Critical Sets of 2-Dimensional Compact Manifolds

Liana Ţopan

Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

In this paper we characterize the critical sets of the sphere and of the closed cylinder. Necessary conditions for a subset of a closed surface to be critical are given in the last part of the paper.

Mathematics Subject Classifications: 57R45, 58K05, 58E05, 58C25
Key words: critical point, critical set.

1 Introduction

Let M be a smooth 2-dimensional manifold and $f : M \to \mathbf{R}$ a smooth function. The point $p \in M$ is called *critical point* of f if, for some chart (U, φ) around p, $\varphi(p)$ is a critical point of the function $f \circ \varphi^{-1} : \varphi(U) \to \mathbf{R}$, i.e. $\text{rank}_{\varphi(p)} f \circ \varphi^{-1} = 0$, or, equivalently, $d_{\varphi(p)}(f \circ \varphi^{-1}) = 0$. Otherwise, p will be a *regular point* of f. The set of all critical points of f is the *critical set* of f and it will be denoted by $C(f)$. A number $y_0 \in \mathbf{R}$ is a *critical value* of f if it is the image of a critical point and a *regular value* if it is the image of a regular point. The set of all critical values of f is called the *bifurcation set* of f and is denoted by $B(f)$. A set $C \subset M$ is called *critical* if it is the critical set of some smooth function $f : M \to \mathbf{R}$; $C = C(f)$. The set C is *properly critical* if f can be chosen to be proper, i.e. the inverse images by f of compact sets are also compact.

In the first part of this paper, we characterize the critical sets of the sphere and of the closed cylinder. In the second part, we give some necessary conditions for a subset of a 2-dimensional surface M of genus g, orientable or not, to be the critical set of some smooth function defined on M.

2 Critical sets of the sphere and of the closed cylinder

Let M be a smooth manifold and let $f : M \to \mathbf{R}$ be a smooth map. If there exists an open set $U \subset M$ such that $C(f) = M \setminus U$, f will be called *boundary-critical Morse function for U*. Inside U, f has no critical points, while outside U, all the points are critical. On connected components of $M \setminus U$, f is constant.

Balkan Journal of Geometry and Its Applications, Vol.8, No.2, 2003, pp. 113-120.

Lemma 2.1 [6] *Open multiply connected subsets of* S^2 *have boundary critical Morse functions.*

Theorem 2.2 *Let* $C \subset S^2$, $C \neq S^2$ *be a closed set. Then* C *is critical if and only if the components of* $S^2 \setminus C$ *are all multiply connected.*

Proof. The case $C = S^2$ is trivial, since S^2 is the critical set of any constant function $f : S^2 \to \mathbf{R}$.

First of all, let us suppose that there is a smooth function $f : S^2 \to \mathbf{R}$ with $C(f) = C$ and that there exists some connected component U of $S^2 \setminus C$, which is simply connected. Let c be a regular value of f in U. Then $f^{-1}(c) \cap U$ is a non-empty compact boundaryless 1-dimensional manifold, hence it consists of Jordan curves J and $f = c$ on J. At some point p inside J, we have either a maximum $f(p) > c$, or a minimum $f(p) < c$, or $f = c$ inside J. In any case, $(df)_p = 0$. Since U is simply connected, $p \in U$, contrary to U being in the complement of the critical set.

To prove the converse, consider a non-empty compact subset K of S^2, such that all the connected components of $S^2 \setminus K$ are multiply connected. Suppose that $S^2 \setminus K$ is the union of its components $U_1, U_2, \ldots$.Since each U_i is connected, there exist $f_i : S^2 \to [0, 1]$ the boundary critical Morse functions for U_i. Every f_j has no critical point inside U_j and it is constant in every connected component of $S^2 \setminus U_j$. Since U_k is connected, then U_k lies entirely in one of the connected components of $S^2 \setminus U_j$, for $j \neq k$, hence f_j is constant in each U_k, for $k \neq j$.

We can choose some constants a_j such that, for any $x \in S^2$, $a_j f_j(x) \leq \dfrac{1}{2^j}$. Consider the function $f : S^2 \to \mathbf{R}$, defined by

$$f(x) = \sum_{j \in \mathbf{N}^*} a_j f_j(x).$$

It is clear that the series defining, f converges uniformly, and it can be integrated term by term. One also has

$$(df)_x = \begin{cases} a_j (df_j)_x & \text{for} \quad x \in U_j \\ 0 \quad \text{for} & x \in K. \end{cases}$$

It follows that $C(f) = K$ and the proof is completed. $\square$

In order to study the critical sets of the closed cylinder $S^1 \times [-1, 1]$, we remark first that, by using the stereographic projection, $S^1 \times [-1, 1]$ is diffeomorphic with a closed annulus (Figure 1).

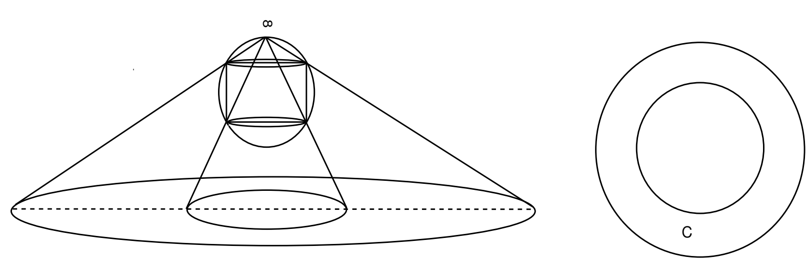

Figure 1

Let C be the closed annulus, with the boundary given by two concentric circles C_1 and C_2 and such that the center of these circles is the origin of a coordinates system of the Euclidean plane.

Consider K a subset of C. We say that K is *interior* to C if $K \cap (C_1 \cup C_2) = \emptyset$. If $K \cap C_1 \neq \emptyset$ (respectively $K \cap C_2 \neq \emptyset$), then K will be called *riparian* to C_1 (respectively to C_2).

Lemma 2.3 *Let $K \subset C$ be an open connected set, riparian to C_1, and D_{C_1} the disk bounded by C_1. Then $K \cup \operatorname{int} D_{C_1} \setminus \{0\}$ is arcwise connected, hence it is connected.*

Proof. Since K is open and connected and C is locally arcwise connected, il follows that K is arcwise connected.

Consider $x \in K$. Then K is riparian to C_1, hence $K \cap C_1 \neq \emptyset$. Choose $x_0 \in K \cap C_1$ and $y \in \operatorname{int} D_{C_1} \setminus \{0\}$. There is an arc $\alpha : [0,1] \to K$, with $\alpha(0) = x$ and $\alpha(1) = x_0$. There is, also, an arc $\beta : [0,1] \to K \cup \operatorname{int} D_{C_1} \setminus \{0\}$ such that $\beta(0) = x_0$ and $\beta(1) = y$. We can choose the arc β such that $\mathrm{Im}\beta \cap K = \{x_0\}$ (Figure 2).

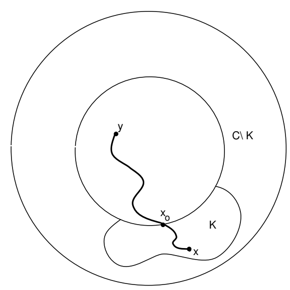

Figure 2

We glue together α and β in order to obtain the arc $\beta \cup \alpha$, joining x and y. It follows that $K \cup \operatorname{int} D_{C_1} \setminus \{0\}$ is arcwise connected . $\square$

In the same way, one obtains:

Lemma 2.4 *Let $K \subset C$ be an open connected set, riparian to C_2. Then $K \cup \operatorname{int}(\operatorname{ext} D_{C_2})$ is arcwise connected, hence it is connected.*

Theorem 2.5 *Let $K \subset C$, $K \neq C$ be a closed set. Then K is critical in C if and only if every connected component of $C \setminus K$ which is interior to C is multiply connected.*

Proof. Let P be a component of $C \setminus K$, interior to C and let us suppose that P would be simply connected. Choose a point $p \in P$. Because K is critical in C, there is a function $f : C \to \mathbf{R}$, with $C(f) = K$. It follows that $f(p)$ is a regular value of the smooth function $f_{|intC}$: $\operatorname{int} C \to \mathbf{R}$.

Since $C \setminus K$ is open in C and P is a connected component of C, it follows that P is open in $\mathbf{R}^2$. Because P is simply connected, $(f_{|intC})^{-1}(f(p)) \cap P$ will contain at least one Jordan curve, which will be entirely included in P.

Conversely, let K be closed in C, so in $\mathbf{R}^2$. One has the obvious relation

$$\mathbf{R}^2 \setminus (K \cup \{0\}) = \operatorname{int}(\operatorname{ext} D_{C_2}) \cup (C \setminus K) \cup (\operatorname{int} D_{C_1} \setminus \{0\}).$$

The union is disjunct and the sets int(extD_{C_2}) and (int D_{C_1})$\setminus\{0\}$ are connected. Then the connected components of $\mathbf{R}^2 \setminus (K \cup \{0\})$ are the connected components of $C \setminus K$ interior to C, the connected component of int $D_{C_1} \setminus \{0\}$ and the connected component of int(ext D_{C_2}). Indeed, if P is a connected component of $\mathbf{R}^2 \setminus (K \cup \{0\})$ which is not a component of $C \setminus K$, it will be riparian to C_1 or to C_2. In the first case, using Lemma 2.3 it follows that $P \cup$ int $D_{C_1} \setminus \{0\}$ is connected, and P lies in the connected component of int D_{C_1}, while for the last case, P will lie in the connected component of int(ext D_{C_2}), according to the Lemma 2.4.

Hence the connected components of $\mathbf{R}^2 \setminus (K \cup \{0\})$ are multiply connected. Since $K \cup \{0\}$ is compact, there is a smooth function $f : \mathbf{R}^2 \to \mathbf{R}$, such that $C(f) = K \cup \{0\}$ [6]. The required map will be $f_{|C} : C \to \mathbf{R}$, since $C(f_{|C}) = K$. □

As a consequence, one has

Theorem 2.6 *Let C be a closed cylinder and $K \subset C$ closed. Then K is critical in C if and only if the connected components of $C \setminus K$ interior to C are multiply connected.*

The interior sets to a closed cylinder are those which do not intersect the boundary of the cylinder, seen as a 2-manifold with boundary.

3 Critical sets of a 2-dimensional surface

Any closed surface is topologically equivalent either to a sphere with p handles (an orientable M_p-type surface), or to a sphere with q Möbius strips glued to it (a non-orientable N_q-type surface).

In any case, a closed surface can be covered by $\mathbf{R}^2$.

Theorem 3.1 [7] *For any positive integers $p \geq 1$, $q > 1$, there is a covering $p : \mathbf{R}^2 \to M_p$ (or $p : \mathbf{R}^2 \to N_q$) which is isomorphic to a covering $p' : \mathbf{R}^2 \to \mathbf{R}^2/\pi_1(M_p)$ (or $p' : \mathbf{R}^2 \to \mathbf{R}^2/\pi_1(N_q)$). For $q = 1$ there is a covering $p_1 : S^2 \to P\mathbf{R}^2$, isomorphic to $p_1' : S^2 \to S^2/\mathbf{Z}_2$.*

Let G be a group which acts free, smoothly and properly on a smooth manifold M. The space M/G can be endowed with a differentiable structure, such that the projection $p : M \to M/G$ is a covering, hence, in particular, a local diffemorphism. One has:

Proposition 3.2 *Let C be critical in M/G. Then $p^{-1}(C)$ is critical in M.*

Proof. Since C is critical in M/G, there is a smooth map $f : M/G \to \mathbf{R}$ such that $C(f) = C$. Take $g = f \circ p : M \to \mathbf{R}$. Then

$$(Tg)_x = (T(f \circ p))_x = (Tf)_{p(x)} \circ (Tp)_x,$$

where $(Tf)_x$ is the tangent map of f at the point x.

Since p is a local diffeomorphism, it follows that $(Tp)_x \neq 0$, $\forall x \in M$. Moreover, the map $(Tp)_x$ is invertible, for all $x \in M$. We have

$$x \in C(g) \Leftrightarrow (Tf)_{p(x)} \circ (Tp)_x = 0.$$

Compose at right by $(Tp)_x^{-1}$, one obtains

$$(Tf)_{p(x)} = 0 \Leftrightarrow p(x) \in C(f) \Leftrightarrow x \in p^{-1}(C(f)) = p^{-1}(C). \quad \square$$

Applying Theorem 3.1 and Proposition 3.2 for $M = \mathbf{R}^2$ and $G = \pi_1(M_p)$, $p \geq 1$, respectively $G = \pi_1(N_q)$, $q > 1$, gives:

Theorem 3.3 *Let C be critical in M_p, $p \geq 1$ (respectively N_q, $q > 1$). Then $p^{-1}(C)$ is critical in $\mathbf{R}^2$, where p is the given by $p : \mathbf{R}^2 \to \mathbf{R}^2/\pi_1(M_p)$ (respectively $p : \mathbf{R}^2 \to \mathbf{R}^2/\pi_1(N_q)$).*

Using the characterization of the critical sets in the plane [6], a set $K \subset M_p$ (respectively $K \subset N_q$), $p^{-1}(K)$ will be critical and unbounded in $\mathbf{R}^2$ if and only if ∞ is arcwise accessible in each simply connected component of $\mathbf{R}^2 \setminus p^{-1}(K)$. One has

$$\mathbf{R}^2 \setminus p^{-1}(K) = C(p^{-1}(K)) = p^{-1}(C(K)) = p^{-1}(M_p \setminus K)\,(\text{respectively}\, p^{-1}(N_q \setminus K)).$$

Theorem 3.4 *Let C be critical in M_p $(p \geq 1)$ (respectively N_q $(q > 1)$). Then ∞ is arcwise accessible in each simply connected component of $p^{-1}(M_p \setminus C)$ (respectively $p^{-1}(N_q \setminus C)$).*

Let us apply Theorem 3.4 for M_2. The group $\pi_1(M_2) = \mathbf{Z} \times \mathbf{Z}$ acts on $\mathbf{R}^2$ as follows:

$$(m, n) \times (x, y) \to (x + m, y + n), \text{ for any } (x, y) \in \mathbf{R}^2 \text{ and for any } (m, n) \in \mathbf{Z} \times \mathbf{Z}.$$

The covering $p : \mathbf{R}^2 \to \mathbf{R}^2/\mathbf{Z}^2$ is isomorphic to the covering

$$p' : \mathbf{R}^2 \to S^1 \times S^1, p'(x, y) = (e^{2\pi i x}, e^{2\pi i y}),$$

the isomorphism being induced by the map

$$h : S^1 \times S^1 \to \mathbf{R}^2/\mathbf{Z}^2, h(e^{2\pi i x}, e^{2\pi i y}) = ([x, y]), 0 \leq x < 1, 0 \leq y < 1,$$

whose inverse is

$$h^{-1} : \mathbf{R}^2/\mathbf{Z}^2 \to S^1 \times S^1, h^{-1}([x, y]) = (e^{2\pi i x}, e^{2\pi i y}).$$

Consider the set $(A = S^1 \times \{1\}) \cup (\{1\} \times S^1)$ and apply the theorem for the covering p'.

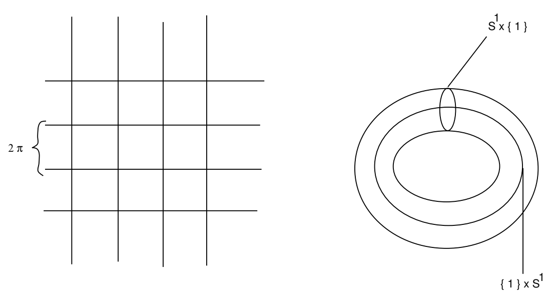

Figure 3

One has

$$p'^{-1}(A) = \{(x, 2k\pi), (2k\pi, y) : k \in \mathbf{Z}, x, y \in \mathbf{R}\}.$$

Since $\mathbf{R}^2 \setminus p'^{-1}(A)$ consists in a system of rectangles, then its components are simply connected. Also, ∞ is arcwise accessible in none of the components of $\mathbf{R}^2 \setminus p'^{-1}(A)$. Hence, $S^1 \times \{1\} \cup \{1\} \times S^1$ cannot be critical in T^2.

Let C be a curve in T^2. The curve C is called *fundamental* if it is not homotopic to a constant curve in T^2. Two curves C_1 and C_2 are called *equivalent* in T^2 if they are homotopic in T^2.

Since $\pi_1(T^2) = \mathbf{Z} \times \mathbf{Z}$, there exist two equivalence classes which generate this fundamental group. These two classes can be represented by two non-equivalent fundamental curves. One can prove that the two classes may be represented by the curves $(S^1 \times \{1\}) \cup (\{1\} \times S^1)$ in T^2. The assertion that the set

$$A = (S^1 \times \{1\}) \cup (\{1\} \times S^1)$$

is not critical in T^2 can be generalized as follows:

Theorem 3.5 *Let C be a closed subset of T^2, which contains two fundamental non-equivalent curves. Then C is not critical in T^2.*

Proof. Let C_1 and C_2 be the two fundamental non-equivalent curves of C. Clearly, $p'^{-1}(C_1 \cup C_2) \subset p^{-1}(C)$. On the other hand, the following relation holds:

$$p'^{-1}(C_1 \cup C_2) = p'^{-1}(C_1) \cup p'^{-1}(C_2).$$

Taking into account the above considerations, we may suppose that $C_1 \approx S^1 \times \{1\}$ and $C_2 \approx S^1 \times \{1\}$. It follows that

$$p'^{-1}(C_1) = \{(x, \alpha(x) + 2k\pi) : x \in \mathbf{R} \,\text{and}\, \alpha : \mathbf{R} \to \mathbf{R} \,\text{is unbounded}\}$$

$$p'^{-1}(C_2) = \{(\beta(y) + 2k\pi, y) : y \in \mathbf{R} \,\text{and}\, \beta : \mathbf{R} \to \mathbf{R} \,\text{is unbounded}\}.$$

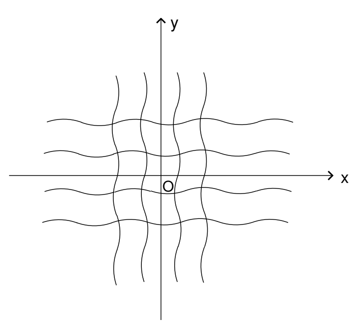

Figure 4

Then any connected component of $\mathbf{R}^2 \setminus p'^{-1}(C_1)$ will be contained in one of the domains bounded by the curves which form $p'^{-1}(C_1)$ and $p'^{-1}(C_2)$, therefore ∞ will be arcwise accessible in none of these components. $\square$

Unfortunately, the necessary conditions for a set C to be critical in a closed surface are not sufficient.

Proposition 3.6 *Consider the torus T^2 and $S^1 \times \{1\} \subset T^2$. Then $p'^{-1}(S^1 \times \{1\})$ is critical in $\mathbf{R}^2$, but $S^1 \times \{1\}$ is not critical in T^2.*

Proof. Since $\mathbf{R}^2 \setminus p'^{-1}(S^1 \times \{1\})$ is the union of parallel strings in $\mathbf{R}^2$, then ∞ is arcwise accessible in each of these strings.

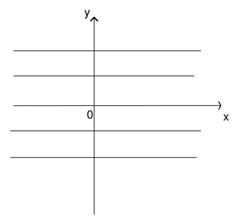

Figure 5

Suppose that $S^1 \times \{1\}$ is the critical set of some differentiable map $f : T^2 \to \mathbf{R}$. Since $S^1 \times \{1\}$ is connected, it follows that $f_{/S^1\times\{1\}}$ is constant, i.e. $f_{/S^1\times\{1\}} = c$, $c \in \mathbf{R}$. But T^2 being compact, there exist $x_1, x_2 \in T^2$ such that $f(x_1) = M = \max\limits_{x\in T^2} f(x)$ and $f(x_2) = m = \min\limits_{x\in T^2} f(x)$. If $M = m$, then f is constant, $f = M$ and $C(f) = T^2$, which is false. Hence, at least one of M and m is non-equal to c. Suppose $M \neq c$. That means that $x_1 \notin S^1 \times \{1\}$. There exists a neighborhood V of x_1 in T^2 such that $x \in V$, $V \cap (S^1 \times \{1\}) = \emptyset$ and x_1 is a maximum of f in V. There exists a diffeomorphism $\rho : V \to U = \rho(V) \subseteq \mathbf{R}^2$.

The map $f \circ \rho^{-1} : U \to \mathbf{R}$ is a local representation of f. Since x_1 is a maximum of f in V, then $\rho(x_1)$ is a maximum of $f \circ \rho^{-1}$ in V. It follows that $\rho(x_1)$ is a critical point of $f \circ \rho^{-1}$ and so will be x_1 for f. But $C(f) = S^1 \times \{1\}$, hence $x_1 \in S^1 \times \{1\}$, which is a contradiction with the fact that $V \cap S^1 \times \{1\} = \emptyset$. Then $S^1 \times \{1\}$ cannot be critical in T^2. $\square$

Acknowledgements. This paper is partially supported by the European Research Training Network "*GEOMETRIC ANALYSIS*".

References

[1] Borisovich, Yu., Bliznyakov, M., Izrailovich, Ya., Fomenko, T., *Introduction to Topology*, Mir Publishers, Moscow, 1985

[2] Ehresman C., *Sur la topologie de certaines variétés algébriques réelles*, Journal de Math, 104 (1939), 362-372

[3] Godbillon, C., *Eléments de topologie algébrique*, Hermann, Paris, 1971

[4] Guillemin V., Pollack A., *Differential Topology*, Prentice Hall, Inc., Englewood Cliffs New-Jersey, 1977

[5] Kirby, R., Siebenmann, L., *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. of Math. Stud., 88, Princeton Univ. Press, Princeton, N.J., 1989

[6] Norton A., Pugh Ch., *Critical Sets in the Plane*, Michigan Journal of Math., 38(1991), 441-459

[7] Stilwell J., *Classical Topology and Combinatorial Group Theory*, Springer-Vergal, New-York Inc., 1980

"Babeş-Bolyai" University
Faculty of Mathematics and Computer Science
Cluj-Napoca, Romania
email: ltopan@math.ubbcluj.ro