A certain complete space-like hypersurface in Lorentz manifolds

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Abstract

In this paper, we find an upper bound of the squared norm of the second fundamental tensor of a complete space-like hypersurface in a Lorentz space form $M_1^m(c)$ satisfying some curvature conditions . Then it gives naturally an extension of some theorems of Cheng and Nakagawa ([3]), Ishihara ([7]), Li ([8]) and Nishikawa ([9]).

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1 Introduction

In connection with the negative settlement of the Berstein problem by Calabi ([2]), Cheng-Yau ([4]) and Chouque-Bruhat et al.([6]) proved the following famous theorem independently.

Theorem A *Let M be a complete space-like Lorentz space form $M_1^{n+1}(c)$, $c > 0$. If M is maximal, then it is totally geodesic.*

On the other hand, complete space-like hypersurface with constant mean curvature in a Lorentz space form $M_1^m(c)$ are investigated by many differential geometers in various view points ; for example Akutagawa ([1]), Cheng and Nakagawa ([3]), Li ([8]), Nishikawa([9]) and Ramanathan ([11]). In this paper, we'll give an upper bound of the sequared norm of the second fundamental form, of complete space-like hypersurface with constant mean curvature in a Lorentz space form $M_1^m(c)$. Namely, the following assertion is our main theorem.

Main Theorem *Let M' be an $(n+1)$-dimensional Lorentz manifold which satisfies the condition $(*)$ and M be a complete space-like hypersurface with constant mean curvature. If M is not maximal and if it satisfies*

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$$2nc_2 + c_1 \ > \ 0,$$

then there exist a positive constant a_1 depending on c_1, c_2, c_3, h and n such that $h_2 \leq a_1$, where the condition $(*)$ means (5.1), (5.2) and (5.3) , h_2 denotes the square norm of the second fundamental form.

As an application of this result, we able to make a generalization of some theorem which are investigated by Cheng and Nakagawa ([3]), Ishihara ([7]), Li ([8]), and Nishikawa ([9]) in new different view point.

2 Definitions

Let $M^{'} = (M^{'}, g^{'})$be a Lorentz manifold with a Lorentz metric $g^{'}$ of signature $(-, +, \cdots, +)$. $M^{'}$ has uniquely defined torsion-free affine connection $\nabla^{'}$ compatible with the metric $g^{'}$. $M^{'}$ is called *locally symmetric* if the curvature tensor $R^{'}$ of $M^{'}$ is parallel, that is, $\nabla^{'} R^{'} = 0$. Let M be a hypersurface immersed in $M^{'}$. M is said to be *space-like* if the Lorentz metric $g^{'}$ of $M^{'}$ induces a Riemannian metric g on M. For a space-like hypersurface M there is naturally defined the second fundamental form (the extrinsic curvature) α of M. M is called *maximal space-like* if the mean(extrinsic) curvature H=Tr α, the trace of α, of M vanishes identically. M is maximal space-like if and only if it is extreme under the variations, with compact support through space-like hypersurfaces, for the induced volume. M is said to be *totally geodesic* (a moment of time symmetry) if the second fundamental form α vanishes identically.

3 Preliminaries

Let M be a space-like hypersurface in a Lorentz $(n+1)$-manifold $M^{'}$=$(M^{'}, g^{'})$. We choose a local field of Lorentz orthonormal frames $e_0, \cdots, e_n$ are tangent to $M^{'}$ such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M. Here and in the sequel the following convention on the range of indices used throughout this paper, unless otherwise stated:

$$i, j, k, \cdots = 1, 2, \cdots, n \qquad \alpha, \beta, \cdots, = 0, 1, 2 \cdots n$$

Let ω_α be its dual frame field so that the Lorentz metric $g^{'}$ can be written as $g^{'} = -\ \omega_0^2 + \sum_i \omega_i^2$. then the connection forms $\omega_{\alpha\beta}$ of $M^{'}$ are characterized by the equations

(3.1)
$$\begin{aligned} d\omega_i &= -\sum_k \omega_{ik} \wedge \omega_k + \omega_{i0} \wedge \omega_0, \\ d\omega_0 &= -\sum_k \omega_{0k} \wedge \omega_k, \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0. \end{aligned}$$

The curvature forms $\Omega^{'}_{\alpha\beta}$ of $M^{'}$ are given by

$$\begin{aligned}(3.2)\qquad \Omega'_{ij} &= d\omega_{ij} + \sum_k \omega_{ik}\wedge\omega_{kj} - \omega_{i0}\wedge\omega_{0j},\\ \Omega'_{0i} &= d\omega_{0i} + \sum_k \omega_{0k}\wedge\omega_{ki},\end{aligned}$$

and we have

$$(3.3)\qquad \Omega'_{\alpha\beta} = -\frac{1}{2}\sum_{\gamma,\delta} R'_{\alpha\beta\gamma\delta}\omega_\gamma\wedge\omega_\delta,$$

where $R'_{\alpha\beta\gamma\delta}$ are components of the curvature tensor R' of M'. We restrict these forms to M.

Then

$$(3.4)\qquad \omega_0 = 0,$$

and the induced Riemannian metric g of M is written as $g = \sum_i \omega_i^2$. From formulas (3.1) $\sim$ (3.4), we obtain the structure equations of M

$$\begin{aligned}(3.5)\qquad d\omega_i &= -\sum_k \omega_{ik}\wedge\omega_k, \quad \omega_{ij}+\omega_{ji}=0,\\ d\omega_{ij} &= -\sum_k \omega_{ik}\wedge\omega_{kj} + \omega_{i0}\wedge\omega_{0j} + \Omega'_{ij},\\ \Omega_{ij} &= d\omega_{ij} + \sum_k \omega_{ik}\wedge\omega_{kj} = -\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l,\end{aligned}$$

where Ω_{ij} and R_{ijkl} denote the curvature forms and the components of the curvature tensor R of M, respectively. We can also write

$$(3.6)\qquad \omega_{i0} = \sum_j h_{ij}\omega_j,$$

where h_{ij} are components of the second fundamental form $\alpha = \sum_{i,j} h_{ij}\omega_i\otimes\omega_j$ of M. Using (3.6) in (3.5) then gives the Gauss formula

$$(3.7)\qquad R_{ijkl} = R'_{ijkl} - (h_{ik}h_{jl} - h_{il}h_{jk}).$$

Let h_{ijk} denote the covariant derivative of h_{ij} so that

$$(3.8)\qquad \sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{kj}\omega_{ki} - \sum_k h_{ik}\omega_{kj}.$$

Then, by exterior differentiating (3.6), we obtain the Codazzi equation

$$(3.9)\qquad h_{ijk} - h_{ikj} = R'_{0ijk}.$$

From the exterior derivative of (3.8) , we define the second covariant derivative of h_{ij} by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ljk}\omega_{li} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}.$$

Then we obtain the Ricci formula

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{3.10}$$

The components of the Ricci tensor S and the scalar curvature r of M are given by

$$\begin{aligned} S_{ij} &= \sum_k R'_{kikj} - hh_{ij} + h^2_{ij}, \\ r &= \sum_{j,k} R_{kjkj} - h^2 + h_2, \end{aligned}$$

where $h = \sum_i h_{ii}$, $h^2_{ij} = \sum_r h_{ir}h_{rj}$ and $h_2 = \sum_j h^2_{jj}$.

Let us now denote the covariant derivative of $R'_{\alpha\beta\gamma\delta}$, as a curvature tensor of M', by $R'_{\alpha\beta\gamma\delta;\eta}$. Then restricting on M, $R'_{0ijk;l}$ is given by

$$R'_{0ijk;l} = R'_{0ijkl} - R'_{0i0k}h_{jl} - R'_{0ij0}h_{kl} - \sum_m R'_{mijk}h_{ml}, \tag{3.11}$$

where R'_{0ijkl} denotes the covariant derivative of R'_{0ijk} as a tensor on M so that

$$\sum_l R'_{0ijkl}\omega_l = dR'_{0ijk} - \sum_l R'_{0ljk}\omega_{li} - \sum_l R'_{0ilk}\omega_{lj} - \sum_l R'_{0ijl}\omega_{lk}.$$

For the sake of brevity, a tensor h^{2m}_{ij} and a function h_{2m} on M, for any integer $m(\geq 2)$, are introduced as follows:

$$\begin{aligned} h^{2m}_{ij} &= \sum_{i_1,\cdots,i_{m-1}} h^2_{ii_1}h^2_{i_1i_2}\cdots h^2_{i_{m-1}j}, \\ h_{2m} &= \sum_i h^{2m}_{ii}. \end{aligned}$$

First of all, let us introduce a fundamental property for the generalized maximal principle due to Omori ([10]) and Yau([13]).

Theorem 3.1 ([10], [13]) *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below on M. Let F be a C^2-function bounded from below on M, then for any $\epsilon > 0$, there exists a point p such that*

$$|\nabla F(p)| < \epsilon, \quad \triangle F(p) > -\epsilon \quad \textit{and} \quad \textit{inf}\ F + \epsilon > F(p).$$

We also know the following result ([5]).

Theorem 3.2 ([5]) *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a polynomial of the variable f with constant coefficients such that*

$$F(f) = c_0 f^n + c_1 f^{n-1} + \cdots + c_k f^{n-k} + c_{k+1}, \tag{3.12}$$

where $n > 1$, $1 \geq n-k \geq 0$ and $c_0 > c_{k+1}$. If a C^2-positive function f satisfies

$$\triangle f \geq F(f),$$

then we have

$$F(f_1) \leq 0,$$

where f_1 denotes the supremum of the given function f.

4 The Laplacian operator

Let M be a space-like hypersurface of an $(n+1)$-dimensional Lorentz manifold $M^{'}$. Then the Laplacian $\triangle h_{ij}$ of the components h_{ij} of α is defined by

$$\triangle h_{ij} = \sum_k h_{ijkk}.$$

From (3.9) we have

$$\triangle h_{ij} = \sum_k h_{kijk} + \sum_k R^{'}_{0ijkk}, \tag{4.1}$$

and from (3.10) it follows that

$$h_{kijk} = h_{kikj} + \sum_m h_{mi} R_{mkjk} + \sum_m h_{km} R_{mijk}. \tag{4.2}$$

By using (3.9), replace h_{kikj} in (4.2) by $h_{kkij} + R^{'}_{0kikj}$ and substitute the right hand side of (4.2) into h_{kijk} in (4.1). Then we get

$$\begin{aligned} \triangle h_{ij} &= \sum_k (h_{kkij} + R^{'}_{0kikj} + R^{'}_{0ijkk}) \\ &+ \sum_k (\sum_m h_{mi} R_{mkjk} + \sum_m h_{km} R_{mijk}). \end{aligned} \tag{4.3}$$

From (3.7), (3.11) and (4.3) we have

$$
\begin{aligned}
(4.4)\qquad \triangle h_{ij} &= \sum_k h_{kkij} + \sum_k R'_{0kik;j} + \sum_k R'_{0ijk;k} \\
&+ \sum_k (h_{kk} R'_{0ij0} + h_{ij} R'_{0k0k}) \\
&+ \sum_{m,k} (h_{mj} R'_{mkik} + 2h_{mk} R'_{mijk} + h_{mi} R'_{mkjk}) \\
&- \sum_{m,k} (h_{mi} h_{mj} h_{kk} + h_{km} h_{mj} h_{ik} - h_{km} h_{mk} h_{ij} - h_{mi} h_{mk} h_{kj}).
\end{aligned}
$$

5 Some curvature conditions

Let M' be an $(n+1)$-dimensional Lorentz manifold and let M be a space-like hypersurface of M'. For a point x in M let $\{e_0, e_1, \cdots, e_n\}$ be a local field of orthonormal frames of M' around of x in such a way that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M and the other is normal to M. Accordingly, $e_1, \cdots, e_n$ are space-like vectors and e_0 is a time-like one. For linearly independent vectors u and v in the tangent space $T_x M'$, by which the non-degenerate plane section is spanned, we denote by $K'(u,v)$ the sectional curvature of the plane section in M' and by R' or $Ric'(u,u)$ the Riemannian curvature tensor on M or the Ricci curvature in the direction of u in M', respectively. Let us denote by ∇' the Riemannian connection on M'. We assume that the ambient space M' satisfies the following three conditions : For some constants c_1, c_2 and c_3

$$
(5.1)\qquad K'(u,v) = \frac{c_1}{n},
$$

for any space-like vector u and time-like vector v,

$$
(5.2)\qquad K'(u,v) \geq c_2,
$$

for any space-like vectors u and v

$$
(5.3)\qquad |\nabla' R'| \leq \frac{c_3}{n}.
$$

When M' satisfies the above conditions (5.1), (5.2) and (5.3), it is said simply for M' to satisfy the $(*)$ condition.

Remark 5.1 *It can be easily seen that* c_3=0, *then the ambient space* M' *is locally symmetric.*

Remark 5.2 *If* M *is maximal, then the condition (5.1) can be replaced by*

$$
(5.4)\qquad Ric'(v,v) \geq c_1
$$

for any time-like vector v.

If $M^{'}$ satisfies the conditions (5.4), (5.2) and (5.3), it is said simply for $M^{'}$ to satisfy the condition $(*^{'})$.

Remark 5.3 *If $M^{'}$ is a Lorentz space form $M_1^{n+1}(c)$ of index 1 and of constant curvature c, then it satisfies the condition $(*^{'})$, where $-\frac{c_1}{n}=c_2=c$.*

Now we assume that the ambient space $M^{'}$ satisfies the condition $(*)$ and the mean curvature of the hypersurface M is constant. Then the Laplacian of the squared norm h_2 of the second fundamental form α of M is given by

$$\begin{aligned}\triangle h_2 &= \triangle(\sum_{i,j} h_{ij}h_{ij}) = 2\sum_{i,j,k}(h_{ijk}h_{ij})_k = 2\sum_{i,j,k}(h_{ijkk}h_{ij} + h_{ijk}h_{ijk}) \\ &= 2|\nabla\alpha|^2 + 2\sum_{i,j,k} h_{ijkk}h_{ij} = 2|\nabla\alpha|^2 + 2\sum_{i,j}(\triangle h_{ij})h_{ij},\end{aligned}$$

where $\nabla\alpha$ is the covariant derivative of the second fundamental form α and $|\nabla\alpha|$ is the norm of $\nabla\alpha$ which is defined by $\sum_{i,j,k} h_{ijk}h_{ijk}$. Hence, by (4.4) and the assumption $\sum_k h_{kkj}=0$, we have

$$\begin{aligned}\triangle h_2 &= 2|\nabla\alpha|^2 + 2\sum_{i,j}\{\sum_k(R^{'}_{0kik;j} + R^{'}_{0ijk;k}) + \sum_k(h_{kk}R^{'}_{0ij0} + h_{ij}R^{'}_{0k0k}) \\ &+ \sum_{k,m}(2h_{km}R^{'}_{mijk} + h_{mj}R^{'}_{mkik} + h_{mi}R^{'}_{mkjk}) - hh^2_{ij} + h_2h_{ij}\}h_{ij}.\end{aligned}$$

Thus we get

$$\begin{aligned}(5.5)\qquad \triangle h_2 &= 2|\nabla\alpha|^2 + 2\sum_{i,j,k} h_{ij}(R^{'}_{0kik;j} + R^{'}_{0ijk;k}) \\ &+ 2(\sum_{i,j} hh_{ij}R^{'}_{0ij0} + h_2\sum_k R^{'}_{0k0k}) \\ &+ 4(\sum_{i,j,k,m} h_{ij}h_{km}R^{'}_{mijk} + \sum_{j,k,m} h^2_{mj}R^{'}_{mkjk}) - 2(hh_3 - h_2^2),\end{aligned}$$

where we have denoted by $h^3_{ij}=\sum h_{ir}h^2_{rj}$ and $h_3 = \sum h^3_{ii}$. Since the matrix $H=(h_{ij})$ can be diagonalized , the component of h_{ij} of H can be expressed by

$$(5.6)\qquad h_{ij} = \lambda_i\delta_{ij},$$

where λ_i is the principle curvature on M. By definition, we see

$$\lambda_i^2 \le h_2 = \sum_i \lambda_i^2,$$

and hence we have

$$-\sqrt{h_2} \leq \lambda_i \leq \sqrt{h_2}, \tag{5.7}$$

$$-h_2 \leq \lambda_i \lambda_j \leq h_2. \tag{5.8}$$

Now, we estimate (5.5) from above. First, we treat with the second term of (5.5). It is seen that we have

$$\begin{aligned} -2\sum_{i,j,k}(R'_{0kik;j} + R'_{0ijk;k})h_{ij} &= -2\sum_{j,k}\lambda_j(R'_{0kjk;j} + R'_{0jjk;k}) \\ &\leq 2\sum_{j,k}|\lambda_j|(|R'_{0kjk;j}| + |R'_{0jjk;k}|). \end{aligned}$$

So by (5.3) and (5.7) we have

$$\textit{the second term of } (5.5) \geq -4c_3\sqrt{h_2}. \tag{5.9}$$

Next, we consider the third term of (5.4). It is estimated as follows:

$$\begin{aligned} 2(\sum_{i,j} hh_{ij}R'_{0ij0} &+ h_2\sum_k R'_{0k0k}) = 2\sum_k(h\lambda_k R'_{0kk0} + h_2\sum_k R'_{0k0k}) \\ &= 2\sum_k(h_2 - h\lambda_k)R'_{0k0k} = 2\sum_k(h_2 - h\lambda_k)\frac{c_1}{n}, \end{aligned}$$

where we have used (5.1). Hence we have

$$\textit{the third term of } (5.5) = \frac{2c_1(nh_2 - h^2)}{n}. \tag{5.10}$$

It is evident that if the ambient space M' is a Lorentz space form $M_1^{n+1}(c)$ of constant curvature c and if the hypersurface M is *maximal*,then it also holds under (5.4), namely if M' satisfies the condition $(*')$, then *the third term of* (5.5) $\geq 2c_1h_2$. Last we estimate the fourth term of (5.5). We have by (5.2)

$$4(\sum_{i,j,k,m} h_{ij}h_{km}R'_{mijk} + \sum_{j,k,m} h_{mj}^2 R'_{mkjk}) = 4\sum_{j,k}(\lambda_j\lambda_k R'_{kjjk} + \lambda_j^2 R'_{kjkj})$$

$$= 4\sum_{j,k}(\lambda_j^2 - \lambda_j\lambda_k)R'_{kjkj} = 2\sum_{j,k}(\lambda_j - \lambda_k)^2 R'_{kjkj} \geq 2c_2\sum_{j,k}(\lambda_j - \lambda_k)^2.$$

Accordingly, we obtain

$$\textit{the fourth term of } (5.5) \geq 4c_2(nh_2 - h^2), \tag{5.11}$$

where we have used the formula

$$\sum_{j,k}(\lambda_j - \lambda_k)^2 = 2n\sum_j \lambda_j^2 - 2\sum_{j,k}\lambda_j\lambda_k = 2n\sum_j \lambda_j^2 - 2(\sum_j \lambda_j)^2$$

and the definitions of $h_2 = \sum_j \lambda_j^2$ and $h^2 = (\sum_j \lambda_j)^2$. Thus, substituting (5.9), (5.10) and (5.11) into (5.5), we can prove the following.

Lemma 5.4 *Let $M^{'}$ be an $(n+1)$-dimensional Lorentz manifold satisfying the condition $(*)$ and M a space-like hypersurface of $M^{'}$. If its mean curvature is constant, then we have*

$$\triangle h_2 \geq -4c_3\sqrt{h}_2 + \frac{2(2nc_2 + c_1)(nh_2 - h^2)}{n} - 2(hh_3 - h_2^2). \tag{5.12}$$

In particular, if M is maximal, we have

$$\triangle h_2 \geq -4c_3\sqrt{h}_2 + 2(2nc_2 + c_1)h_2 + 2h_2^2.$$

Also, if $M^{'}=M_1^{n+1}(c)$, then we obtain

$$\triangle h_2 \geq 2c(nh_2 - h^2) - 2(hh_3 - h_2^2).$$

6 Proof of Main Theorem

Let $M^{'}$ be an $(n+1)$-dimensional Lorentz manifold and let M be a complete hypersurface of $M^{'}$ with constant mean curvature. Assume that the ambient space satisfies the condition $(*)$. The condition $(*)$ is defined by (5.1), (5.2) and (5.3). Now, by (5.12) in Lemma 5.1 the function h_2 satisfies

$$\triangle h_2 \geq -4c_3\sqrt{h}_2 + frac2(2nc_2 + c_1)(nh_2 - h^2)n - 2(hh_3 - h_2^2).$$

Moreover, we obtain

$$-2hh_3 = -2h\sum_i h_{ii}^3 = -2h\sum_j \lambda_j^3 \geq -2h\sum_j \sqrt{h}_2^{\;3} = -2nhh_2\sqrt{h}_2, \tag{6.1}$$

from which together with (5.12) it follows that

$$\triangle h_2 \geq -4c_3\sqrt{h}_2 + 2(2nc_2 + c_1)(h_2 - \frac{h^2}{n}) - 2nhh_2\sqrt{h}_2 + 2h_2^2. \tag{6.2}$$

Now we define a non-negative function f by $f^2 = h_2$. Then it turns out to be

$$\triangle f^2 \geq 2[f^4 - nhf^3 + (2nc_2 + c_1)f^2 - 2c_3 f - \frac{h^2}{n}(2nc_2 + c_1)]. \tag{6.3}$$

Proof of the Main Theorem

Let $\lambda_1, \cdots, \lambda_n$ be principal curvatures on M. The Ricci tensor S_{ij} is expressed by

$$S_{ij} = \sum_k (R'_{kikj} - h_{ij}h_{kk} + h_{ik}h_{jk}).$$

So we have

$$S_{jj} \geq (n-1)c_2 - h\lambda_j + \lambda_j^2 \geq (n-1)c_2 - \frac{h^2}{4},$$

which yields the Ricci curvature of M is bounded from below. For the function f defined by $f^2 = h_2$, by (6.3) we have

$$\triangle f^2 \geq F(f^2),$$

where the function $F(x)$ is defined by

$$F(x) = 2[x^2 - nhx^{\frac{3}{2}} + (2nc_2 + c_1)x - 2c_3x^{\frac{1}{2}} - \frac{h^2}{n}(2nc_2 + c_1)].$$

By comparing with (3.12), we get

$n = 2$, $n - k = \frac{1}{2}$, $c_0 = 2$, $c_{k+1} = -\frac{2h^2(2nc_2+c_1)}{n}$, where we have used $2nc_2 + c_1 > 0$. Now we are able to apply Theorem 3.2 to the function f^2. Then we obtain

$$F(f_1^2) \leq 0, \tag{6.4}$$

where f_1^2 denotes the supremum of the given function f^2.

We define the function $y = y(x)$ of the variable x by

$$y = y(x) = x^4 - nhx^3 + (2nc_2 + c_1)x^2 - 2c_3x - \frac{h^2}{n}(2nc_2 + c_1).$$

By the assumption $2nc_2+c_1 > 0$ and the fact that the hypersurface is not maximal, the algebraic equation $y(x) = 0$ with constant coefficients has positive roots because $y(0) < 0$ and it converges to infinity as x tends to infinity. We denote by $\sqrt{a}_1$ $(a_1 > 0)$ the minimal root among the positive roots. So it depends only on the constant coefficients, namely, it depends on c_1, c_2, c_3, h and n, and by definition we see that

$$y|[0, \sqrt{a}_1) < 0.$$

From the above equation together with (6.4) it follows that we have $0 \leq f_1 \leq \sqrt{a}_1$. Since the squared norm h_2 of the second fundamental form is given by $h_2 = f^2$, we have

$$sup\ h_2 = f_1^2 \leq a_1.$$

So we get the conclusion. □

If the hypersurface M is maximal, then we have by (6.3)

$$\triangle f^2 \geq 2\{f^4 + (2nc_2 + c_1)f^2 - 2c_3 f\} = F(f^2),$$

where a non-negative function f is defined by $f^2 = h_2$. By a similar method to the proof of our Main Theorem, we have

$$F(f_1^2) \leq 0, \tag{6.5}$$

where f_1 denotes the supremum of the function f. We define a function y of the variable x by

$$y = y(x) = x\{x^3 + (2nc_2 + c_1)x - 2c_3\}.$$

By the direct calculus, there exists a unique positive root of the equation $y(x) = 0$, say $\sqrt{a_1}$, if $c_3 > 0$.

Corollary 6.1 *Let $M^{'}$ be an $(n+1)$-dimensional Lorentz manifold which satisfies the condition $(*)$ and let M be a complete space-like maximal hypersurface. If $M^{'}$ is not locally symmetric, then there exists a positive constant a_1 depending on $c_1, c_2, c_3 (> 0), h$ and n such that $h_2 \le a_1$.*

Remark 6.2 *Corollary 6.1 was proved by Li ([8]) under the additional condition* $c_3^2 + \frac{(2nc_2+c_1)^3}{27} < 0$.

Remark 6.3 *In the case where the ambient space is locally symmetric and it satisfies the condition $(*^{'})$, the constant a_1 is the positive root of the algebraic equation*

$$F(x^2) = x^2\{x^2 + (2nc_2 + c_1)\} = 0,$$

which yields that if $2nc_2 + c_1 \ge 0$, then $F|(0,\infty) > 0$, which means that we have no positive roots. In the case where $2nc_2 + c_1 < 0$ there exists a unique positive root of the equation $y(x) = 0$, say $\sqrt{a_1}$. In the first case, considering (6.5) we have $f_1 = 0$. By definition of $f_1 = supf$ for the non-negative function f, we see that f vanishes identically on M. It yields that M is totally geodesic. So if it satisfies $2nc_2 + c_1 < 0$, then we have $a_1 = -(2nc_2 + c_1)$. This result was derived by Li ([8]). The first assertion of Corollary 6.1 was also proved by Nishikawa([9]). In particular, when $M^{'} = H_1^{n+1}(c)$, this reduces to Ishihara's theorem ([7]).

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