Remarks on the cohomological classification of certain Fréchet bundles

George Galanis and Efstathios Vassiliou

Dedicated to the memory of Grigorios TSAGAS (1935-2003)

Abstract

We discuss the classification of certain infinite dimensional fiber bundles modelled on Fréchet spaces. First we consider *vector* bundles (over a Banach manifold X) of fiber type a Fréchet space $\mathbb{F}$, obtained as the limit of a projective system of Banach vector bundles. Such bundles are classified by the cohomology set $H^1(X, \underline{H}^0(\mathbb{F}))$, with coefficients in the sheaf of germs of $H^0(\mathbb{F})$-valued smooth maps on X, where $H^0(\mathbb{F})$ is an appropriate topological Fréchet group replacing the pathological $GL(\mathbb{F})$. An analogous classification is proved for Fréchet *principal* bundles whose structural group can be realized as a projective limit of Banach Lie groups.

Mathematics Subject Classification: 58B20, 58B25, 55R15, 55N30.
Key words: Fréchet vector and principal bundles, projective limits, cohomology set, cohomological classification.

Introduction

In our earlier papers [2], [3], and [10] we have studied the geometry of certain Fréchet vector and principal bundles, arising as projective limits of corresponding ordinary Banach bundles. The consideration of such bundles is necessitated by the fact that arbitrary Fréchet bundles present serious difficulties, especially when dealing with problems regarding differential equations and/or the structural group $GL(\mathbb{F})$, if $\mathbb{F}$ is the Fréchet fiber type. In this way, equations, structural groups and various geometric features such as connections, parallel translations, holonomy homomorphisms etc. reduce to projective limits of corresponding entities in the category of Banach bundles, where the classical gadgetry can be fully applied. In particular, the pathological group $GL(\mathbb{F})$, which does not admit even a satisfactory topological group structure, is replaced by a projective limit of appropriate Banach-Lie groups, denoted by $H^0(\mathbb{F})$, which can be also thought of as a generalized Fréchet-Lie group (see Section 1 for details).

In the present note we are concerned with the cohomological classification of vector and principal Fréchet bundles of the aforementioned type. More precisely, in Section

Balkan Journal of Geometry and Its Applications, Vol.9, No.2, 2004, pp. 23-31.

2 we prove that the isomorphism classes of such vector bundles, with base (a Banach manifold) X and of fiber type a Fréchet space $\mathbb{F}$, are in bijective correspondence with the 1st cohomology set $H^1(X, \underline{H}^0(\mathbb{F}))$ with coefficients in the sheaf of germs of $H^0(\mathbb{F})$-valued (generalized) smooth maps on X (Theorem 2.3).

It is worth noticing that the preceding classification relies heavily on the fact that the isomorphisms of vector bundles considered are also assumed to be projective limits of ordinary isomorphisms of Banach bundles.

In Section 3 we obtain an analogous classification for principal bundles whose structural group is a projective limit of Banach-Lie groups (Theorem 3.3). This choice of structural group ensures that the bundles themselves and *all* the isomorphisms between them are necessarily projective limits.

We conclude this note by relating $H^0(\mathbb{F})$-principal bundles with vector bundles of fiber type $\mathbb{F}$.

1 Preliminaries

For the reader's convenience, we summarize here the basic notions needed throughout this note, referring for more details to [2], [3].

Let $\mathbb{F}$ be a Fréchet space whose topology is determined by a family of seminorms $\{p_i\}_{i\in\mathbb{N}}$ with $p_1 \leq p_2 \leq \cdots$. Then, $\mathbb{F}$ can be realized as a projective limit of Banach spaces, i.e., $\mathbb{F} \cong \varprojlim\{\mathbb{E}^i; \rho^{ji}\}_{i,j\in\mathbb{N}}$, where $\mathbb{E}^i$ are the completions of the quotients $\mathbb{F}/\mathrm{K}er(p_i)$ (see [9]). We note that indices referring to projective systems are written as superscripts in order to avoid confusion with indices relative to local charts used later on.

With these notations we define the group

$$H^0(\mathbb{F}) := \big\{(g^i)_{i\in\mathbb{N}} \in \textstyle\prod_{i=1}^{\infty} GL(\mathbb{E}^i) \,\big|\, \varprojlim g^i \text{ exists}\big\}.$$

Since it can be realized as the projective limit of the Banach-Lie groups

$$H_0^i(\mathbb{F}) := \big\{(g^1, g^2, \ldots, g^i) \in \textstyle\prod_{j=1}^{i} \mathcal{L}is(\mathbb{E}^j) \,\big|\, \rho^{jk} \circ g^j = g^k \circ \rho^{jk} (i \geq j \geq k \geq 1)\big\},$$

after the identification $(g^i)_{i\in\mathbb{N}} \cong \big((g^1), (g^1, g^2), \ldots\big)$, $H^0(\mathbb{F})$ is a topological group with the inverse limit topology. Moreover, $H^0(\mathbb{F})$ considered as embedded in the Fréchet space

$$H(\mathbb{F}) := \big\{(g^i)_{i\in\mathbb{N}} \in \textstyle\prod_{i=1}^{\infty} \mathcal{L}(\mathbb{E}^i) \,\big|\, \varprojlim g^i \text{ exists}\big\}$$

is called a *generalized Fréchet-Lie group*.

The group $H^0(\mathbb{F})$ plays a fundamental role in the study of the Fréchet vector bundles that can be obtained as projective limits of corresponding Banach bundles. Indeed, assume that

$$\{(E^i, \pi^i, X); \phi^{ji}\}_{i,j\in\mathbb{N}}$$

is a projective system of Banach vector bundles satisfying the following conditions:

(PLVB 1) The corresponding fiber types $\mathbb{E}^i$ form a projective system.

(PLVB 2) There exists an open covering $\{U_\alpha\}_{\alpha\in I}$ of X and respective trivializations $\{(U_\alpha, \tau_\alpha^i)\}_{\alpha\in I}$ of E^i's such that the limit $\varprojlim_{i\in\mathbb{N}} \tau_\alpha^i$ exists for each index $\alpha \in I$.

Then, the projective limit $E := \varprojlim E^i$ of this system can be endowed with the structure of a smooth Fréchet vector bundle of fiber type $\mathbb{F} = \varprojlim \mathbb{E}^i$ over the same base X. The differentiability considered here is meant in the sense of [7], [8]. More precisely, the pairs $\{(U_\alpha, \varprojlim \tau_\alpha^i)\}_{a\in I}$, obtained by condition (PLVB 2), form a trivializing cover of E. As corresponding transition functions we consider the mappings

$$T^*_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow H^0(\mathbb{F}) : x \mapsto (T^i_{\alpha\beta}(x))_{i\in\mathbb{N}}, \tag{1.1}$$

where $T^i_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{E}^i)$ are the usual transition functions of E^i. Note that $(T^*_{\alpha\beta})$ are naturally continuous maps, with respect to the inverse limit topology of $H^0(\mathbb{F})$, and smooth if they are considered as taking values in $H(\mathbb{F})$.

Therefore, the structural group of the bundle E is now $H^0(\mathbb{F})$, which replaces the general linear group $GL(\mathbb{F})$. The relation between the new transition functions (1.1) and the ordinary ones

$$T_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(\mathbb{F}) : x \mapsto \big(\varprojlim_{i\in\mathbb{N}} \tau^i_{\alpha,x}\big) \circ \big(\varprojlim_{i\in\mathbb{N}} \tau^i_{\beta,x}\big)^{-1},$$

where $\tau^i_{\alpha,x}$ stands for the restriction of τ^i_α on the fiber over x, is given by

$$\varepsilon \circ T^*_{\alpha\beta} = T_{\alpha\beta}; \qquad \alpha, \beta \in I. \tag{1.2}$$

Here ε is the continuous mapping

$$\varepsilon : H^0(\mathbb{F}) \longrightarrow GL(\mathbb{F}) : (g^i)_{i\in\mathbb{N}} \mapsto \varprojlim g^i,$$

connecting the previous two groups of $\mathbb{F}$.

It is essential to note that the transition functions (1.1) fully characterize the bundles under discussion, for if an arbitrary Fréchet vector bundle E has transition functions $\{T_{\alpha\beta}\}$ factorizing in the form of (1.2), then E can be always realized as a projective limit of Banach vector bundles (see [2, Theorem 1.4]).

This approach allows to circumvent many difficulties posed by the pathological structure of $GL(\mathbb{F})$ when one tries to study the geometry of Fréchet vector bundles by employing the "classical" methods that have been proven successful up to the case of finite dimensional and Banach bundles. In this respect see, e.g., [2], [10].

On the other hand, *principal* Fréchet bundles are easier to handle, since the assumption that their structural groups are projective limits of Banach-Lie groups suffices to recapture every element of the bundle as a projective limit. More precisely, if (P, G, X, π) is a Fréchet principal bundle, where the base X is a *Banach* manifold and the structural group G a projective limit of Banach-Lie groups $G = \varprojlim G^i$, then:

- There always exists a projective system of Banach principal bundles $\{(P^i, G^i, X, \pi^i)\}_{i\in\mathbb{N}}$ whose limit is isomorphic to P;
- Every connection on P can be realized as a projective limit of connections on (P^i).

Based on these results, we can prove most of the main geometric properties of the bundles under consideration, e.g. the existence of connections and parallel displacements, Cartan's equations, the existence of frame bundles etc., whose validity is not ensured in the case of arbitrary Fréchet bundles (see for details [3]).

2 Classification of Fréchet vector bundles

Let $\mathcal{V}_X(\mathbb{B})$ be the set of isomorphism classes of finite dimensional or Banach vector bundles of fiber type $\mathbb{B}$, over a manifold X. As is well known (see, e.g., [5]), this set is equivalent to the 1st cohomology set of X with coefficients in the sheaf of germs of smooth $GL(\mathbb{B})$-valued maps on X; that is,

$$\mathcal{V}_X(\mathbb{B}) \;\cong\; H^1(X, \underline{GL}(\mathbb{B})).$$

However, by what have been said in the Introduction, the previous result fails in the case of arbitrary Fréchet bundles of fiber type a Fréchet space $\mathbb{F}$.

The aim of this section is to obtain an analogous cohomological classification for the Fréchet vector bundles that can be realized as projective limits of Banach bundles. To this end, let $\{(E^i, \pi^i, X); \phi^{ji}\}_{i,j\in\mathbb{N}}$ and $\{(\widetilde{E}^i, \widetilde{\pi}^i, X); \widetilde{\phi}^{ji}\}_{i,j\in\mathbb{N}}$ be two projective systems of *Banach* vector bundles, of fiber type $\mathbb{E}^i$, $i \in \mathbb{N}$, satisfying conditions (PLVB 1), (PLVB.2). We denote by

$$(E = \varprojlim E^i, \pi = \varprojlim \pi^i, X), \quad (\widetilde{E} = \varprojlim \widetilde{E}^i, \widetilde{\pi} = \varprojlim \widetilde{\pi}^i, X)$$

the corresponding limit bundles with fiber type the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}^i$. Concerning isomorphisms between E and $\widetilde{E}$ we obtain the following Lemmas, which are essential for the proof of the main Theorem 2.3 below.

Lemma 2.1. *If $\{g^i : E^i \to \widetilde{E}^i\}_{i\in\mathbb{N}}$ is a projective system of vector bundle isomorphisms, then the corresponding limit*

$$g := \varprojlim g^i : E \longrightarrow \widetilde{E}$$

is a Fréchet vector bundle isomorphism.

Proof. The mapping g is a smooth bijection in the sense of [7], [8] as a projective limit of smooth bijective mappings (see also [2]).

On the other hand, g preserves the fibers of the bundles under consideration since

$$\widetilde{\pi} \circ g = (\varprojlim \widetilde{\pi}^i) \circ (\varprojlim g^i) = \varprojlim (\widetilde{\pi}^i \circ g^i) = \varprojlim \pi^i = \pi.$$

Moreover, if we consider the trivializations $(U, \varprojlim \tau^i)$ and $(U, \varprojlim \widetilde{\tau}^i)$ of E and $\widetilde{E}$ respectively (without loss of generality we can take the same open cover U of X), then the mapping

$$F : U \longrightarrow \mathcal{L}(\mathbb{F}) : x \mapsto \widetilde{\tau}_x \circ g_x \circ \tau_x^{-1}$$

can be factored in the following way:

$$F = \varepsilon \circ G,$$

with

$$G : U \longrightarrow H(\mathbb{F}) : x \mapsto (\widetilde{\tau}^i_x \circ g^i_x \circ (\tau^i_x)^{-1})_{i\in\mathbb{N}}$$

and

$$\varepsilon : H(\mathbb{F}) \longrightarrow \mathcal{L}(\mathbb{F}) : (g^i)_{i\in\mathbb{N}} \mapsto \varprojlim g^i.$$

Indeed, for every $x \in U$, we have that

$$\begin{aligned}(\varepsilon \circ G)(x) &= \varprojlim(\widetilde{\tau}^i_x \circ g^i_x \circ (\tau^i_x)^{-1}) \\ &= \varprojlim \widetilde{\tau}^i_x \circ \varprojlim g^i_x \circ \varprojlim (\tau^i_x)^{-1} \\ &= \widetilde{\tau}_x \circ g_x \circ \tau_x^{-1} \\ &= F(x).\end{aligned}$$

As a result, taking into account that each component

$$G^i : U \longrightarrow \mathcal{L}(\mathbb{E}^i) : x \mapsto \widetilde{\tau}^i_x \circ g^i_x \circ (\tau^i_x)^{-1}$$

is smooth, we conclude that G and F are also smooth.

Analogously we prove that the map

$$U \longrightarrow \mathcal{L}(\mathbb{F}) : x \mapsto \tau_x \circ g_x^{-1} \circ (\widetilde{\tau}_x)^{-1}$$

is smooth, which completes the proof. □

Lemma 2.2. *Any vector bundle isomorphism of the form $g = \varprojlim g^i : E \to \widetilde{E}$ corresponds bijectively to a family $\{h_\alpha : U_\alpha \to H^0(\mathbb{F})\}_{\alpha\in I}$ of smooth mappings, where $\{U_\alpha\}_{\alpha\in I}$ is an open covering of (the common base) X, such that*

$$\widetilde{T}^*_{\alpha\beta}(x) = h_\alpha(x) \circ T^*_{\alpha\beta}(x) \circ h_\beta^{-1}(x), \tag{2.1}$$

if $\{T^*_{\alpha\beta} : U_\alpha \cap U_\beta \to H^0(\mathbb{F})\}$ and $\{\widetilde{T}^*_{\alpha\beta} : U_\alpha \cap U_\beta \to H^0(\mathbb{F})\}$ are the transition functions of E and $\widetilde{E}$ respectively, defined in Section 1.

Proof. Since each $g^i : E^i \to \widetilde{E}^i$ is an isomorphism between Banach bundles, there exists a family of smooth mappings

$$h^i_\alpha : U_\alpha \longrightarrow GL(\mathbb{E}^i); \qquad \alpha \in I,$$

such that

$$h^i_\alpha(x) = \widetilde{\tau}^i_{\alpha,x} \circ g^i_x \circ (\tau^i_{\alpha,x})^{-1},$$

where $(U_\alpha, \tau^i_\alpha)_{\alpha\in I}$, $(U_\alpha, \widetilde{\tau}^i_\alpha)_{\alpha\in I}$ are trivializing coverings of E^i and $\widetilde{E}^i$ respectively. These coverings can be chosen so that condition (PLVB 2) be satisfied. Then, for any $i, j \in \mathbb{N}$ with $j \geq i$, we see that

$$\begin{gathered}\rho^{ji} \circ h^j_\alpha(x) = \rho^{ji} \circ \widetilde{\tau}^j_{\alpha,x} \circ g^j_x \circ (\tau^j_{\alpha,x})^{-1} = \widetilde{\tau}^i_{\alpha,x} \circ \widetilde{\phi}^{ji} \circ g^j_x \circ (\tau^j_{\alpha,x})^{-1} = \\ \widetilde{\tau}^i_{\alpha,x} \circ g^i_x \circ \phi^{ji} \circ (\tau^j_{\alpha,x})^{-1} = \widetilde{\tau}^i_{\alpha,x} \circ g^i_x \circ (\tau^i_{\alpha,x})^{-1} \circ \rho^{ji} = h^i_\alpha(x),\end{gathered}$$

where $\rho^{ji} : \mathbb{E}^j \to \mathbb{E}^i$ are the connecting morphisms of the projective system $(\mathbb{E}^i)$. As a result, the linear isomorphism

$$h_\alpha(x) := \varprojlim h^i_\alpha(x) : \mathbb{F} \longrightarrow \mathbb{F}$$

($\mathbb{F} = \varprojlim \mathbb{E}^i$) exists for each $x \in U_\alpha$, thus we may define the (smooth) mapping

$$h_\alpha : U_\alpha \longrightarrow H^0(\mathbb{F}) : x \mapsto \left(h^1_\alpha(x), \left(h^1_\alpha(x), h^2_\alpha(x)\right), \dots\right). \tag{2.2}$$

On the other hand, for any $\alpha, \beta \in I$, we have that

$$\begin{aligned}\widetilde{T}^*_{\alpha\beta}(x) \circ h_\beta(x) \circ T^*_{\beta\alpha}(x) = \left(\widetilde{T}^i_{\alpha\beta}(x) \circ h^i_\beta(x) \circ T^i_{\beta\alpha}(x)\right)_{i\in\mathbb{N}} = \\ \left(\widetilde{\tau}^i_{\alpha,x} \circ (\widetilde{\tau}^i_{\beta,x})^{-1} \circ h^i_\beta(x) \circ \tau^i_{\beta,x} \circ (\tau^i_{\alpha,x})^{-1}\right)_{i\in\mathbb{N}} = \\ \left(\widetilde{\tau}^i_{\alpha,x} \circ g^i_x \circ (\tau^i_{\alpha,x})^{-1}\right)_{i\in\mathbb{N}} = \left(h^i_\alpha(x)\right)_{i\in\mathbb{N}} = h_\alpha(x)\end{aligned}$$

Conversely, any family of smooth mappings as in (2.2), satisfying also the compatibility condition (2.1), gives rise, for each $i \in \mathbb{N}$, to a corresponding family of smooth mappings (relative to the bundle E^i)

$$h^i_\alpha : U_\alpha \longrightarrow GL(\mathbb{E}^i); \qquad \alpha \in I,$$

so that

$$\widetilde{T}^i_{\alpha\beta}(x) \circ h^i_\beta(x) \circ T^i_{\beta\alpha}(x) = h^i_\alpha(x); \qquad \alpha \in I,$$

holds true for every $x \in U_\alpha$, where $\{T^i_{\alpha\beta}\}$ and $\{\widetilde{T}^i_{\alpha\beta}\}$ are the usual transition functions of E^i and $\widetilde{E}^i$, respectively. Therefore, we define a bundle isomorphism $g^i : E^i \to \widetilde{E}^i$ such that

$$h^i_\alpha(x) = \widetilde{\tau}^i_{\alpha,x} \circ g^i_x \circ (\tau^i_{\alpha,x})^{-1}.$$

Then, for any $j \geq i$, we easily check that $\widetilde{\phi}^{ji} \circ g^j = g^i \circ \phi^{ji}$ and, in virtue of Lemma 2.1, the mapping $g = \varprojlim g^i : E \to \widetilde{E}$ exists and determines a vector bundle isomorphism. □

Note. According to a well-known terminology, two cocycles satisfying (2.1) are said to be *cohomologous*.

Within the category of vector bundles with fiber the Fréchet space $\mathbb{F}$ and base X, we single out the ones which are projective limits of Banach bundles in the sense of Section 1. Considering the obvious equivalence relation induced by the isomorphisms that can be realized as projective limits, we obtain the quotient space $\mathcal{V}^{pl}_X(\mathbb{F})$. Then, we are in a position to prove the following cohomological classification theorem.

Theorem 2.3. *Equality*

$$\mathcal{V}^{pl}_X(\mathbb{F}) = H^1(X, \underline{H}^0(\mathbb{F}))$$

holds true within a bijection.

Proof. Lemmas 2.1 and 2.2 allow now us to follow the classical pattern. As a matter of fact, if $E = \varprojlim E^i$ and $\widetilde{E} = \varprojlim \widetilde{E}^i$ are isomorphic bundles by means of the limit isomorphism $g = \varprojlim g^i$, then the transition functions $\{T^*_{\alpha\beta} : U_\alpha \cap U_\beta \to H^0(\mathbb{F})\}$ and $\{\widetilde{T}^*_{\alpha\beta} : U_\alpha \cap U_\beta \to H^0(\mathbb{F})\}$ of E and $\widetilde{E}$, respectively, are cohomologous by means of the family (h_α) obtained in Lemma 2.2.

Therefore, to $[E] \in \mathcal{V}^{pl}_X(\mathbb{F})$ we assign the class $[(T^\star_{\alpha\beta})] \in H^1(X, \underline{H}^0\mathbb{F}))$. This is a well defined bijection according to Lemma 2.2 and the constructions of Section 1. □

Remark 2.4. From the preceding results it is clear that the cohomological classification of Fréchet vector bundles that are projective limits of Banach bundles is based on two main considerations:

1) The general linear group $GL(\mathbb{F})$ (of a Fréchet space $\mathbb{F}$) is replaced by the group $H^0(\mathbb{F})$, which serves now as the structural group of our bundles.

2) The isomorphisms of our vector bundles are also projective limits of isomorphisms of ordinary Banach bundles.

Both are very natural since we wish to remain within the category of projective limits of Banach vector bundles.

3 Classification of Fréchet principal bundles

As explained in the Introduction and the last part of Section 1, in the case of principal bundles it is enough to assume that only the structural groups involved are projective limits of Banach Lie groups. As a matter of fact, we have already proven (see [3, Theorem 2.1]) the following.

Lemma 3.1. *Let (P, G, X, π) be a Fréchet principal bundle over a Banach base X with structural group a projective limit of Banach Lie groups $G \cong \varprojlim\{G^i; \phi^{ji}\}$, $(i, j \in \mathbb{N})$. Then, the bundle P can be thought of as a projective limit of Banach bundles over X with structural groups G^i, i.e.,*

$$P \cong \varprojlim \left\{(P^i, G^i, X, \pi^i); (f^{ji}, \phi^{ji})\right\},$$

where $f^{ji} : P^j \to P^i$ and $\phi^{ji} : G^j \to G^i$ are the connecting morphisms of the projective systems $(P^i)_{i\in\mathbb{N}}$ and $(G^i)_{i\in\mathbb{N}}$, respectively.

Consequently, we obtain

Proposition 3.2. *Let $P = \varprojlim P^i$ and $\widetilde{P} = \varprojlim \widetilde{P}^i$ be G-principal bundles over X, as in Lemma 3.1. Then, every bundle isomorphism $g : P \to \widetilde{P}$ is a projective limit of bundle isomorphisms, i.e., $g = \varprojlim g^i$, with $g^i : P^i \to \widetilde{P}^i$.*

Proof. Working as in the case of Banach principal bundles (see, e.g., [1, N° 6.4.4]), we check that an isomorphism $g : P \to \widetilde{P}$ is fully determined by a family of smooth mappings $\{h_\alpha : U_\alpha \to G\}_{\alpha\in I}$ satisfying the conditions

$$\begin{aligned} g(\sigma_\alpha(x)) &= \widetilde{\sigma}_\alpha(x) \cdot h_\alpha(x); \qquad x \in U_\alpha, \\ \widetilde{g}_{\alpha\beta}(x) &= h_\alpha(x) \cdot g_{\alpha\beta}(x) \cdot h_\beta^{-1}(x); \qquad x \in U_\alpha \cap U_\beta, \end{aligned} \tag{3.1}$$

where $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ (resp. $\{\widetilde{g}_{\alpha\beta}\}$) are the transition functions and $\{\sigma_\alpha : U_\alpha \to P\}$ (resp. $\{\widetilde{\sigma}_\alpha\}$) the natural sections of P (resp. $\widetilde{P}$), over a common trivializing covering $\{U_\alpha\}$ of X.

Since these transition functions $(g_{\alpha\beta})$ and $(\widetilde{g}_{\alpha\beta})$, as well as the natural sections (σ_α) and $(\widetilde{\sigma}_\alpha)$ coincide with the projective limits of their counterparts on P^i and $\widetilde{P}^i$ respectively, we check that, for each $i \in \mathbb{N}$, the family

$$\{\phi^i \circ h_\alpha : U_\alpha \to G^i\}_{\alpha\in I}, \tag{3.2}$$

where $\phi^i : G \to G^i$ are the canonical projections of the projective limit $G = \varprojlim G^i$, satisfies the analog of (3.1). Thus, (3.2) determines an isomorphism $g^i : P^i \xrightarrow{\simeq} \widetilde{P}^i$. Moreover, we see that

$$\phi^{ji} \circ g^j = g^i \circ \phi^{ji},$$
$$\phi^i \circ g = g^i \circ \phi^i,$$

for all $i, j \in \mathbb{N}$ with $j \geq i$. Thus $g = \varprojlim g^i$ and the proof is complete. □

We denote by $\mathcal{P}_X(G)$ the set of isomorphism classes of Fréchet principal bundles over a *Banach* base X, with structural group a projective limit of Banach Lie groups $G = \varprojlim G^i$. Then, based on Proposition 3.2 and working analogously to the case of vector bundles, we obtain the following classification of Fréchet principal bundles.

Theorem 3.3. *If $\underline{G}$ is the sheaf of germs of G-valued smooth maps on X, then*

$$\mathcal{P}_X(G) = H^1(X, \underline{G}),$$

within a bijection.

Corollary 3.4. *Theorems 2.3 and 3.3 imply that*

$$\mathcal{P}_X(H^0(\mathbb{F})) \cong \mathcal{V}_X^{pl}(\mathbb{F}).$$

The last result essentially shows that any $H^0(\mathbb{F})$-principal bundle identifies with the generalized bundle of frames of a vector bundle as in Section 2. We have already dealt with generalized frame bundles in our previous paper [10].

Acknowledgement. The first author has been partially supported by the Greek State Scholarships Foundation and the second by the Special Research Grant 70/4/3410, University of Athens.

References

[1] N. Bourbaki, *Varietés différentielles et analytiques, Fascicule de résultats*, §§ 1-7, Hermann, 1967.

[2] G. Galanis, *Projective Limits of vector bundles*, Portugal. Math. **55** (1998), 11-24.

[3] G. Galanis, *On a type of Fréchet principal bundles over Banach bases*, Period. Math. Hungar. **35** (1997), 15-30

[4] W. Greub, S. Halperin, R. Vanstone, *Connections, Curvature and Cohomology*, Vol. II, Academic Press, 1973.

[5] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, 1966.

[6] S. Lang, *Differential manifolds*, Addison-Wesley, 1972.

[7] J. A. Leslie, *On a differential structure for the group of diffeomorphisms*, Topology **46** (1967), 263-271.

[8] J. A. Leslie, *Some Frobenious theorems in Global Analysis*, J. Diff. Geom. **42** (1968), 279-297.

[9] H. H. Schaeffer, *Topological Vector Spaces*, Springer-Verlag, 1980.

[10] E. Vassiliou, G. Galanis, *A generalized frame bundle for certain Fréchet vector bundles and linear connections*, Tokyo J. Math. **20** (1997), 129-137.

Authors' addresses:

G. Galanis
Hellenic Naval Academy, Hatzikyriakion,
GR–185 39 Piraeus, Greece
e-mail address: ggalanis@snd.edu.gr

E. Vassiliou
University of Athens, Department of Mathematics,
Panepistimiopolis, GR–157 84 Athens, Greece
e-mail address: evassil@cc.uoa.gr