Value Problems for Differential Forms on C^1-Domains

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

Existence and uniqueness value boundary problems for differential forms in C^1-domains are investigated.

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Introduction

In this paper we resolve boundary value problems for differential forms in a bounded C^1-domain Ω of $\mathbf{R}^n$, $3 \leq n$. The obtained results are an extension of some theorems established by C. Miranda in [8] for differential forms in $C^{2,\alpha}$-domains of $\mathbf{R}^n$.

The fundamental result is Theorem 2.1:

Let $F_{s-1} \in C^1_{s-1}(\Omega)$ be a closed form with interior nontangential trace $F^-_{s-1} \in L^p_{s-1}(\partial\Omega)$. If

$$\int_{\tau^i_{s-1}} F^-_{s-1} = 0, \qquad i = 1, ..., R^-_{s-1},$$

where $([\tau^i_{s-1}])_{1 \leq i \leq R^-_{s-1}}$ is a base of C^1-differentiable singular homology space $\mathcal{H}_{s-1}(\partial\Omega)$ and R^-_{s-1} is the $(s-1)$-th Betti number of Ω, then there exists a form $U_{s-2} \in C^1_{s-2}(\Omega)$, whose coefficients are in $W^{1,p}(\Omega)$, such that

$$(*) \qquad dU_{s-2} = F_{s-1} \qquad \text{in } \Omega.$$

This result is obtained using a formula of Bidal-de Rham (see (1.10)). The use of this formula changes the existence problem of a differential form that satisfies $(*)$ in a suitable Neumann problem for harmonic forms already studied in [15].

Using a regularity result for the solutions of the homogeneous Neumann problem for harmonic forms established in [15] (see Theorem 2.1) and the continuity hypothesis for F_{s-1} in $\overline{\Omega}$, we obtain a uniqueness theorem (see Theorem 2.4). With this aim, first we prove an extension theorem (see Theorem 1.2) and then, as a consequence, we

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deduce a Stokes' Theorem (see Theorem 1.3) for differential s-forms of $\widetilde{C}^1_s(\overline{\Omega})$ (see Preliminaries).

Throughout this work we use the definitions and the results of [13] concerning differential forms and singular homology and cohomology groups of the C^1-manifold $\partial\Omega$.

1 Preliminaries

In this section we summarize basic concepts, notations, and results that will be used throughout the paper.

We assume that Ω is a bounded and connected C^1-domain of $\mathbf{R}^n$, $3 \le n$. Thus, (see [16]), there exist an increasing sequence $(\Omega_h)_{h\in\mathbf{N}}$ of C^∞-domains, $\Omega_h \subset \Omega$, such that $\Omega_h \to \Omega$ in C^1 according to Nečas (see [7] p. 85) and a sequence $(\Lambda_h)_{h\in\mathbf{N}}$ of C^1-diffeomorphisms $\Lambda_h : \partial\Omega \to \partial\Omega_h$ such that

$$\lim_h \sup_{Q\in\partial\Omega} |Q - \Lambda_h(Q)| = 0. \tag{1.1}$$

Furthermore, there is a finite covering $(B_r)_{1\le r\le m}$ of $\partial\Omega$ by open spheres $B_r = B(Q_r, \delta)$ with center $Q_r \in \partial\Omega$ and radius δ, such that, for $r = 1, \dots, m$

$$B(Q_r, 2\delta) \cap \partial\Omega = \{(x, x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} \,:\, x_n = \xi_r(x)\} \cap B(Q_r, 2\delta), \tag{1.2}$$

and

$$B(Q_r, 2\delta) \cap \partial\Omega_h = \{(x, x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} \,:\, x_n = \xi_{rh}(x)\} \cap B(Q_r, 2\delta), \tag{1.3}$$

where $\xi_r \in C^1_0(\mathbf{R}^{n-1})$, $\xi_r(0) = \dfrac{\partial \xi_r}{\partial x_l}(0) = 0$ $(l = 1, \dots, n-1)$, $\xi_{rh} \in C^\infty_0(\mathbf{R}^{n-1})$, and

$$\lim_h \|\xi_{rh} - \xi_r\|_{C^1_0(\mathbf{R}^{n-1})} = 0. \tag{1.4}$$

Let now

$$\widetilde{x}_r = (x, \xi_r(x)) \in \partial\Omega \cap B(Q_r, 2\delta) \to x \in \mathbf{R}^{n-1} \tag{1.5}$$

and

$$\widetilde{x}_{hr} = (x, \xi_{hr}(x)) \in \partial\Omega_h \cap B(Q_r, 2\delta) \ \to x \in \mathbf{R}^{n-1}. \tag{1.6}$$

For $l, i = 1, \dots, n-1$,

$$\lim_h \frac{\partial(\widetilde{x}_{rh} \circ \Lambda_h \circ \widetilde{x}_r^{-1})_i}{\partial x_l}(x) = \delta_{il} \tag{1.7}$$

uniformly in $U_r = \widetilde{x}_r(\partial\Omega \cap B(Q_r, 2\delta))$, where $(\widetilde{x}_{rh} \circ \Lambda_h \circ \widetilde{x}_r^{-1})_i$ is the i-th coordinate of the function $\widetilde{x}_{rh} \circ \Lambda_h \circ \widetilde{x}_r^{-1}$.

Let $U_s = \sum_{i\in N^n_s} a_i dX_i$ be a form defined in Ω (respectively in $\mathbf{R}^n \setminus \overline{\Omega}$). [1]

If $U_s \in C^2_s(\Omega)$ we set

[1] $N^n_s = \{i = (i_1, \dots, i_s) \in \mathbf{N}^s : 1 \le i_1 < \dots < i_s \le n\}$; if $i = (i_1, \dots, i_s) \in \mathbf{N}^n_s$, $dX_i = dX_{i_1} \wedge \dots \wedge X_{i_s}$.

(1.8) $$\delta U_s = (dU_s)^* \quad \text{and} \quad \Delta U_s = d\delta U_s,$$

where dU_s is the exterior derivative of U_s and $*$ is the Hodge's operator. U_s is said to be *closed* (*harmonic*, respectively) in Ω iff $dU_s = 0$ ($\Delta U_s = 0$, respectively) in Ω. Furthermore U_s is said to be *derived* in Ω iff there exists a form V_{s-1} such that

(1.9) $$dV_{s-1} = U_s \quad \text{in} \quad \Omega.$$

and V_{s-1} is called the *primitive of* U_s.

Thus we have the following identity of Bidal-de Rham (see (60) in [8])

(1.10) $$d\delta U_s^* + (-1)^n \delta\delta U_s = (-1)^{n(s+1)} \sum_{i\in \mathbf{N}_s^n} \Delta a_i \; dX_i,$$

where $\Delta a_i = \sum_{l=1}^{n} \frac{\partial^2 a_i}{\partial x_l^2}$.

We denote with $\widetilde{C}_s^1(\overline{\Omega})$ the space of the forms $U_s \in C_s^1(\Omega)$ such that each coefficient of U_s and of dU_s is in $C^0(\overline{\Omega})$.

Given $k \in \mathbf{N}$ and $1 < p < \infty$, we denote with $D_s^{k,p}(\Omega)$ the space of the forms U_s such that each coefficient of U_s is in $W^{k,p}(\Omega)$.

We say that U_s has *interior* (*exterior,* respectively) *nontangential trace in* $L_s^p(\partial\Omega)$ iff, for any $i \in \mathbf{N}_s^n$, a_i has interior nontangential trace a_i^- (exterior nontangential trace a_i^+, respectively) in $L^p(\partial\Omega)$. The form [2]

(1.11) $$U_s^- = \sum_{i\in \mathbf{N}_s^n} a_i^- dX_i(Q)$$

(respectively the form

(1.12) $$U_s^+ = \sum_{i\in \mathbf{N}_s^n} a_i^+ dX_i(Q))$$

is called *interior* (*exterior,* respectively) *nontangential trace* of U_s.

If $U_s \in D^{1,p}(\Omega)$, the form

(1.13) $$\mathrm{Tr}(U_s) = \sum_{i\in \mathbf{N}_s^n} \mathrm{Tr}(a_i) dX_i(Q)$$

is considered, where the mapping $\mathrm{Tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is the continuous extension to $W^{1,p}(\Omega)$ of the mapping restriction defined initially on $C^\infty(\overline{\Omega})$.

If $U_s \in C_s^0(\Omega)$ and U_s has interior nontangential trace in $L_s^p(\partial\Omega)$, then (see Theorem 2.3 in [13])

(1.14) $$\lim_h \Lambda_h^*(U_{sh}) = U_s^- \quad \text{in} \quad L_s^p(\partial\Omega),$$

where U_{sh} is the restriction of U_s on $\partial\Omega_h$.

We denote with $\mathcal{N}_s^{1,p}(\Omega)$ the space of the forms $U_s \in C_s^1(\Omega)$ such that each coefficient of U_s and dU_s has interior nontangential trace in $L^p(\partial\Omega)$.

The following result holds:

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[2] $dX_i(Q)$ is the restriction to $\partial\Omega$ of dX_i, hence $dX_i(Q) = j^* dX_i$, where $j : \partial\Omega \to \mathbf{R}^n$ is the inclusion map.

Theorem 1.1. *If* $U_s \in C^0_s(\Omega) \cap D^{1,p}_s(\Omega)$ *and* $\Phi_{n-s-1} \in C^0_{n-s-1}(\partial\Omega)$, *then*

$$\lim_h \int_{\partial\Omega_h} U_s \wedge \Lambda_h^{-1*}(\Phi_{n-s-1}) = \int_{\partial\Omega} \mathrm{Tr}(U_s) \wedge \Phi_{n-s-1}. \tag{1.15}$$

Proof. By using a partition of unity $(\varphi_r)_{1\le r\le m}$, corresponding to the covering $(B_r)_{1\le r\le m}$ of $\partial\Omega$ described above, in order to prove (1.15) it will suffice to show that

$$\lim_h \int_{\mathbf{R}^{n-1}} \widetilde{x}_{rh}^{-1*}(\varphi_r U_{sh} \wedge \Lambda_h^{-1*}(\Phi_{n-s-1})) = \int_{\mathbf{R}^{n-1}} \widetilde{x}_r^{-1*}(\varphi_r \mathrm{Tr}(U_s) \wedge \Phi_{n-s-1}). \tag{1.16}$$

for all $r = 1, \dots, m$. For simplicity of notation in the following we omit the index r.

Let

$$U_s = \sum_{i\in\mathbf{N}_s^{n-1}} a_i dX_i + \sum_{i\in\mathbf{N}_{s-1}^{n-1}} a_{in} dX_i \wedge dX_n \quad \text{and} \quad \widetilde{x}^{-1*}(\varphi\Phi_{n-s-1})) = \sum_{i\in\mathbf{N}_{n-s-1}^{n-1}} b_i dx_i.$$

Thus $\widetilde{x}_h^{-1*}(\varphi U_{sh} \wedge \Lambda_h^{-1*}(\Phi_{n-s-1}))$ is a $(n-1)$-form on $\mathbf{R}^{n-1}$. Its coefficients are the sum of a finite number of terms like

$$\varphi(x, \xi_h(x)) a_i(x, \xi_h(x)) b_j(f_h(x)) |\frac{\partial f_{hj}}{\partial x_j}(x)|$$

or terms like

$$\varphi(x, \xi_h(x)) a_{in}(x, \xi_h(x)) b_j(f_h(x)) |\frac{\partial f_{hj}}{\partial x_j}(x)| \frac{\partial \xi_h}{\partial x_l}(x),$$

where

$$f_h = \widetilde{x} \circ \Lambda_h^{-1} \circ \widetilde{x}_h^{-1} = (f_{h1}, \dots, f_{h(n-1)}), \text{ and } |\frac{\partial f_{hj}}{\partial x_j}| = det \frac{\partial(f_{hj_1}, \cdots, f_{hj_{n-s-1}})}{\partial(x_{j_1}, \cdots, x_{j_{n-s-1}})}.$$

Hence, since $spt(\varphi) \subset B(Q, \delta)$, the integrals in (1.16) are integrals on the compact set

$$K = \{x \in \mathbf{R}^{n-1} : |x| \le \delta\}.$$

We observe that using Theorem 4.5, p. 85 in [10] we have

$$\lim_h a_i(x, \xi_h(x)) = \mathrm{Tr}(a_i)(x, \xi(x)) \ \text{ in } L^p(K),$$

while, using (1.1) and (1.7), we have

$$\lim_h \varphi(x, \xi_h(x)) b_j(f_h(x)) |\frac{\partial f_{hj}}{\partial x_j}(x)| = \varphi(x, \xi(x)) b_j(x) \,\text{and}\, \lim_h \frac{\partial \xi_h}{\partial x_j}(x) = \frac{\partial \xi}{\partial x_j}(x) \tag{1.17}$$

uniformly in K. Therefore, since the sequence of functions that appear in the left side of (1.17) is uniformly bounded in K, by the Dominated Convergence Theorem we obtain the proof .

□

Corollary. *Assume $U_s \in C^1_s(\Omega) \cap D^{1,p}_s(\Omega)$. If dU_s has interior nontangential trace in $L^p_{s+1}(\partial\Omega)$, then $Tr(U_s) \in W^{1,p}_s(\partial\Omega)$ and*

$$d\mathrm{Tr}(U_s) = (dU_s)^- \qquad a.e. \quad on \quad \partial\Omega. \tag{1.18}$$

Proof. It is sufficient to show that

$$\int_{\partial\Omega} (dU_s)^- \wedge \Phi_{n-s-2} = (-1)^s \int_{\partial\Omega} \mathrm{Tr}(U_s) \wedge d\Phi_{n-s-2}$$

for all $\Phi_{n-s-2} \in \widetilde{C}^1_{n-s-2}(\partial\Omega)$ (see n.1 in [13]), that is to say, because of (1.14) and since $(dU_s)_h = d(U_{sh})$,

$$\lim_h \int_{\partial\Omega} \Lambda^*_h(dU_{sh}) \wedge \Phi_{n-s-2} = (-1)^s \int_{\partial\Omega} \mathrm{Tr}(U_s) \wedge d\Phi_{n-s-2}.$$

Hence, it is enough to observe that

$$\begin{aligned}\int_{\partial\Omega} \Lambda^*_h(dU_{sh}) \wedge \Phi_{n-s-2} &= \int_{\partial\Omega_h} dU_s \wedge \Lambda^{-1*}_h(\Phi_{n-s-2}) \\ &= (-1)^s \int_{\partial\Omega_h} U_s \wedge \Lambda^{-1*}_h(d\Phi_{n-s-2}),\end{aligned}$$

and to apply Theorem 1.1.

□

We obtain also the following Theorems

Theorem 1.2. *If $U_s \in \widetilde{C}^1_s(\overline{\Omega})$, then there exists an open set Ω' of $\mathbf{R}^n$ such that $\overline{\Omega} \subset \Omega'$ and there exists a form $\overline{U}_s \in \widetilde{C}^1_{0,s}(\Omega')$ such that $\overline{U}_s$ and $d\overline{U}_s$ are extensions of U_s and dU_s respectively.*

Proof. Let $U_s = \sum_{i\in\mathbf{N}^n_s} a_i dX_i \in \widetilde{C}^1_s(\overline{\Omega})$ and let $dU_s = \sum_{j\in\mathbf{N}^n_{s+1}} b_j dX_j$. Then we have

$$(\forall i \in \mathbf{N}^n_s)\,(a_i \in C^0(\overline{\Omega}) \cap C^1(\Omega)) \quad \text{and} \quad (\forall j \in \mathbf{N}^n_{s+1})\,(b_j \in C^0(\overline{\Omega})).$$

In Ω it results

$$(\forall j \in \mathbf{N}^n_{s+1})\,(b_j = \sum_{k=1}^{s+1} (-1)^k \frac{\partial a_{\widehat{j}^k}}{\partial X_{j_k}}),$$

where $\widehat{j}^k = (j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{s+1})$ when $j = (j_1, \ldots, j_{s+1})$.

Arguing in a manner similar to the proof of Theorem 54 XV of [11], we prove the existence of an open set Ω' of $\mathbf{R}^n$ such that $\overline{\Omega} \subset \Omega'$ and, for $i \in \mathbf{N}^n_s$ and $j \in \mathbf{N}^n_{s+1}$, the existence of functions $\overline{a}_i \in C^0_0(\Omega') \cap C^1(\Omega' \setminus \partial\Omega)$ and $\overline{b}_j \in C^0_0(\Omega')$ extensions of a_i and b_j, respectively, such that

$$\overline{b}_j = \sum_{k=1}^{s+1} (-1)^k \frac{\partial \overline{a}_{\widehat{j}^k}}{\partial X_{j_k}} \quad \text{in } \Omega' \setminus \partial\Omega. \tag{1.19}$$

Let, then

$$\overline{U}_s = \sum_{i\in\mathbf{N}^n_s} \overline{a}_i dX_i \quad \text{and} \quad \overline{V}_{s+1} = \sum_{j\in\mathbf{N}^n_{s+1}} \overline{b}_j dX_j.$$

We have that $\overline{U}_s \in C^0_{0,s}(\Omega')$ and $\overline{V}_{s+1} \in C^0_{0,s+1}(\Omega')$. In order to prove that $\overline{U}_s$ is regular in Ω', using the Lemma 16.d p. 105 in [17], it suffices to show the existence of a sequence $(\overline{U}^p_s)_{p\in\mathbf{N}}$ in $C^\infty_s(\Omega')$ such that

$$\overline{U}^p_s \to \overline{U}_s \quad \text{and} \quad d\overline{U}^p_s \to \overline{V}_{s+1} \tag{1.20}$$

uniformly in every compact subset of Ω'. Clearly we may suppose $\overline{U}_s$ and $\overline{V}_{s+1}$ defined in $\mathbf{R}^n$. Let now $(\rho_p)_{p\in\mathbf{N}}$ be a sequence of mollifiers and let

$$\overline{U}^p_s = \rho_p * \overline{U}_s = \sum_{i\in\mathbf{N}^n_s} \rho_p * \overline{a}_i dX_i$$

and

$$\overline{V}^p_{s+1} = \rho_p * \overline{V}_{s+1} = \sum_{j\in\mathbf{N}^n_{s+1}} \rho_p * \overline{b}_j dX_j,$$

where $*$ is the usual convolution product between functions.

First we obtain that $\overline{U}^p_s \in C^\infty_{0,s}(\mathbf{R}^n)$ and $\overline{V}^p_{s+1} \in C^\infty_{0,s+1}(\mathbf{R}^n)$, and from Proposition IV.21, in [1], it follows that

$$\overline{U}^p_s \to \overline{U}_s \quad \text{and} \quad \overline{V}^p_{s+1} \to \overline{V}_{s+1}$$

uniformly on compact sets of $\mathbf{R}^n$. We finish by proving the second formula in (1.20). For this it is sufficient to show that

$$\sum_{k=1}^{s+1} (-1)^k \frac{\partial}{\partial X_{j_k}} (\rho_p * \overline{a}_{\widehat{j^k}}) = \rho_p * \overline{b}_j$$

or, for Proposition IV.2 in [1], that

$$\sum_{k=1}^{s+1} (-1)^k (\frac{\partial}{\partial X_{j_k}} \rho_p) * \overline{a}_{\widehat{j^k}} = \rho_p * \overline{b}_j$$

for all $p \in \mathbf{N}$ and $j \in \mathbf{N}^n_{s+1}$. We observe that, as a consequence of (1.19), we have

$$\begin{aligned}\sum_{k=1}^{s+1} (-1)^k \frac{\partial}{\partial Y_{j_k}} (\rho_p(X-Y)\cdot \overline{a}_{\widehat{j^k}}(Y)) \quad &= \sum_{k=1}^{s+1} (-1)^{k+1} (\frac{\partial}{\partial X_{j_k}} \rho_p)(X-Y)\cdot \overline{a}_{\widehat{j^k}}(Y) \\ &+\rho_p(X-Y)\cdot \overline{b}_j(Y)\end{aligned}$$

for $X \in \mathbf{R}^n$ and $Y \in \mathbf{R}^n \setminus \partial\Omega$. Thus $\sum_{k=1}^{s+1} (-1)^k \frac{\partial}{\partial Y_{j_k}} (\rho_p(X-Y)\cdot \overline{a}_{\widehat{j^k}}(Y))$ is in $L^1(\mathbf{R}^n)$. Since

$$\begin{aligned}&\int_{\mathbf{R}^n} \sum_{k=1}^{s+1} (-1)^k \frac{\partial}{\partial Y_{j_k}} (\rho_p(X-Y)\cdot \overline{a}_{\widehat{j^k}}(Y))dY = \int_\Omega \sum_{k=1}^{s+1} (-1)^k \frac{\partial}{\partial Y_{j_k}} (\rho_p(X-Y)\cdot \overline{a}_{\widehat{j^k}}(Y))dY \\ &+ \int_{\mathbf{R}^n\setminus\Omega} \sum_{k=1}^{s+1} (-1)^k \frac{\partial}{\partial Y_{j_k}} (\rho_p(X-Y)\cdot \overline{a}_{\widehat{j^k}}(Y))dY = I_1 + I_2,\end{aligned}$$

to obtain (1.20) we need to prove

$$I_1 + I_2 = 0. \tag{1.21}$$

Now, since (see Preliminares)

$$\begin{aligned} I_1 &= \lim_h \sum_{k=1}^{s+1} (-1)^k \int_{\Omega_h} \frac{\partial}{\partial Y_{j_k}} (\rho_p(X-Y) \cdot \overline{a}_{\widehat{j}^k}(Y)) dY \\ &= \sum_{k=1}^{s+1} (-1)^k \lim_h \int_{\partial\Omega_h} \rho_p(X-Y) \cdot \overline{a}_{\widehat{j}^k}(Y) dY \\ &= \sum_{k=1}^{s+1} (-1)^k \lim_h \int_{\partial\Omega} \rho_p(X - \Lambda_h(Y)) \cdot \overline{a}_{\widehat{j}^k}(\Lambda_h(Y)) \omega_h(Y) dY, \end{aligned}$$

where $\omega_h : \partial\Omega \to \mathbf{R}_+$ is the function of [17]: Theorem 1.12, by Dominated Convergence Theorem it follows that

$$I_1 = \sum_{k=1}^{s+1} (-1)^k \int_{\partial\Omega} \rho_p(X-Y) \cdot \overline{a}_{\widehat{j}^k}(Y) dY. \tag{1.22}$$

In the same manner we prove that

$$I_2 = -\sum_{k=1}^{s+1} (-1)^k \int_{\partial\Omega} \rho_p(X-Y) \cdot \overline{a}_{\widehat{j}^k}(Y) dY \tag{1.23}$$

taking a sequence $(\Omega'_h)_{n\in\mathbf{N}}$ of C^∞-domains of $\mathbf{R}^n$ with $\Omega \subset \Omega'_h$ and a sequence of diffeomorphisms $\Lambda'_h : \partial\Omega \to \partial\Omega'_h$ such that $\Omega \to \Omega'_h$ in C^1 according to Nečas (see [10] and [16]) and $\lim_h \sup_{Q\in\partial\Omega} |Q - \Lambda'_h(Q)| = 0$. From this and (1.22) we have (1.21). Hence the thesis.

□

Theorem (Stokes) 1.3. *If $U_s \in \widetilde{C}^1_s(\overline{\Omega})$, then*

$$\int_{\partial\Gamma_{s+1}} U_s = \int_{\Gamma_{s+1}} dU_s \tag{1.24}$$

for all chain $\Gamma_{s+1} \subset \overline{\Omega}$.

Proof. Let Ω' be an open set of $\mathbf{R}^n$ such that $\overline{\Omega} \subset \Omega'$ and let $\overline{U}_s$, $d\overline{U}_s$ be regular forms in Ω' such that the coefficients of $\overline{U}_s$ and of $d\overline{U}_s$ are respectively the extensions of the corresponding coefficients of U_s and dU_s (see Theorem 1.2). Then, by the definition of regular forms, (1.24) is obtained.

□

2 Existence and Uniqueness Theorems

Let $1 < s < n$. Following [8] we introduce the form

$$\omega_{n-s}(X,Y) = \sum_{i\in\mathbf{N}^n_{n-s}} \frac{1}{|X-Y|^{n-2}} dX_i dY_i \tag{2.1}$$

in two variables $(X,Y) \in \mathbf{R}^n \times \mathbf{R}^n$ (see [4], Section. 7). By (2.1), if $X, Y \in \mathbf{R}^n, X \neq Y$ we obtain

$$d_X \delta_X \omega_{n-s}(X,Y) = (-1)^{n-s} d_Y \delta_X \omega_{n-s-1}(X,Y) \tag{2.2}$$

and

$$\delta_X \omega_{s-1}(X,Y) = (-1)^{n(s-1)-1} \delta_Y \omega_{n-s}(X,Y). \tag{2.3}$$

Let $([\tau^j_{s-1}],[\gamma^l_{s-1}])_{\substack{1\le j\le R^-_{s-1}\\ 1\le l\le R^+_{s-1}}}$, $([t^j_{s-1}])_{1\le j\le R^-_{s-1}}$ and let $([c^l_{s-1}])_{1\le l\le R^+_{s-1}}$ be bases of C^1-differentiable singular homology spaces $\mathcal{H}_{s-1}(\partial\Omega)$, $\mathcal{H}_{s-1}(\Omega)$, and $\mathcal{H}_{s-1}(\overline{\mathbf{R}}^n \setminus \overline{\Omega})$ respectively, verifying the following conditions (see n. 5 in [13])

$$\tau^j_{s-1} \sim 0 \quad \text{in} \quad \overline{\mathbf{R}}^n \setminus \Omega \quad \text{and} \quad \gamma^l_{s-1} \sim 0 \quad \text{in} \quad \overline{\Omega}$$

and

$$t^j_{s-1} \sim \tau^j_{s-1} \quad \text{in} \quad \overline{\Omega} \quad \text{and} \quad c^l_{s-1} \sim \gamma^l_{s-1} \quad \text{in} \quad \overline{\mathbf{R}}^n \setminus \Omega,$$

for $l = 1,\ldots,R^+_{s-1}$ and $j = 1,\ldots,R^-_{s-1}$ (see n. 5 in [13]).

Let now $1 < p < \infty$. The following results hold:

Theorem 2.1. *Let $F_{s-1} \in C^1_{s-1}(\Omega)$ be a closed form with interior nontangential trace in $L^p_{s-1}(\partial\Omega)$ such that*

$$\int_{\tau^i_{s-1}} F^-_{s-1} = 0 \qquad i = 1,...,\mathbf{R}^-_{s-1}. \tag{2.4}$$

Then there exists $U_{s-2} \in C^1_{s-2}(\Omega) \cap D^{1,p}_{s-2}(\Omega)$ verifying
i) $dU_{s-2} = F_{s-1}$ *in* Ω,
ii) $\mathrm{Tr}(U_{s-2}) \in W^{1,p}_{s-2}(\partial\Omega)$ *and* $d\mathrm{Tr}(U_{s-2}) = F^-_{s-1}$ *a.e. on* $\partial\Omega$,
iii) $dU^*_{s-2} = 0$ *in* Ω.

Proof. Let B_{n-s} be a generic harmonic form in Ω, such that δB_{n-s} has interior nontangential trace in $L^p_{s-1}(\partial\Omega)$. Since F^*_{s-1} and dB_{n-s} are forms of $C^1_{n-s+1}(\Omega)$ with interior nontangential trace in $L^p_{n-s+1}(\partial\Omega)$, from Theorem 2.7 in [13] it follows that they are in $L^p_{n-s+1}(\Omega)$. Then we can put

$$G_{s-1}(X) = \frac{(-1)^{ns-1}}{k_n} \int_\Omega \omega_{s-1}(X,Y) \wedge (F^*_{s-1} + (-1)^{(n-1)(s-1)} dB_{n-s})(Y). \tag{2.5}$$

According to Theorem 77.VI in [9] this form is in $C^2_{s-1}(\Omega) \cap D^{2,p}_{s-1}(\Omega)$. Let then

$$U_{s-2} = \delta G^*_{s-1} \quad \text{and} \quad V_{n-s} = (-1)^n \delta G_{s-1}. \tag{2.6}$$

From (1.10), using Theorem 77.VI in [9], we obtain that

$$dU_{s-2} + \delta V_{n-s} = F_{s-1} + \delta B_{n-s} \ \text{a.e. in } \Omega. \tag{2.7}$$

Then in order to prove the thesis it is sufficient to show that there exists B_{n-s} such that

$$\delta V_{n-s} = \delta B_{n-s} \quad \text{a.e. in } \Omega. \tag{2.8}$$

Now (2.3) and (2.6) imply that

$$\begin{aligned} V_{n-s}(X) &= \frac{1}{k_n}\int_\Omega \delta_Y \omega_{n-s}(X,Y) \wedge (F^*_{s-1} + (-1)^{(n-1)(s-1)} dB_{n-s})(Y) \\ &= \frac{1}{k_n}\int_\Omega d_Y \omega_{n-s}(X,Y) \wedge (F_{s-1} + \delta B_{n-s})(Y) \\ &= \frac{1}{k_n}\lim_{h\to\infty}\int_{\Omega_h} d_Y \omega_{n-s}(X,Y) \wedge (F_{s-1} + \delta B_{n-s})(Y). \end{aligned}$$

Hence, since F_{s-1} and δB_{n-s} are closed forms in Ω, from Theorem 11, p.121 in [10] we have

$$V_{n-s}(X) = \frac{1}{k_n}\lim_{h\to\infty}\int_{\partial\Omega_h} \omega_{n-s}(X,Y) \wedge (F_{s-1} + \delta B_{n-s})(Y).$$

Moreover, because F_{s-1} and δB_{n-s} have interior nontangential trace in $L^p_{s-1}(\partial\Omega)$, by (1.14) it follows that

$$V_{n-s}(X) = H_{n-s}(X) + \frac{1}{k_n}\int_{\partial\Omega} \omega_{n-s}(X,Y) \wedge \Phi_{s-1}(Y), \tag{2.9}$$

where

$$H_{n-s}(X) = \frac{1}{k_n}\int_{\partial\Omega} \omega_{n-s}(X,Y) \wedge F^-_{s-1}(Y) \tag{2.10}$$

and

$$\Phi_{s-1} = (\delta B_{n-s})^-. \tag{2.11}$$

If we require that the form V_{n-s} satisfies (2.8), it results that

$$\delta B_{n-s}(X) = \delta H_{n-s}(X) + \frac{1}{k_n}\int_{\partial\Omega} \delta_X \omega_{n-s}(X,Y) \wedge \Phi_{s-1}(Y). \tag{2.12}$$

From this, if we take into account Theorem 1 in [14], it follows that a.e. on $\partial\Omega$

$$(\delta H_{n-s})^- = (\frac{1}{2}I - \frac{1}{k_n}T_{s-1})(\Phi_{s-1}), \tag{2.13}$$

where I is the identity operator on $L^p_{s-1}(\partial\Omega)$ and T_{s-1} is a compact operator on the same space. Consider now the the homogeneous transposed equation of (2.13)

$$\widetilde{T}(\psi_{n-s}) = \frac{k_n}{2}\Psi_{n-s} + T_{n-s}(\Psi_{n-s}) = 0. \tag{2.14}$$

In order to show that (2.13) has solution in $L^p_{s-1}(\partial\Omega)$, we prove that for all $i = 1, \ldots, R^-_{s-1}$

$$\int_{\partial\Omega} \Psi^i_{n-s} \wedge (\delta H_{n-s})^- = 0, \tag{2.15}$$

where $(\Psi^i_{n-s})_{1\le i\le R^+_{n-s}}$ is a base of $\mathrm{Ker}(\widetilde{T}_{n-s})$ verifying the conditions (see Theorem 7 in [14]):

$$\int_{\overline{\tau}^l_{n-s}} \Psi^i_{n-s} = 0 \quad \text{and} \quad \int_{\overline{\gamma}^j_{n-s}} \Psi^i_{n-s} = \delta_{ij} \tag{2.16}$$

for $l = 1, \ldots, R^-_{n-s}$ and for $i, j = 1, \ldots, R^+_{n-s}$. Here $(\overline{\tau}^l_{n-s}, \overline{\gamma}^j_{n-s})_{\substack{1\le l\le R^-_{n-s}\\ 1\le j\le R^+_{n-s}}}$ is the dual fundamental system of the fundamental system $(\tau^j_{s-1}, \gamma^l_{s-1})_{\substack{1\le j\le R^-_{s-1}\\ 1\le l\le R^+_{s-1}}}$. We now observe that using Theorem 1 in [14] we obtain

$$(\delta H_{n-s})^- = \frac{1}{2} F^-_{s-1} + \frac{1}{k_n} T_{s-1}(F^-_{s-1}). \tag{2.17}$$

Hence from (2.14) we have

$$\begin{aligned}\int_{\partial\Omega} \Psi^i_{n-s} \wedge (\delta H_{n-s})^- &= \frac{1}{2}\int_{\partial\Omega} \Psi^i_{n-s} \wedge F^-_{s-1} + \frac{1}{k_n}\int_{\partial\Omega} \Psi^i_{n-s} \wedge T_{s-1}(F^-_{s-1})\\ = \frac{1}{2}\int_{\partial\Omega} \Psi^i_{n-s} \wedge F^-_{s-1} &- \frac{1}{k_n}\int_{\partial\Omega} T_{n-s}(\Psi^i_{n-s}) \wedge (F^-_{s-1}) = \int_{\partial\Omega} \Psi^i_{n-s} \wedge F^-_{s-1}.\end{aligned} \tag{2.18}$$

Thus, since Ψ^i_{n-s} and F^-_{s-1} are closed forms in $W^{1,r}_{n-s}(\partial\Omega)$ (for all $r > 1$) and in $W^{1,p}_{s-1}(\partial\Omega)$ respectively, by applying Theorem 1.7 in [14], using (2.4) and (2.16), it follows that

$$\begin{aligned}\int_{\partial\Omega} \Psi^i_{n-s} \wedge F^-_{s-1} &= (-1)^{(s-1)(n-s)} \sum_{j=1}^{R^-_{s-1}} \int_{\tau^j_{s-1}} F^-_{s-1} \int_{\overline{\gamma}^j_{n-s}} \Psi^i_{n-s}\\ &+ \sum_{l=1}^{R^+_{s-1}} \int_{\gamma^l_{s-1}} F^-_{s-1} \int_{\overline{\tau}^l_{n-s}} \Psi^i_{n-s} = 0.\end{aligned}$$

Furthermore, using (2.18) we obtain that (2.15) is verified.

Let now Φ_{s-1} be a solution of (2.13) in $L^p_{s-1}(\partial\Omega)$. Thus, from (2.17) and Theorem 2 in [14], since $F^-_{s-1} \in W^{1,p}_{s-1}(\partial\Omega)$ is a closed form, it results that also δH_{n-s} is a closed form in $W^{1,p}_{s-1}(\partial\Omega)$. Hence, arguing in a manner similar to the proof of the Theorem 1.4 in [15], we have that Φ_{s-1} is a closed form in $W^{1,p}_{s-1}(\partial\Omega)$. Let finally

$$B_{n-s}(X) = H_{n-s} + \frac{1}{k_n}\int_{\partial\Omega} \omega_{n-s}(X,Y) \wedge \Phi_{s-1}(Y). \tag{2.19}$$

Thus using (11) in [14], we have that B_{n-s} is a harmonic form. Moreover in Ω it is

$$\delta B_{n-s}(X) = \delta H_{n-s}(X) + \frac{1}{k_n}\int_{\partial\Omega} \delta_X \omega_{n-s}(X,Y) \wedge \Phi_{s-1}(Y).$$

Furthermore by Theorem 1 in [14] δB_{n-s} has interior nontangential trace in $L^p_{s-1}(\partial\Omega)$ and

$$(\delta B_{n-s})^- = (\delta H_{n-s})^- + \frac{1}{2}\Phi_{s-1} + \frac{1}{k_n} T_{s-1}(\Phi_{s-1}).$$

From this, being Φ_{s-1} a solution of (2.13), we deduce that $(\delta B_{n-s})^- = \Phi_{s-1}$ and i) is satisfied.

ii) and iii) follow from Corollary of Theorem 1.1 and from (2.6) respectively. □

By applying Lemma 2.1 of [15], from (2.6) we obtain the following
Corollary. *If F_{s-1} is a form that satisfies the hypotheses of* Theorem 2.1, *then the form*

$$U_{s-2}(X) = \frac{(-1)^{n+s}}{k_n}\int_\Omega \delta_X\omega_{n-s+1}(X,Y)\wedge(F_{s-1}(Y)+\delta B_{n-s}(Y)) \tag{2.20}$$

is a primitive of F_{s-1}, where B_{n-s} is the form defined in (2.19) *and Φ_{s-1} is a solution of* (2.13).
Theorem 2.2. *If F_{s-1} is a form that satisfies the hypotheses of* Theorem 2.1, *and $dF^*_{s-1}\in L^p_{n-s+2}(\Omega)$, then there exists a primitive U_{s-2} of F_{s-1} in $C^2_{s-2}(\Omega)\cap D^{1,p}_{s-2}(\Omega)$.*

Proof. Arguing in a manner similar to the proof of (2.9), from (2.20) it follows that

$$\begin{aligned} U_{s-2}(X) = \quad & -\frac{1}{k_n}\int_\Omega \omega_{s-2}(X,Y)\wedge dF^*_{s-1}(Y) \\ & +\frac{1}{k_n}\int_{\partial\Omega}\omega_{s-2}(X,Y)\wedge((F^*_{s-1})^-(Y)+(-1)^{(n-1)(s-1)}(dB_{n-s})^-(Y)) \end{aligned}$$

and applying Theorem 1.2 of [15] and Theorem 77.VI of [9], we obtain our claim. □

Corollary. *If F_{s-1} satisfies the hypotheses of* Theorem 2.1 *and F^*_{s-1} is a closed form in Ω, then there exists a primitive U_{s-2} of F_{s-1} in $C^\infty_{s-2}(\Omega)\cap\mathcal{N}^{1,p}_{s-2}(\Omega)$.*
Theorem 2.3. *If $F_{s-1}\in C^0_{s-1}(\overline{\Omega})$ is a closed form and satisfies* (2.4), *then for all $p>n$ there exists a primitive U_{s-2} of F_{s-1} in $C^0_{s-2}(\overline{\Omega})\cap D^{1,p}_{s-2}(\Omega)$.*

Proof. Let $p>n$. Theorem 2.1 implies that there exists a primitive U_{s-2} of F_{s-1} in $D^{1,p}_{s-2}(\Omega)$. By Theorem 77.VI of [9] it follows that $U_{s-2}\in C^{0,\mu}(\overline{\Omega})$, where $\mu = 1-\dfrac{n}{p}$. □

Our purpose is to obtain the following uniqueness Theorem:
Theorem 2.4. *Let F_{s-1} and G_{n-s+3} be closed forms in $C^0_{s-1}(\overline{\Omega})\cap C^1_{s-1}(\Omega)$ and in $C^0_{n-s+3}(\overline{\Omega})\cap C^1_{n-s+3}(\Omega)$, respectively, such that*

$$\int_{\tau^l_{s-1}} F_{s-1} = 0 \quad \textit{and} \quad \int_{\tau^j_{n-s+3}} G_{n-s+3} = 0, \tag{2.21}$$

for all $l=1,\dots,R^-_{s-1}$ and for all $j=1,\dots,R^-_{n-s+3}$. Let $L_{s-2}\in\widetilde{C}^1_{s-2}(\partial\Omega)$ be such that

$$dL_{s-2} = F_{s-1} \text{ on } \partial\Omega \tag{2.22}$$

and

$$\int_{\gamma^i_{s-2}} L_{s-2} = \int_{\Gamma^i_{s-1}} F_{s-1} \tag{2.23}$$

for $i=1,\dots,R^+_{s-2}$, where $\gamma^i_{s-2}=\partial\Gamma^i_{s-1}$. Then we have

i) *if $R^-_{n-s+2}=0$, there exists a unique form $U_{s-2}\in C^1_{s-2}(\Omega)$ with an interior nontangential trace in $L^p_{s-2}(\partial\Omega)$ such that*

$$dU_{s-2} = F_{s-1} \text{ in } \Omega, \qquad dU^*_{s-2} = G_{n-s+3} \text{ in } \Omega \tag{2.24}$$

and

$$U^-_{s-2} = L_{s-2} \quad \text{on} \quad \partial\Omega; \tag{2.25}$$

ii) *if* $R^-_{n-s+2} > 0$, *then for any sequence* $(\alpha_i)_{1\le i\le R^-_{n-s+2}}$ *in* $\mathbf{R}$ *there exists a unique form* $U_{s-2} \in C^1_{s-2}(\Omega)$ *with interior nontangential trace in* $L^p_{s-2}(\partial\Omega)$ *satisfying* (2.24), (2.25), *and*

$$\int_{t^i_{n-s+2}} U^*_{s-2} = \alpha_i, \tag{2.26}$$

for all $i = 1, \dots, R^-_{n-s+2}$.

Proof. By applying Theorems 2.1 and 2.3 to forms F_{s-1} and G_{n-s+3}, it results that there exist two forms $\overline{U}_{s-2} \in C^0_{s-2}(\overline{\Omega}) \cap C^1_{s-2}(\Omega)$ and $\overline{V}_{n-s+2} \in C^0_{n-s+2}(\overline{\Omega}) \cap C^1_{n-s+2}(\Omega)$, such that in Ω

$$d\overline{U}_{s-2} = F_{s-1}, \qquad d\overline{U}^*_{s-2} = 0 \tag{2.27}$$

and

$$d\overline{V}_{n-s+2} = G_{n-s+3}, \qquad d\overline{V}^*_{n-s+2} = 0. \tag{2.28}$$

Let $K_{n-s+1} \in C^2_{n-s+1}(\Omega) \cap \mathcal{N}^{1,p}_{n-s+1}(\Omega)$ and $M_{s-3} \in C^2_{s-3}(\Omega) \cap \mathcal{N}^{1,p}_{s-3}(\Omega)$ be two arbitrary harmonic forms in Ω. Thus if we let

$$\overline{W}_{s-2} = \overline{U}_{s-2} + (-1)^{s(n-1)}\overline{V}^*_{n-s+2} \tag{2.29}$$

and

$$U_{s-2} = \overline{W}_{s-2} + \delta K_{n-s+1} + dM_{s-3}, \tag{2.30}$$

in Ω, it follows that $\overline{W}_{s-2} \in \widetilde{C}^1_s(\overline{\Omega}), dU_{s-2} = F_{s-1}$ and $dU^*_{s-2} = G_{n-s+3}$ in Ω.

We want to show that there exist K_{n-s+1} and M_{s-3} such that U_{s-2} satisfies (2.25) and (2.26). In order to do this let

$$M_{s-3}(X) = \sum_{k=1}^{R^-_{n-s+2}} \beta_k \int_{\partial\Omega} \omega_{s-3}(X,Y) \wedge (\Psi^k_{n-s+2})(Y), \tag{2.31}$$

where $(\beta_k)_{1\le k\le R^-_{n-s+2}}$ is an arbitrary sequence in $\mathbf{R}$ and $(\Psi^k_{n-s+2})_{1\le k\le R^-_{n-s+2}}$ is a base of $\mathrm{Ker}(\widetilde{T}'_{n-s+2})$ such that for $h, i = 1, \dots, R^-_{n-s+2}$ and for $l = 1, \dots, R^+_{n-s+2}$ we have (see: Theorem 8 in [14])

$$\int_{\tau^h_{n-s+2}} \Psi^i_{n-s+2} = \delta_{ih} \qquad \text{and} \qquad \int_{\gamma^l_{n-s+2}} \Psi^i_{n-s+2} = 0. \tag{2.32}$$

Since Ψ^i_{n-s+2} belongs to $\mathrm{Ker}(\widetilde{T}'_{n-s+2})$, Theorem 4 in [14] implies that Ψ^i_{n-s+2} is a closed form in $W^{1,q}_{n-s+2}(\partial\Omega)$ for any $q > 1$. Using Theorem 72.V in [9] we have that M_{s-3} is a continuous form in $\mathbf{R}^n$. Moreover, by Theorems 1 and 2 in [14], it follows that M_{s-3} is harmonic form in $\mathbf{R}^n \setminus \partial\Omega$ with interior and exterior nontangential trace in $L^q_{n-s+2}(\partial\Omega)$ for any $q > 1$, and $(\delta M_{s-3})^+ = 0$. Furthermore, from Theorem 2.1 in [13] it follows that $\delta M_{s-3} \in L^q_{n-s+2}(\mathbf{R}^n \setminus \Omega)$ for any $q > 1$. By applying Theorem 11, p. 121 in [10], we have

$$\int_{\mathbf{R}^n\setminus\Omega} dM_{s-3} \wedge \delta M_{s-3} = \int_{\partial\Omega} M_{s-3} \wedge (\delta M_{s-3})^+ = 0.$$

We deduce that $dM_{s-3} = 0$ in $\mathbf{R}^n \setminus \Omega$. Thus, for all $\Phi_{n-s+1} \in \widetilde{C}_{n-s+1}(\partial\Omega)$, it results that

$$\int_{\partial\Omega} (dM_{s-3})^- \wedge \Phi_{n-s+3} = \int_{\partial\Omega} M_{s-3} \wedge d\Phi_{n-s+3} = \int_{\partial\Omega} (dM_{s-3})^+ \wedge \Phi_{n-s+3} = 0.$$

Hence, by Lemma 3.1 in [13], we obtain $(dM_{s-3})^- = 0$. From this, if we require that the form U_{s-2}, defined in (2.30), satisfies the boundary condition (2.25), we have

$$(\delta K_{n-s+1})^- = L_{s-2} - \overline{W}_{s-2} \quad \text{on } \partial\Omega. \tag{2.33}$$

Let $1 < p < \infty$. From (2.22), (2.27), and (2.28) it follows that $L_{s-2} - \overline{W}_{s-2}$ is a closed form of $W^{1,p}_{s-2}(\partial\Omega)$, instead from Theorem 1.3 and (2.23) we have

$$\int_{\gamma^i_{s-2}} (L_{s-2} - \overline{W}_{s-2}) = 0$$

for any $i = 1, \dots, R^+_{s-2}$. Hence, using Theorem 1.5 in [15] we obtain that there exists a form $K_{n-s+1} \in C^2_{n-s+1}(\Omega) \cap \mathcal{N}^{1,p}_{n-s+1}(\Omega)$, harmonic in Ω and satisfying (2.33). Now, since using (2.30) and (2.31)

$$\begin{aligned} U^*_{s-2}(X) &= \overline{W}^*_{s-2}(X) + (-1)^{ns-1} dK_{n-s+1}(X) \\ &\quad + \sum_{k=1}^{R^+_{n-s+2}} \beta_k \int_{\partial\Omega} \delta_X \omega_{s-3}(X,Y) \wedge (\Psi^k_{n-s+2})(Y), \end{aligned}$$

we prove the existence of a sequence $(\beta_k)_{1\le k\le R^+_{n-s+3}}$ in $\mathbf{R}$ such that U^*_{s-2} satisfies (2.26). For this purpose, arguing in a manner similar to the proof of the Theorem (2.2) in [15], it is sufficient to show the existence of a sequence $(\beta_k)_{1\le k\le R^+_{n-s+3}}$ in $\mathbf{R}$ such that

$$\alpha_i = \int_{t^i_{n-s+2}} (\overline{W}^*_{s-2} + (-1)^{(n-1)s} dK_{n-s+1}) + \beta_i \int_{Y\in c^i_{s-2}} \delta_Y \int_{X\in t^i_{n-s+2}} \omega_{n-s+2} \tag{2.34}$$

for all $i = 1, \dots, R^-_{n-s+2}$. This is easily seen to be true, because by (27) in [8], it is

$$\int_{Y\in c^i_{s-2}} \delta_Y \int_{X\in t^i_{n-s+2}} \omega_{n-s+2} \neq 0.$$

Thus the existence of a solution of the problem is obtained. Finally to show the uniqueness of the solution it is sufficient to prove that the homogeneous problem associated to the above problem is the zero form. Thus, let U_{s-2} be a solution of the homogeneous problem. Hence U_{s-2} and U^*_{s-2} are closed forms in Ω, with interior nontangential trace of class $L^p_{s-2}(\partial\Omega)$, $U^-_{s-2} = 0$, and

$$\int_{t^i_{n-s+2}} U^*_{s-2} = 0 \tag{2.35}$$

for all $i = 1, \dots, R^-_{n-s+2}$. We now prove that for $i = 1, \dots, R^-_{n-s+2}$

$$\int_{\tau^i_{n-s+2}} U^*_{s-2} = 0. \tag{2.36}$$

Let $l = 1, \dots, R^-_{n-s+2}$. If $\Phi_{s-3} \in \widetilde{C}^1_{s-3}(\partial\Omega)$ is a closed form associated to τ^i_{n-s+2}. Using Definition 5.3 in [13] and (1.14) we have

$$\begin{aligned} \int_{\tau^l_{n-s+2}} U^*_{s-2} &= \int_{\partial\Omega} U^*_{s-2} \wedge \Phi_{s-3} = \lim_h \int_{\partial\Omega} \Lambda^*_h(U^*_{s-2}) \wedge \Phi_{s-3} \\ &= \lim_h \int_{\partial\Omega_h} U^*_{s-2} \wedge \Lambda^{-1*}_h(\Phi_{s-3}) = \lim_h \int_{\tau^l_{(n-s+2)h}} U^*_{s-2} \end{aligned} \tag{2.37}$$

where $\tau^l_{(n-s+2)h} = \Lambda_h(\tau^l_{n-s+2})$. Since $\tau^l_{(n-s+2)h}$ is a C^1-differentiable cycle on $\partial\Omega_h$, $\overline{\Omega}_h \subset \Omega$ and $([t^i_{n-s+2}])_{1 \le i \le R^-_{n-s+2}}$ is a base of $\mathcal{H}_{n-s+2}(\Omega)$, there exists a sequence $(\alpha^h_i)_{1 \le i \le R^-_{n-s+2}}$ in $\mathbf{R}$ and a cycle $c^l_{(n-s+3)h}$ C^1-differentiable in Ω, such that

$$\tau^l_{(n-s+2)h} = \sum_{i=1}^{R^-_{n-s+2}} \alpha^h_i t^i_{n-s+2} + \partial c^l_{(n-s+3)h}.$$

Thus, using (2.35), since U^*_{s-2} is a closed form in Ω, we have that

$$\int_{\tau^l_{(n-s+2)h}} U^*_{s-2} = \sum_{i=1}^{R^-_{n-s+2}} \alpha^h_i \int_{t^i_{n-s+2}} U^*_{s-2} + \int_{\partial c^l_{(n-s+2)h}} U^*_{s-2} = 0.$$

Moreover using (2.37), it follows (2.36). Hence U^*_{s-2} verifies the hypotheses of Corollary of Theorem 2.2. Thus there exists a form

$$P_{n-s+1} \in C^0_{n-s+1}(\overline{\Omega}) \cap \mathcal{N}^{1,p}_{n-s+1}(\Omega)$$

such that

$$dP_{n-s+1} = U^*_{s-2}. \tag{2.38}$$

From this, since $dU_{s-2} = 0$ in Ω and $U^-_{s-2} = 0$ on $\partial\Omega$, we obtain that P_{n-s+1} is a solution of the homogeneous Neumann problem. Thus $dP_{n-s+1} = 0$ in Ω (see Theorem 2.1 in [15]) and, hence, from (2.38), $U_{s-2} = 0$ in Ω.

□

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