Second Order Conditions for Extrema of Functionals Defined on Regular Surfaces

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

Here we study second-order conditions for a functional of the form

$$\int_V J(x, u, \nabla u) dV$$

which achieves extrema in a convenient Banach space. V denotes a type of compact subset of a smooth surface for which a Maximum Principle holds. The method consists of finding the second derivative of the functional and generalizing the notion of conjugate point to the boundary of V. The results include proofs of general Legendre's and Jacobi's necessary conditions as well as a complete set of sufficient conditions for the existence of maxima and minima. Also, we provide an interesting example showing the application of the theory to the case of the conformal prescribed Gaussian curvature deformation functional.

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1 Introduction

For the moment, V is a compact subset of an oriented, simple, regular surface whose interior $\imath V$ is connected and whose boundary ∂V is simple, close and enough regular, including interior spheres at each of its points. Later on, we will impose on V an additional condition, namely, a hypothesis of the so-called "Maximum Principle when a supersolution exists". $C_0(V)$, $C(V)$, $C_0^1(V)$, $C^1(V)$ and $C_0^2(V)$, $C^2(V)$ denote the usual Banach spaces of functions vanishing or not on ∂V furnished with their corresponding supremum norms. We assume some familiarity of the reader with the Classical Differential Geometry of Surfaces (as in [1], for example) and with the basics of Calculus of Variations, for instance, the following result, cf. [2].

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Lemma 1.1 *Let U be an open subset of a Banach space B and $F : U \to \mathbf{R}$ a twice differentiable functional at $u \in U$ such that $dF(u) = 0$ and $d^2F(u) \neq 0$. Then, F achieves a local minimum (maximum) at the critical point u if and only if there exists a $c > 0$ ($c < 0$) such that $d^2F(u)(h) \geq (\leq)\ c\|h\|^2$, for all $h \in B$, i. e., $d^2F(u)$ is a definite positive (negative) quadratic functional.*

In a previous work [3], it was proved the rightness of next theorem.

Theorem 1.2 (Euler-Lagrange) *If a functional $F : C_0^1(V) \to \mathbf{R}$ has the form $F(u) = \int_V J(x, u, \nabla u)dV$, where J is enough differentiable, then F is differentiable and its first derivative is given by*

$$dF(u)h = \int_V [\partial_u J - div(\partial_{\nabla u} J)]hdV.$$

Henceforth, at any local extremum u of F the Euler-Lagrange equation

$$\partial_u J - div(\partial_{\nabla u} J) = 0$$

holds in V.

The point of departure for what follows is to compute the second derivative of such a functional. Then, this derivative is used to establish necessary and sufficient conditions for the existence of maxima or minima. In the first place, we can generalize Legendre's necessary condition from the Classical Calculus of Variations, cf. [2].

Theorem 1.3 (Legendre) *Let $F : C_0^1(V) \to \mathbf{R}$, $F(u) = \int_V J(x, u, \nabla u)dV$, be an enough differentiable functional. Then, its second derivative is given by*

$$d^2F(u)(h) = \int_V (\nabla h^t P \nabla h + Qh^2)dV,$$

where the tensor field $P = \frac{1}{2}\partial^2_{\nabla u \nabla u} J$ and the scalar field $Q = \frac{1}{2}(\partial^2_{u^2} J - div(\partial^2_{u \nabla u} J))$. Moreover, if F achieves a minimum (maximum) at a critical point u, then P is definite nonnegative (nonpositive).

Next, we need a general and convenient definition of conjugate point.

Definition 1.4 (Conjugate points to ∂V) *Given a quadratic functional $d^2F(u) : C_0^2(V) \to \mathbf{R}$ of the form $d^2F(u)(h) = \int_V (\nabla h^t P \nabla h + Qh^2)dV$, we say that $\imath V$ does not have conjugate points to ∂V with respect to $d^2F(u)$ if no one of the non-trivial solutions (if any), $h \in C_0^2(V) \setminus \{h \equiv 0\}$, of Jacobi's differential equation $-\mathrm{div}(\mathrm{P}\nabla \mathrm{h}) + \mathrm{Qh} = 0$ vanishes in $\imath V$.*

This allows us to get a general version of other classical theorem on quadratic functionals.

Theorem 1.5 *If $\imath V$ does not have conjugate points to ∂V with respect to $d^2F(u)(h) = \int_V (\nabla h^t P \nabla h + Qh^2)dV$, where the tensor field P is definite positive (negative) in V, then $d^2F(u)$ is definite positive (negative).*

Clearly, Jacobi's equation is elliptic if and only if P is definite positive or negative and so, it is not surprising that the analytic ground for more elaborated conditions turns out to be the existence and uniqueness theory for elliptic partial differential equations as well as the Maximum Principles, cf. [4]. In particular, we are interested in those Principles which do not depend at all on the sign of the independent term in the elliptic operator.

Lemma 1.6 (Maximum principle for nonpositive functions) *If L is an elliptic operator and $u \in C^2(\imath V) \cap C(V)$ satisfies $Lu = 0$ in $\imath V$ and and $u \leq 0$ in V, then either $u < 0$ in $\imath V$, or $u \equiv 0$. Moreover, if ∂V satisfies an interior sphere condition at y and $u \in C^1(\imath V \cup \{y\})$ with $u < u(y) = 0$ in V, then $\partial u/\partial \nu < 0$ at y in any direction ν pointing into an interior sphere.*

In order to obtain a general version of Jacobi's condition, we need to assume the validity of the following Maximum Principle.

Lemma 1.7 (Maximum Principle if a positive supersolution exists) *Suppose L is elliptic and there is a $w \in C^2(V)$ with $w > 0$ and $Lw \geq 0$ in V. If $u \in C^2(\imath V) \cap C(V)$ satisfies $Lu = 0$ in $\imath V$, then either there exists a constant $r \in \mathbf{R}$ such that $u \equiv rw$, or u/w does not attain a nonnegative maximum in $\imath V$.*

Accordingly, from now on we will demand our domain V to fulfil the supersolution hypothesis of Lemma 1.7. The existence of such a supersolution can be accomplished, for example, by demanding V to be narrow or, alternatively, by assuring the existence of a positive eigenfunction for Jacobi's operator, cf. [5]. The notion of conjugate point together with this Maximum Principle lead to a deeper general necessary condition.

Theorem 1.8 (Jacobi) *Assume $u \in C_0^2(V)$ is a local minimum (maximum) point of a differentiable enough functional $F : C_0^2(V) \to \mathbf{R}$ of the form*

$$F(u) = \int_V J(x, u, \nabla u)dV, d^2F(u) \neq 0$$

and the tensor field $2P = \partial^2_{\nabla u \nabla u} J$ is definite positive (negative) at this extremum. Then, $\imath V$ does not have conjugate points to ∂V with respect to $d^2F(u)$.

Corollary 1.9 *Let $d^2F(u)(h) = \int_V (\nabla h^t P \nabla h + Qh^2)dV$ be the second derivative of the functional at some $u \in C_0^2(V)$ and suppose $P = P(x, u(x))$ is definite positive (negative) in V. $d^2F(u)$ is definite positive (negative) if and only if $\imath V$ does not have conjugate points to ∂V with respect to $d^2F(u)$.*

To conclude, we put together the conditions of Euler, Legendre and Jacobi to obtain a general theorem that gives a set of sufficient conditions for the existence of an extremum.

Theorem 1.10 *Let $F : C_0^2(V) \to \mathbf{R}$ be a twice differentiable functional of the form*

$$F(u) = \int_V J(x, u, \nabla u) dV.$$

Suppose also $u \in C_0^2(V)$ fulfills the following conditions:

1. *u is a solution of the Euler-Lagrange equation $\partial_u J - div(\partial_{\nabla u} J) = 0$;*
2. *u makes the tensor field $P(x, u(x))$ definite positive (negative);*
3. *$\imath V$ does not contain conjugate points to ∂V with respect to $d^2F(u)$.*

Then, F attains a local minimum (maximum) at u.

2 Proofs of theorems and corollaries

Proof. (Theorem 1.3.) An easy calculation shows that

$$d^2F(u)(h) = \frac{1}{2} \int_V \left[\frac{\partial^2 J}{\partial u^2} h^2 + 2h \langle \partial^2_{u \nabla u} J, \nabla h \rangle + \nabla h^t [\partial^2_{\nabla u \nabla u} J] \nabla h \right] dV.$$

Next, usual integration by parts transforms the second term of this integral into

$$\int_V 2h \langle \partial^2_{u \nabla u} J, \nabla h \rangle dV = - \int_V h^2 \mathrm{div}(\partial^2_{\mathrm{u} \nabla \mathrm{u}} \mathrm{J}) \mathrm{dV}.$$

Substituting this equation in the second derivative yields

$$d^2F(u)(h) = \int_V (\nabla h^t P \nabla h + Q h^2) dV,$$

where $P = \frac{1}{2} [\partial^2_{\nabla u \nabla u} J]$ and $Q = \frac{1}{2} \left(\frac{\partial^2 J}{\partial u^2} - \mathrm{div}(\partial^2_{\mathrm{u} \nabla \mathrm{u}} \mathrm{J}) \right)$.

Now, if P is negative at $x_0 \in V$, the continuity of P implies that

$$v^t P(x) v < -\beta \|v\|^2, \quad \forall v \in T_x(V),$$

for some $\beta > 0$ and for all x in a geodesic ball of radius α centered at x_0. Now, we consider the radially symmetric function $h \in C_0^1(V)$ defined by

$$h(x) = \begin{cases} \sin^2 \frac{\pi r}{\alpha} & \text{if } r \in [0, \alpha] \\ 0 & \text{otherwise,} \end{cases}$$

where r is the geodesic distance from x_0 to x. In this way,

$$\begin{aligned} \int_V (\nabla h^t P \nabla h + Q h^2) dV &< -\beta \mathrm{max} \left(\frac{\pi^2}{\alpha^2} \sin^2 (2 \frac{\pi \mathrm{r}}{\alpha}) \right) \pi \alpha^2 + \mathrm{max} |\mathrm{Q}| \pi \alpha^2 \\ &< -\beta \pi^3 + \mathrm{max} |\mathrm{Q}| \pi \alpha^2. \end{aligned}$$

Finally, making α small enough, we obtain that the second derivative is negative. This is an obstruction for u to be a minimum. Word by word, the proof for a maximum is accomplished. □

Proof. (Theorem 1.5.) With a non-trivial solution $\widehat{h}$ to Jacobi's equation we construct the vector field $w = -P\nabla\widehat{h}/\widehat{h}$ in $\imath V$. A straightforward computation shows this field satisfies "Riccati's" equation

$$Q + \mathrm{div w} = \mathrm{w^t P^{-1} w}.$$

Hence, for any $h \in C_0^2(V)$, we can complete a perfect square as follows.

$$\begin{aligned}
\nabla h^t P \nabla h + Qh^2 + \mathrm{div(wh^2)} &= \nabla h^t P \nabla h + Qh^2 + \mathrm{(div w)h^2} + \mathrm{\langle w, 2h\nabla h\rangle} \\
&= \nabla h^t P \nabla h + 2h\langle \nabla h, w\rangle + (Q + \mathrm{div w})\mathrm{h^2} \\
&= (\nabla h + hP^{-1}w)^t P(\nabla h + hP^{-1}w).
\end{aligned}$$

The usual procedure of integration by parts together with the fact that h is null on ∂V yield

$$\int_V \mathrm{div(wh^2)dV} = 0.$$

In this way, the integral reduces to

$$d^2F(u)(h) = \int_V (\nabla h^t P \nabla h + Qh^2)dV = \int_V (\nabla h + hP^{-1}w)^t P(\nabla h + hP^{-1}w)dV.$$

Since P is definite positive, $d^2F(u)(h) = 0$ implies that $\nabla h + hP^{-1}w = 0$. The unique solution to this equation in $C_0^2(V)$ is $h \equiv 0$. This shows the functional $d^2F(u)$ is definite positive. □

Remark 2.1 *For the rest of the proofs below, we make repeated use of the additional condition on the surface domain V and the Maximum Principles.*

Proof. (Theorem 1.9.) If $\imath V$ has a conjugate point to ∂V, Jacobi's problem

$$-\mathrm{div(P\nabla h) + Qh = 0,\ h \in C_0^2(V) \setminus \{h \equiv 0\}},$$

possesses a solution h vanishing at some point in $\imath V$. We claim this is impossible. For, we consider the family of functionals

$$d_t F(u)(h) = t\int_V (\nabla h^t P \nabla h + Qh^2)dV + (1-t)\int_V \langle \nabla h, \nabla h\rangle dV,$$

$t \in [0, 1]$. The members of this family are all definite positive after Lemma 1.1. Also, after rewriting

$$d_t F(u)(h) = \int_V (\nabla h^t (tP + (1-t)I)\nabla h + tQh^2)dV,$$

we notice that the associated family of Jacobi's elliptic equations is given by

$$-\mathrm{div}\big((\mathrm{tP}+(1-\mathrm{t})\mathrm{I})\nabla \mathrm{h}\big)+\mathrm{tQh}=0,\ \mathrm{t}\in[0,1].$$

From now on, we look at these equations as a unique equation with parameter t whose solutions are denoted by $h(x,t),(x,t)\in V\times[0,1]$. To $t=0$, it corresponds Dirichlet's functional $d_0F(u)(h)=\int_V\langle\nabla h,\nabla h\rangle dV$ whose Jacobi's equation is just but Laplace's equation $-\mathrm{div}(\nabla \mathrm{h})=-\Delta \mathrm{h}=0$. From the Weak Maximum Principle, it follows that $\imath V$ does not have conjugate points to ∂V with respect to $d_0^2F(u)$. To $t=1$, it corresponds the functional under consideration. Our claim on the absence of conjugate points to ∂V with respect to $d_1^2F(u)$ relies on the fact that it cannot appear conjugate points as t varies smoothly from 0 to 1 (homotopy invariance).

Let $h(x,1)$ be the nontrivial solution with $h(x_0,1)=0$ at some point $x_0\in\imath V$, conjugate to ∂V with respect to $d_1^2F(u)$. We must also have $\nabla h(x_0,1)\neq 0$. Indeed, the only choice in Lemma 1.7 is $r=0$, which leads to the excluded possibility $h(x,1)\equiv 0$. Therefore, $h(x,1)/w$ does not achieve a nonnegative maximum in $\imath V$ and since $w>0$, so does $h(x,t)$ and $\nabla h(x_0,1)=0$ is not possible. This suggests to study a differentiable structure for the set

$$N=\{(x,t)\in\imath V\times[0,1]\mid h(x,t)=0\}.$$

After our assumption, $N\neq\emptyset$. The Implicit Function Theorem implies then that N is a curve $x(t)=(x_1(t),x_2(t))$ near $(x_0,1)$. Since $\nabla h(x,t)\neq 0$ in N for the same reasons above, we consider its continuation in $V\times[0,1]$:

1. The curve cannot stop suddenly inside $\imath V\times(0,1)$ because this would contradict the smooth dependence of $h(x,t)$ on t.

2. It cannot return to $\imath V\times\{1\}$ as this would yield $\nabla h=0$ at some point and this never happens.

3. The curve cannot reach $\partial V\times[0,1]$ as, in an interior sphere with $h>0$, Lemma 1.3 forbids $\partial h/\partial\nu=0$ in any interior direction ν.

4. From above, N cannot hit $\imath V\times\{0\}$.

To sum up, the existence of a conjugate point results in a contradiction. The proof for a maximum runs similarly. □

Proof. (Corollary 1.9.) It follows directly from Theorem 1.5. □

Proof. (Theorem 1.10.) It is an immediate consequence of Lemma 1.1 and Corollary 1.9. □

3 Application to the conformal prescribed Gauss curvature problem

The conformal Gaussian curvature deformation functional on a compact Riemannian surface V with connected interior, cf. [7], is defined to be

$$F(u)=\int_V\left(\frac{1}{2}\langle\nabla u,\nabla u\rangle-\frac{K}{2}e^{2u}+ku\right)dV.$$

The associated Euler-Lagrange equation is, therefore, given by

$$-\mathrm{div}(\nabla u) - Ke^{2u} + k = -\Delta u - Ke^{2u} + k = 0.$$

A straight-forward computation yields

$$d^2F(u)(h) = \int_V (\nabla h^t P \nabla h + Qh^2) dV,$$

where the tensor field $P = \frac{1}{2}(\partial^2_{\nabla u \nabla u} J) = I$ (constant and equal to the identity matrix, which is certainly positive definite) and

$$Q = \frac{1}{2}\left[\partial^2_{uu} J - \mathrm{div}(\partial^2_{u\nabla u} J)\right] = -Ke^{2u}.$$

Now, if the prescribed curvature function $K \leq 0$ in V, $d^2F(u)$ is definite positive and any solution of the Euler-Lagrange equation (critical point) is a minimum.

The problem is not that easy if we allow $K(x) > 0$ at some $x \in V$. However, a simple sufficient condition can be stated. Let u be a critical point of the Gaussian conformal functional and suppose that

$$u_{max} < -\log\sqrt{C_p |K|_{max}}$$

holds with the notations $|K|_{max} = \max\{|K(x)| : x \in V\}, u_{\max} = \max\{u(x) : x \in V\}$ and C_p is Poincaré's constant (Lemma 3.1 below). Then, u is a minimum. Certainly,

$$\left| \int_V Qh^2 dV \right| \leq C_p |K|_{max} e^{2u_{max}} \int_V \langle \nabla h, \nabla h \rangle dV$$

implies that

$$d^2F(u)(h) \geq (1 - C_p |K|_{max} e^{2u_{max}}) \int_V \langle \nabla h, \nabla h \rangle dV \geq 0.$$

We have used the following well-known result, cf. [6].

Lemma 3.1 (Poincaré's Inequality) *For any $h \in C_0^2(V)$, we have*

$$\int_V h^2 dV \leq C_p \int_V \langle \nabla h, \nabla h \rangle dV,$$

for some positive constant C_p.

It is perhaps more interesting to notice that the assumption for V of a positive eigenfunction with eigenvalue λ for Jacobi's operator $-\Delta h - Ke^{2u}h$ implies a Maximum Principle for $Q = -K(x)e^{2u(x)} < \lambda$, cf. [5].

4 Concluding remarks

Most (if not all) of the results presented above do not really need the functions to be identically zero on the boundary of the domain. It is important to point out that the given definition of conjugate point is appropriate because it applies to unordered sets as well to ordered sets. The reason for this is that it is based upon a boundary value problem instead of an initial value problem. Last but not least, we remark again that the generalization of Jacobi's condition presented above depends on a domain satisfying a Maximum Principle and interior sphere conditions at each of its boundary points. These are indeed quite restrictive technical assumptions.

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