A Remarkable Transformation Group on Cotangent Bundle

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

The present paper deals with the almost symplectic structure on T^*M. The set of all almost symplectic d-linear connections are determined for the case when the nonlinear connection is arbitrary and its structure is discussed. The important invariants are determined for the transformation group of almost symplectic d-linear connections corresponding to the same nonlinear connection N. The problem of integrability of the almost symplectic structure on T^*M is solved by means of these invariants, obtaining two single integrability typs I,II and one combined type $\varepsilon I + II$, where $\varepsilon \neq 0$ is real number.

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1 Introduction

The geometrical structures on T^*M have imposed themselves naturally, once the cotangent bundle geometry was approachead. Moreover, the necessity on studying them has been put forward by R.Miron, who pointed out the connection between the metrical structures on T^*M and the geometry of [4, 5, 6] Hamilton spaces.

The study of the cotangent bundle geometry is important also because it provides a natural geometrical structure for the Gauge theories of theoretical physics.

The the cotangent bundle geometry has been studied by R.Miron, S.Watana- be and S.Ikeda in [9], by K.Yano and S.Ishihara in [12], by Gh.Atanasiu and F.Klepp in [2], C.Udrişte and O.Şandru [11], by R.Miron, D.Hrimiuc, H.Shimada and S.Sabău in [8], and others.

Concerning the terminology and notations, we use those from [7].

Let M be a real C$^\infty$-differentiable manifold with dimension n, and let (T^*M, π^*, M) be its cotangent bundle.

If (x^i) is a local coordinates system on a domain U of a chart on M, the induced system of coordonates on ${\pi^*}^{-1}(U)$ is $(x^i, p_i), (i = 1, ..., n)$.

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Let N be a nonlinear connection on T^*M, with the coefficients $N_{ij}(x,p)$.

We consider on T^*M an almost symplectic structure A:

$$A(x,p) = \frac{1}{2} a_{ij}(x,p) dx^i \wedge dx^j + \widetilde{a}^{ij}(x,p) \delta p_i \wedge \delta p_j, \tag{1.1}$$

where $\{dx^i, \delta p_i\}, (i = 1, ..., n)$ is the dual basis of $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i}\}$, and $(a_{ij}(x,p), \widetilde{a}^{ij}(x,p))$ is a pair of given d-tensor fields on T^*M, of the type (0,2), and (2,0) respectively, each of them alternate and nondegenerate.

We asociate to the lift A the Obata's operators:

$$\begin{cases} \Phi^{ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j - a_{sj} a^{ir}), \Phi^{*ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j + a_{sj} a^{ir}), \\ \widetilde{\Phi}^{ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j - \widetilde{a}_{sj} \widetilde{a}^{ir}), \widetilde{\Phi}^{*ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j + \widetilde{a}_{sj} \widetilde{a}^{ir}). \end{cases} \tag{1.2}$$

Obata's operators have the same properties as the ones associated with a Finsler space [7].

The results obtained in the particular case of the normal d-linear connections support the findings of G.Atanasiu in his paper [1].

2 Almost symplectic d-linear connections on T^*M

Definition 2.1 *A d-linear connection, D^*, on T^*M, with local coefficients $D^*\Gamma(N) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^{\,i}{}_j{}^k, C_i{}^{jk})$, for which:*

$$\begin{cases} a_{ij|k} = 0, \ a_{ij}|^k = 0, \\ \widetilde{a}^{ij}{}_{|k} = 0, \ \widetilde{a}^{ij}|^k = 0, \end{cases} \tag{2.1}$$

where $|$ and $|$ denote the h-and v-covariant derivatives with respect to D^*,is called almost symplectic d-linear connections on T^*M, or compatible whith the almost symplectic structure A, (1.1) and is denoted by: $D^*\Gamma(N)$.

We shall determine the set of all almost symplectic d-linear connections on T^*M.

Let $\overset{0}{N}$ be an other nonlinear connection on T^*M, with the coefficients $\overset{0}{N}_{ij}(x,p)$.

Let $\overset{0}{D}{}^* \Gamma(\overset{0}{N}) = (\overset{0}{H}{}^i{}_{jk}, \overset{0}{\tilde{H}}{}_i{}^j{}_k, \overset{0}{\tilde{C}}{}^i{}_j{}^k, \overset{0}{C}{}_i{}^{jk})$ be the local coefficients of a fixed d-linear connection, $\overset{0}{D}{}^*$, on T^*M. Then any d-linear connection, D^*, on T^*M , can be expresed in the form:

$$\begin{cases} N_{ij} = \overset{0}{N}_{ij} - A_{ij}, \\ H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + A_{lk} \overset{0}{\tilde{C}}{}^i{}_j{}^l - B^i{}_{jk}, \\ \tilde{H}_i{}^j{}_k = \overset{0}{\tilde{H}}{}_i{}^j{}_k + A_{lk} \overset{0}{C}{}_i{}^{jl} - \tilde{B}_i{}^j{}_k, \\ \tilde{C}^i{}_j{}^k = \overset{0}{\tilde{C}}{}^i{}_j{}^k - \tilde{D}^i{}_j{}^k, \\ C_i{}^{jk} = \overset{0}{C}{}_i{}^{jk} - D_i{}^{jk}, \\ A_{\substack{ij \overset{0}{|} k}} = 0, \end{cases} \tag{2.2}$$

where $(A_{ij}, B^i{}_{jk}, \tilde{B}_i{}^j{}_k, \tilde{D}^i{}_j{}^k, D_i{}^{jk})$ are components of the difference tensor fields of $D^*\Gamma(N)$ from $\overset{0}{D}{}^*\,\Gamma(\overset{0}{N})$, [3].

Theorem 2.1 *Let $\overset{0}{D}{}^*$ be a given d-linear connection on T^*M, with local coefficients $\overset{0}{D}{}^*\,\Gamma(\overset{0}{N}) = (\overset{0}{H}{}^i{}_{jk}, \overset{0}{\tilde{H}}{}_i{}^j{}_k, \overset{0}{\tilde{C}}{}^i{}_j{}^k, \overset{0}{C}{}_i{}^{jk})$. The set of all almost symplectic d-linear connections on T^*M, with local coefficients $D^*\Gamma(N) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k,$
$C_i{}^{jk})$ is given by:*

$$
(2.3)\qquad \begin{cases}
N_{ij} = \overset{0}{N}_{ij} - X_{ij}, \\
H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + X_{lk}\,\overset{0}{\tilde{C}}{}^i{}_j{}^l + \frac{1}{2}a^{ir}(a_{rj\overset{0}{|}k} + a_{rj}\overset{0}{|}{}^l\, X_{lk}) + \Phi^{ir}_{mj}X^m{}_{rk}, \\
\tilde{H}_i{}^j{}_k = \overset{0}{\tilde{H}}{}_i{}^j{}_k + X_{lk}\,\overset{0}{C}{}_i{}^{jl} - \frac{1}{2}\widetilde{a}_{ir}(\widetilde{a}^{rj}{}_{\overset{0}{|}k} + \widetilde{a}^{rj}\overset{0}{|}{}^l\, X_{lk}) + \widetilde{\Phi}^{mj}_{ir}\tilde{X}_m{}^r{}_k, \\
\tilde{C}^i{}_j{}^k = \overset{0}{\tilde{C}}{}^i{}_j{}^k + \frac{1}{2}a^{ir}a_{rj}\overset{0}{|}{}^k + \Phi^{ir}_{mj}\tilde{Y}^m{}_r{}^k, \\
C_i{}^{jk} = \overset{0}{C}{}_i{}^{jk} - \frac{1}{2}\widetilde{a}_{ir}\widetilde{a}^{rj}\overset{0}{|}{}^k + \widetilde{\Phi}^{mj}_{ir}Y_m{}^{rk}, \\
X_{ij\overset{0}{|}k} = 0,
\end{cases}
$$

where $\overset{0}{|}$ and $\overset{0}{\Big|}$ denote the h-and v-covariant derivatives with respect to $\overset{0}{D}{}^*$, and $X_{ij}, X^i{}_{\mathrm{j}k}, \tilde{X}_i{}^j{}_k, \tilde{Y}{}^i{}_j{}^k, Y_i{}^{jk}$ are arbitrary tensor fields on T^*M.

Particular cases:

1. If $X_{ij} = X^i{}_{\mathrm{j}k} = \tilde{X}_i{}^j{}_k = \tilde{Y}{}^i{}_j{}^k = Y_i{}^{jk} = 0$, in Theorem 2.1 we have:

Theorem 2.2 *Let $\overset{0}{D}{}^*$ be a given d-linear connection on T^*M. Then the following d-linear connection K^*, with local coefficients $K^*\Gamma(N) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k,$
$C_i{}^{jk})$ given by (2.4) is an almost symplectic d-linear connection on T^*M.*

$$
(2.4)\qquad \begin{cases}
H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + \frac{1}{2}a^{ir}a_{rj\overset{0}{|}k}, \\
\tilde{H}_i{}^j{}_k = \overset{0}{\tilde{H}}{}_i{}^j{}_k - \frac{1}{2}\widetilde{a}_{ir}\widetilde{a}^{rj}{}_{\overset{0}{|}k}, \\
\tilde{C}^i{}_j{}^k = \overset{0}{\tilde{C}}{}^i{}_j{}^k + \frac{1}{2}a^{ir}a_{rj}\overset{0}{|}{}^k, \\
C_i{}^{jk} = \overset{0}{C}{}_i{}^{jk} - \frac{1}{2}\widetilde{a}_{ir}\widetilde{a}^{rj}\overset{0}{|}{}^k,
\end{cases}
$$

where $\overset{0}{|}$ and $\overset{0}{\Big|}$ denote the h-and v-covariant derivatives with respect to the given d-linear connection, $\overset{0}{D}{}^*$, on T^*M.

2. If we take an almost symplectic d-linear connection on T^*M (e.g. K^*) as $\overset{0}{D}{}^*$, in Theorem 2.1., we have:

Theorem 2.3 *Let $\overset{0}{D}{}^*$ be a fixed almost symplectic d-linear connection on T^*M, with local coefficients: $\overset{0}{D}{}^*\ \Gamma(\overset{0}{N}) = (\overset{0}{H}{}^i{}_{jk}, \overset{0}{\tilde{H}}{}_i{}^j{}_k, \overset{0}{\tilde{C}}{}^i{}_j{}^k, \overset{0}{C}{}_i{}^{jk})$. The set of all almost symplectic d-linear connections on T^*M, with local coefficients: $D^*\Gamma(N) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i{}^{jk})$ is given by:*

$$
(2.5)\qquad \begin{cases}
N_{ij} = \overset{0}{N}_{ij} - X_{ij}, \\
H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + X_{lk}\, \overset{0}{\tilde{C}}{}^i{}_j{}^l + \Phi^{ir}_{mj} X^m{}_{rk}, \\
\tilde{H}_i{}^j{}_k = \overset{0}{\tilde{H}}{}_i{}^j{}_k + X_{lk}\, \overset{0}{C}{}_i{}^{jl} + \tilde{\Phi}^{mj}_{ir} \tilde{X}_m{}^r{}_k, \\
\tilde{C}^i{}_j{}^k = \overset{0}{\tilde{C}}{}^i{}_j{}^k + \Phi^{ir}_{mj} \tilde{Y}^m{}_r{}^k, \\
C_i{}^{jk} = \overset{0}{C}{}_i{}^{jk} + \tilde{\Phi}^{mj}_{ir} Y_m{}^{rk}, \\
X_{ij\overset{0}{|}k} = 0,
\end{cases}
$$

where $\overset{0}{|}$ and $\overset{0}{\bigg|}$ denote the h-and v-covariant derivatives with respect to $\overset{0}{D}{}^*$, and $X_{ij}, X^i{}_{jk}, \tilde{X}_i{}^j{}_k, \tilde{Y}{}^i{}_j{}^k, Y_i{}^{jk}$ are arbitrary tensor fields on T^*M.

3. If we take $X_{ij} = 0$, in Theorem 2.3 we obtain:

Theorem 2.4 *Let $\overset{0}{D}{}^*$ be a fixed almost symplectic d-linear connection on T^*M, with local coefficients: $\overset{0}{D}{}^*\ \Gamma(\overset{0}{N}) = (\overset{0}{H}{}^i{}_{jk}, \overset{0}{\tilde{H}}{}_i{}^j{}_k, \overset{0}{\tilde{C}}{}^i{}_j{}^k, \overset{0}{C}{}_i{}^{jk})$. The set of all almost symplectic d-linear connections on T^*M, corresponding to the same nonlinear connection N, with local coefficients:$D^*\Gamma(N) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i{}^{jk})$ is given by:*

$$
(2.6)\qquad \begin{cases}
H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + \Phi^{ir}_{mj} X^m{}_{rk}, \\
\tilde{H}_i{}^j{}_k = \overset{0}{\tilde{H}}{}_i{}^j{}_k + \tilde{\Phi}^{mj}_{ir} \tilde{X}_m{}^r{}_k, \\
\tilde{C}^i{}_j{}^k = \overset{0}{\tilde{C}}{}^i{}_j{}^k + \Phi^{ir}_{mj} \tilde{Y}^m{}_r{}^k, \\
C_i{}^{jk} = \overset{0}{C}{}_i{}^{jk} + \tilde{\Phi}^{mj}_{ir} Y_m{}^{rk},
\end{cases}
$$

where $X^i{}_{jk}, \tilde{X}_i{}^j{}_k, \tilde{Y}{}^i{}_j{}^k, Y_i{}^{jk}$ are arbitrary tensor fields on T^*M.

3 The group of transformations of almost symplectic d-linear connections on T^*M

We study the transformations $D^*\Gamma(N) \to \bar{D}^*\Gamma(N)$ of the almost symplectic d-linear connections on T^*M, corresponding to the same nonlinear connection N.

If we replace $\overset{0}{D}{}^*\ \Gamma(\overset{0}{N})$ and $D^*\Gamma(N)$ in Theorem 2.3 by $D^*\Gamma(N)$ and $\bar{D}^*\Gamma(N)$, respectively, two almost symplectic d-linear connections, we obtain:

Theorem 3.1 *Two almost symplectic d-linear connections $D^*, \bar{D}^*$ with local coefficients $D^*\Gamma(N) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i{}^{jk})$ and*
$\bar{D}^*\Gamma(\bar{N}) = (\bar{H}^i{}_{jk}, \overline{\tilde{H}}_i{}^j{}_k, \overline{\tilde{C}}^i{}_j{}^k, \bar{C}_i{}^{jk})$ *respectively, are related as follows:*

$$\begin{cases} \bar{H}^i{}_{jk} = H^i{}_{jk} + \Phi^{ir}_{sj} X^s{}_{rk}, \\ \overline{\tilde{H}}_i{}^j{}_k = \tilde{H}_i{}^j{}_k + \widetilde{\Phi}^{sj}_{ir} \tilde{X}_s{}^r{}_k, \\ \overline{\tilde{C}}^i{}_j{}^k = \tilde{C}^i{}_j{}^k + \Phi^{ir}_{sj} \tilde{Y}^s{}_r{}^k, \\ \bar{C}_i{}^{jk} = C_i{}^{jk} + \widetilde{\Phi}^{sj}_{ir} Y_s{}^{rk}, \end{cases} \tag{3.1}$$

where $X^i{}_{jk}, \tilde{X}_i{}^j{}_k, \tilde{Y}{}^i{}_j{}^k, Y_i{}^{jk}$are arbitrary tensor fields on T^*M.

Conversely, given the tensor fields $X^i{}_{jk}, \tilde{X}_i{}^j{}_k, \tilde{Y}{}^i{}_j{}^k, Y_i{}^{jk}$ the above (3.1) is thought to be a transformation of an almost symplectic d- linear connection D^* to an almost symplectic d- linear connection $\bar{D}^*$.

We shall denote this transformation by $t(X^i{}_{jk}, \tilde{X}_i{}^j{}_k, \tilde{Y}{}^i{}_j{}^k, Y_i{}^{jk})$.
Thus we have:

Theorem 3.2 *The set $\mathcal{G}_{as}$ of all transformations $t(X^i{}_{jk}, \tilde{X}_i{}^j{}_k, \tilde{Y}^i{}_j{}^k, Y_i{}^{jk})$ given by (3.1) is a transformation group of the set of all almost symplectic d-linear connections corresponding to the same nonlinear connection N, on T^*M together with the mapping product: $t(X'{}^i{}_{jk}, \tilde{X}'{}_i{}^j{}_k, \tilde{Y}'{}^i{}_j{}^k, Y'{}_i{}^{jk}) \circ t(X^i{}_{jk}, \tilde{X}_i{}^j{}_k, \tilde{Y}^i{}_j{}^k,$*
$Y_i{}^{jk}) = (X^i{}_{jk} + X'{}^i{}_{jk}, \tilde{X}_i{}^j{}_k + \tilde{X}'{}_i{}^j{}_k, \tilde{Y}^i{}_j{}^k + \tilde{Y}'{}^i{}_j{}^k, Y_i{}^{jk} + Y'{}_i{}^{jk})$.

This group, $\mathcal{G}_{as}$, is an abelian group wich is isomorphic to the additive group of quadruples of tensor fields $(\Phi^{ir}_{sj} X^s{}_{rk}, \widetilde{\Phi}^{sj}_{ir} \tilde{X}_s{}^r{}_k, \Phi^{ir}_{sj} \tilde{Y}^s{}_r{}^k, \widetilde{\Phi}^{sj}_{ir} Y_s{}^{rk})$.

We determine the invariants of the group $\mathcal{G}_{as}$.
We denote with:

$$t^k{}_{ij} = \mathcal{A}_{jk}\{\frac{\partial N_{ij}}{\partial p_k}\}, \tag{3.2}$$

where $\mathcal{A}_{jk}\{...\}$ denotes the alternate summation.
Since R_{ijk} and the tensor field $t^k{}_{ij}$ depend on N only, they are invariants of the group $\mathcal{G}_{as}$.

We make some notations:

$$\begin{cases} T^*_{ijk} = \mathcal{S}_{ijk}\{a_{im} T^m_{jk}\}, \\ S^{*ijk} = \mathcal{S}_{ijk}\{\widetilde{a}^{im} S_m{}^{jk}\}, \\ R^{*k}{}_{ij} = \mathcal{S}_{ijk}\{\widetilde{a}^{km} R_{mij}\}, \\ \chi_{ij}{}^k = \mathcal{A}_{ij}\{a_{im} \widetilde{C}^m_j{}^k\}, \\ \nu^j{}_i{}^k = \mathcal{A}_{jk}\{\widetilde{a}^{mj} P_{mi}{}^k\}, \end{cases} \tag{3.3}$$

where $\mathcal{S}_{ijk}\{\ldots...\}$ denotes the cyclic summation.
By direct calculations we have:

Theorem 3.3 *The tensor fields $t^k{}_{ij}$, R^{*k}_{ij}, $R{}_{ijk}$, $T^*{}_{ijk}$, S^{*ijk}, χ^k_{ij}, $\nu^j{}_i{}^k$ are invariants of the group $\mathcal{G}_{as}$.*

Theorem 3.4 *Let N be a nonlinear connection on T^*M. The invariant T^*_{ijk} (resp. S^{*ijk}) vanishes if and only if there exists an almost symplectic d-linear connection, D^*, having the local coefficients $D^*\Gamma(N) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i{}^{jk})$, with $T^i_{jk} = 0$ (resp. $S_i^{jk} = 0$).*

Proof. Let ${X^i}_{jk} = \alpha {T^i}_{jk}, \alpha \in \mathbf{R}$ in (3.1), then we have:
${\bar{T}^i}_{jk} = {T^i}_{jk} + \frac{\alpha}{2}\mathcal{A}_{jk}\{\Phi^{ir}_{mj}{T^m}_{rk}\} = (1+\frac{3}{2}\alpha){T^i}_{jk} - \frac{\alpha}{2}a^{ir}T^*_{rjk}$.
Taking $\alpha = -\frac{2}{3}, T^*_{ijk} = 0$ implies ${\bar{T}^i}_{jk} = 0$.
The converse is evident.
The statement about S^{*ijk} is proved in the same way.

Theorem 3.5 *a_{ij} does not depend on p, if and only if ${\chi_{ij}}^k = 0$.*

Proof. From $a_{ij}|^k = 0$ we have $\frac{\partial a_{ij}}{\partial p_k} - {\chi_{ij}}^k = 0$.

4 The integrability of an almost symplectic structure in the cotangent bundle

Let $\Lambda^k(T^*M)$ be the $\mathcal{F}$-module of all k-forms on the cotangent bundle $(T^*(M), \pi^*, M)$ where $\mathcal{F}$ is the ring of all differentiable functions on $T^*(M)$. If N is a given nonlinear connection, then $\{dx^i, \delta p_i\}$ is a local basis of $\Lambda^1(T^*(M))$, which is dual to $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i}\}$.

If $f \in \mathcal{F}$, then the 1-form df is written as:

$$df = \frac{\delta f}{\delta x^i}dx^i + \frac{\partial f}{\partial p_i}\delta p_i, \tag{4.1}$$

and the exterior differential of δp_i is given by:

$$d(\delta p_i) = \frac{1}{2}R_{ijk}dx^k \wedge dx^j - \frac{\partial N_{ij}}{\partial p_k}\delta p_k \wedge dx^j. \tag{4.2}$$

In general, $\omega \in \Lambda^2(T^*(M))$ is written in the form:

$$\omega = \frac{1}{2}\tilde{a}_{ij}dx^i \wedge dx^j + {\tilde{b}_i}^{\ j}dx^i \wedge \delta p_j + \frac{1}{2}\tilde{c}^{ij}\delta p_i \wedge \delta p_j, \tag{4.3}$$

where $\tilde{a}_{ij} = -\tilde{a}_{ji},\ \tilde{c}^{ij} = -\tilde{c}^{ji}$.

The exterior differential $d\omega$ is given by:

$$\begin{aligned} d\omega = {} & \tfrac{1}{6}\overset{1}{\omega}_{ijk}\ dx^i \wedge dx^j \wedge dx^k + \tfrac{1}{2}\overset{2}{\omega}{}_{ij}^{\ \ k}\ dx^i \wedge dx^j \wedge \delta p_k + \\ & + \tfrac{1}{2}\overset{3}{\omega}{}_i^{\ jk}\ dx^i \wedge \delta p_j \wedge \delta p_k + \tfrac{1}{6}\overset{4}{\omega}{}^{ijk}\ \delta p_i \wedge \delta p_j \wedge \delta p_k, \end{aligned} \tag{4.4}$$

where:

$$\begin{cases} \overset{1}{\omega}_{ijk} = S_{ijk}\{\frac{\delta \tilde{a}_{ij}}{\delta x^k} + {\tilde{b}_i}^{\ m}R_{mjk}\}, \\ \overset{2}{\omega}{}_{ij}^{\ \ k} = \frac{\partial \tilde{a}_{ij}}{\partial p_k} + \tilde{c}^{km}R_{mij} + \mathcal{A}_{ij}\{\frac{\delta {\tilde{b}_j}^{\ k}}{\delta x^i} - {\tilde{b}_i}^{\ m}\frac{\partial N_{mj}}{\partial p_k}\}, \\ \overset{3}{\omega}{}_i^{\ jk} = \frac{\delta \tilde{c}^{jk}}{\delta x^i} + \mathcal{A}_{jk}\{\frac{\partial {\tilde{b}_i}^{\ j}}{\delta p_k} - \tilde{c}^{km}\frac{\partial N_{mi}}{\partial p_j}\}, \\ \overset{4}{\omega}{}^{ijk} = S_{ijk}\{\frac{\partial \tilde{c}^{ij}}{\partial p_k}\}. \end{cases} \tag{4.5}$$

Proposition 4.1 *If a d-linear connection D^* is given on T^*M, with local coefficients: $D^*\Gamma(N) = ({H^i}_{jk}, {\tilde{H}_i}^{\ j}{}_k, {\tilde{C}^i}{}_j{}^k, {C_i}^{jk})$, then the coefficients $\overset{1}{\omega}_{ijk}, \overset{2}{\omega}{}_{ij}^{\ \ k}, \overset{3}{\omega}{}_i^{\ jk}, \overset{4}{\omega}{}^{ijk}$ have the following expressions:*

$$
(4.6)\quad \begin{cases} \overset{1}{\omega}_{ijk} = S_{ijk}\{\tilde{a}_{ij|k} + \tilde{a}_{im}T^m{}_{jk} + \tilde{b}_i{}^m R_{mjk}\}, \\ \overset{2}{\omega}_{ij}{}^k = \tilde{a}_{ij}|^k + \tilde{b}_m^k T^m{}_{ji} + \tilde{c}^{km}R_{mij} + \mathcal{A}_{ij}\{\tilde{b}_j{}^k{}_{|i} + \tilde{a}_{im}\tilde{C}^m_j{}^k + \tilde{b}_i^m \tilde{P}_{mj}^k\}, \\ \overset{3}{\omega}_i{}^{jk} = \tilde{c}^{jk}{}_{|i} - \tilde{b}_i{}^m S_m{}^{jk} + \mathcal{A}_{jk}\{\tilde{b}_i^j|^k + \tilde{b}_m^j \tilde{C}_i^m{}^k + \tilde{c}^{mj}\tilde{P}_{mi}{}^k\}, \\ \overset{4}{\omega}{}^{ijk} = \mathcal{S}_{ijk}\{\tilde{c}^{ij}|^k + \tilde{c}^{im}S_m{}^{jk}\}. \end{cases}
$$

Definition 4.1 A 2-form $\omega \in \Lambda^2(T^*M)$, for which the matrix A=$\begin{pmatrix} \tilde{a}_{ij} & \tilde{b}_i^j \\ -\tilde{b}_i^j & \tilde{c}^{ij} \end{pmatrix}$ is nondegenerate, is called integrable if: $d\omega$=0.

Theorem 4.1 *A 2-form $\omega \in \Lambda^2(T^*M)$, for which the matrix A is nondegenerate, is integrable, if and only if the tensor fields* $\overset{1}{\omega}_{ijk} = 0, \overset{2}{\omega}_{ij}{}^k = 0, \overset{3}{\omega}_i{}^{jk} = 0,$ *and* $\overset{4}{\omega}{}^{ijk} = 0$

Assume that a nonlinear connection N on T^*M is given, then an almost symplectic structure on the base manifold M is lifted to a 2-form ω on T^*M thus: we consider the following 2-forms ω of two single types I, II and one combined type εI+II, $\varepsilon \in R^*$:
I: $\omega = \frac{1}{2}a_{ij}dx^i \wedge dx^j$; II: $\omega = \frac{1}{2}\tilde{a}^{ij}\delta p_i \wedge \delta p_j$,
εI+II: $\omega = \varepsilon a_{ij}dx^i \wedge dx^j + \frac{1}{2}\tilde{a}^{ij}\delta p_i \wedge \delta p_j$,

Proposition 4.2 *Each 2-form ω of the type $\varepsilon I + II$ is nondegenerate and defines an almost symplectic structure on T^*M.*

Proposition 4.3 *The coefficients $\overset{1}{\omega}_{ijk}, \overset{2}{\omega}_{ij}{}^k, \overset{3}{\omega}_i{}^{jk}, \overset{4}{\omega}{}^{ijk}$ of the exterior differential of the 2-form given in Proposition 4.2 are invariants of the group $\mathcal{G}_{as}$ and are given in the following form*

εI+II: $\overset{1}{\omega}_{ijk} = \varepsilon T^*_{ijk}, \overset{2}{\omega}_{ij}{}^k = \widetilde{a}^{km}R_{mij} + \varepsilon \chi_{ij}{}^k$,
$\overset{3}{\omega}_i{}^{jk} = \nu^j{}_i{}^k, \qquad \overset{4}{\omega}{}^{ijk} = S^{*ijk}$.

Proof. Using the relations (4.6) and (3.3) we calculate the coefficients of the exterior differentials of the 2-forms of the type I and II:
I: $\overset{1}{\omega}_{ijk} = T^*_{ijk}$, $\overset{2}{\omega}_{ij}{}^k = \chi_{ij}{}^k$, $\overset{3}{\omega}_i{}^{jk} = \quad \overset{4}{\omega}{}^{ijk} = 0$,
II: $\overset{1}{\omega}_{ijk} = 0$, $\overset{2}{\omega}_{ij}{}^k = \widetilde{a}^{km}R_{mij}$, $\overset{3}{\omega}_i{}^{jk} = \nu^j{}_i{}^k, \qquad \overset{4}{\omega}{}^{ijk} = S^{*ijk}$.

Definition 4.2 *An almost symplectic structure on a differentiable manifold M is called integrable of the type $\varepsilon I + II$, if there exists an almost symplectic d-linear connection D^* on T^*M such that the corresponding lifted 2-form on T^*M is integrable.*

Then from Theorems (3.4), (4.1) and from Definition 4.2 we have:

Theorem 4.2 *An almost symplectic structure on a differentiable manifold M is integrable of the type $\epsilon I + II$ if and only if there exists an almost symplectic d-linear connection D^* on T^*M with local coefficients $D^*\Gamma(N) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k,$ $C_i{}^{jk})$ satisfying the following conditions:*

εI+II: $T^i_{\ jk} = S_i^{\ jk} = 0$, $\widetilde{a}^{km} R_{mij} + \varepsilon \chi_{ij}^{\ \ k} = 0, \nu^j_{\ i}{}^k = 0$.

Theorem 4.3 *An almost symplectic structure on a differentiable manifold M, integrable of the type $\varepsilon I + II, \varepsilon \in R^*$, does not depend on p if and only if $R_{ijk} = 0$.*

The proof follows from $\widetilde{a}^{km} R_{mij} + \varepsilon \chi_{ij}^{\ \ k} = 0$ and Theorem 3.5.

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