mathbb{C}\}$ *such that*

$$\varphi_{pt}(\psi^{[a]})\psi^{[b]} = \delta_{a,b} \quad at \quad pt.$$

At the point pt, there exists a spin matrix $\mathrm{e}^{\Omega} \in \Gamma(pt, \mathrm{SPIN}_{\mathbb{E}^4})$ *satisfying* $\psi^{[a]} = \mathrm{e}^{-\Omega}\Psi^{[a]}$ $(a = 1,2,3,4)$. *We define* $\psi^{(i)} := \mathrm{e}^{-\Omega}\Psi^{(i)}$ *and* $\overline{\psi}^{(i)} := \overline{\Psi}^{(i)}\mathrm{e}^{-\Omega}$ $(i = 1,2,3,4)$ *at the point. Then the following relation holds:*

$$g_{S,\alpha,\beta}\overline{\psi}^{(i)}[\iota_g(\sigma^\beta))]\psi^{(i)} = \partial_{s^\alpha}x^i, \quad at \quad pt, \quad (not\ summed\ over\ i). \tag{4.18}$$

Remark 4.2.

1. As our theory is a local theory, this theorem can be extended to any surfaces immersed in $\mathbb{E}^4$.
2. If the surface is conformal, this theorem represents the generalized Weierstrass relation in $\mathbb{E}^4$ given by Konopelchenko [10] and Pedit and Pinkall [22] as mentioned in [15].
3. This can be easily generalized to a k submanifold S^k immersed in the n-Euclidean space $\mathbb{E}^n$. In the above statements, the index "$a = 1,\cdots,4$"should be replaced to "$a = 1,2,\cdots,2^{[n/2]}$", "i"to "$i = 1,\cdots,n$", "$\alpha = 1,\cdots,k$", and "$\dot\alpha = k+1,\cdots,n$" [20].

Proof. Since $\not{D}_{S\hookrightarrow\mathbb{E}^4}$ is the four rank first order differential operator and has no singularity over S due to the construction, a germ of its kernel in $\Gamma(pt, \mathbb{C}^4_{\mathbb{E}^4})$ is given by four dimensional vector space at each point of S. Since $\not{D}_{S\hookrightarrow\mathbb{E}^4}$ is defined as an endomorphism of $\mathrm{Ker}(p_q)$ and $\mathrm{Ker}(p_q)$ contains the zero section, the germ of kernel of the Dirac operator, $\mathrm{Ker}(\not{D}_{S\hookrightarrow\mathbb{E}^4})$, is a submodule of $\mathrm{Ker}(p_q)|_S$. Let $\not{\mathbb{D}}^\perp := \gamma^{\dot\alpha}\partial_{\dot\alpha}$ at S. From the construction, we have

$$\not{D}_{S\hookrightarrow\mathbb{E}^4} + \not{\mathbb{D}}^\perp = \eta_{\mathrm{sa}}(\not{D}_{\{d\xi\},u})|_S.$$

Hence the solution of $\not{D}_{S\hookrightarrow\mathbb{E}^4}$ becomes a solution of $\eta_{\mathrm{sa}}(\not{D}_{\{d\xi\},u})|_S$. Noting Remark 4.1, $\tilde\varphi_{pt}$ is an isomorphism and $\mathcal{H}^{pt}_{p_q}$ gives SO(4)-representation. Thus we prove it. ☐

5 The submanifold Dirac operator over S in $\mathbb{E}^4$ with torsion

For the transformations (2.7), we have the relation,

$$\begin{pmatrix}\Gamma^\beta{}_{3\alpha}\\ \Gamma^\beta{}_{4\alpha}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\tilde\Gamma^\beta{}_{3\alpha}\\ \tilde\Gamma^\beta{}_{4\alpha}\end{pmatrix}.$$

Now let us choose ϑ in $\{\theta \in [0,2\pi]\}$ as

$$\Gamma^\alpha{}_{3\alpha}\sin\vartheta + \Gamma^\alpha{}_{4\alpha}\cos\vartheta = 0,$$

and define

$$\hat{\Gamma}^{\alpha}{}_{3\alpha} := \Gamma^{\alpha}{}_{3\alpha}\cos\vartheta + \Gamma^{\alpha}{}_{4\alpha}\sin\vartheta, \quad \hat{\Gamma}^{\alpha}{}_{4\alpha} = 0, \quad \hat{\Gamma}^{3}{}_{\alpha 4} := \partial_\alpha \vartheta.$$

We call $\hat{\Gamma}^{3}{}_{\alpha 4}$ *torsion* in this system in the sense of the Frenet-Serret relation. It obeys the relation,

$$\Gamma^{\alpha}{}_{3\alpha} = \hat{\Gamma}^{\alpha}{}_{3\alpha}\cos\vartheta, \quad \Gamma^{\alpha}{}_{4\alpha} = -\hat{\Gamma}^{\alpha}{}_{3\alpha}\sin\vartheta.$$

The Dirac operator (4.17) can be expressed by

$$\not{D}_{S\hookrightarrow\mathbb{E}^4} = \gamma^\alpha \nabla_\alpha + \frac{1}{2}\gamma^3 \hat{\Gamma}^{\alpha}{}_{3\alpha} \mathrm{e}^{\sigma^{34}\vartheta},$$

where $\sigma^{34} := \gamma^3\gamma^4$. This type operator appears in [12]. For the gauge transformation,

$$\not{D}^{\vartheta}_{S\hookrightarrow\mathbb{E}^4} := \mathrm{e}^{-\sigma^{34}\vartheta}\, \not{D}_{S\hookrightarrow\mathbb{E}^4} \mathrm{e}^{\sigma^{34}\vartheta},$$

we have

$$\not{D}^{\vartheta}_{S\hookrightarrow\mathbb{E}^4} = \gamma^\alpha(\nabla_\alpha + \gamma_\alpha \sigma^{34}\hat{\Gamma}^{3}{}_{\alpha 4}) + \frac{1}{2}\gamma^3\hat{\Gamma}^{\alpha}{}_{3\alpha}.$$

Here $\gamma_\alpha := g_{S\alpha,\beta}\gamma^\beta$. We call this operator *gauged submanifold Dirac operator.* This is a generalization of the Takagi-Tanzawa Schrödinger operator in [25].

The main result of the article follows:

Theorem 5.1. *Fix a point pt in S expressed by the Cartesian coordinates (x^i). The germs in $\Gamma(pt, \mathbb{C}^4_S)$ satisfying the gauged submanifold Dirac equation,*

$$\sqrt{-1}\,\not{D}^{\vartheta}_{S\hookrightarrow\mathbb{E}^4}\psi = 0 \quad at \quad pt,$$

is given by $\{b_a\psi^{[a]} \mid a = 1,2,3,4,\ b_a \in \mathbb{C}\}$ such that

$$\varphi_{pt}(\psi^{[a]})\psi^{[b]} = \delta_{ab}.$$

There exists a spin matrix $\mathrm{e}^{\hat{\Omega}} \in \Gamma(pt, \mathrm{SPIN}_{\mathbb{E}^4})$ satisfying $\psi^{[a]} = \mathrm{e}^{-\hat{\Omega}}\Psi^{[a]}, (a = 1,2,3,4)$. By defining $\psi^{(i)} := \mathrm{e}^{-\hat{\Omega}}\Psi^{(i)}$ and $\overline{\psi}^{(i)} := \overline{\Psi}^{(i)}\mathrm{e}^{-\hat{\Omega}}$, $(i = 1,2,3,4)$, the following relation holds:

$$g_{S,\alpha,\beta}\overline{\psi}^{(i)}[\iota_g(\sigma^\beta))]\psi^{(i)} = \partial_{s^\alpha}x^i \quad \textit{(not summed over } i\textit{)}. \tag{5.19}$$

Proof. Let $\mathrm{e}^{-\hat{\Omega}} = \mathrm{e}^{-\Omega}\mathrm{e}^{-\sigma^{34}}$. Then we use the proof of Theorem 4.1. □

Remark 5.1.

1. When we extend the gauged submanifold Dirac operator to that over a compact surface, there appears a problem whether ϑ can be globally defined or not. $\hat{\Gamma}^{3}{}_{\alpha 4}$ appears as an associated gauge field.
2. If $\hat{\Gamma}^{\alpha}{}_{3\alpha}$ is constant case, it can be regarded as a mass of the Dirac particle and further the torsion plays a role of the $U(1)$-gauge field. This is very interesting from physical viewpoint. Even though in the string theory, the extra dimensions are connected with gauge fields, it is surprising that the gauge field appears as the torsion of the submanifold.
3. It is not difficult to extend our theory to that in k-submanifold in $\mathbb{E}^n$. Then it is expected that there appears as a $\mathrm{SO}(n-k)$-gauge field.

4. The determinants of the submanifold Dirac operators are related to the geometrical properties as in [11, 12, 14]. It should be expected that the gauged submanifold Dirac operator also brings us to the data of submanifold such as index theorem.

5. The Dirac operator of a conformal surface in $\mathbb{E}^3$ are related to the extrinsic string as in [14, 16]. The gauged submanifold Dirac operator might also be connected with the extrinsic string.

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