$d-$Connections Compatible with Homogeneous Metric on the Cotangent Bundle

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

In this paper is studied the cotangent bundle $\widetilde{T^*M} = T^*M \backslash \{0\}$ with a 0-homogeneous lift $\overset{*}{\mathbf{G}}$. The connection compatible with the homogeneous metric is determined.

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1 Introduction

Let (T^*M, π^*, M) be the cotangent bundle, where M is a C^∞-differentiable, real n-dimensional manifold. If (U, φ) is a local chart on M and (x^i) are the coordinates of a point $p \in M$, $p \in \varphi^{-1}(x) \in U$, then a point $u \in \pi^{*-1}(U)$, $\pi^*(u) = p$ has the coordinates (x^i, p_i), $(i = \overline{1,n})$. The natural basis of the module $\mathcal{X}(\mathcal{T}^*\mathcal{M})$ is given by $(\partial_i = \dfrac{\partial}{\partial x^i}, \partial^r = \dfrac{\partial}{\partial p_r})$. Given a nonlinear connection N on T^*M ([1]) there exist a single system of functions $N_{ia}(x,p)$ such that $\delta_k = \partial_k + N_{ka}(x,p)\partial^a$, $(a = \overline{1,n})$ and (δ_k, ∂^a) is a local basis of $\mathcal{X}(\mathcal{T}^*\mathcal{M})$, which is called the adapted basis to N. We have the dual basis $(dx^i, \delta p_a = dp_a - N_{ka}(x,p)dx^k)$. For $X \in \mathcal{X}(\mathcal{T}^*\mathcal{M})$ is obtained a unique decompostion $X = hX + vX$, $hX \in H$, $vX \in V$, (V is the vertical distribution) and for $\omega \in \mathcal{X}^*(\mathcal{T}^*\mathcal{M})$ we have $\omega = h\omega + v\omega$, where $(h\omega)(X) = \omega(hX)$, $(v\omega)(X) = \omega(vX)$. In the adapted basis (δ_k, ∂^a) we have $X = X^i\delta_i + X_a\partial^a$ and $\omega = \omega_i dx^i + \omega^a \delta p_a$. The homogeneous lift of the Riemannian and Finslerian metrics on the tangent bundle have been studied by Acad. Radu Miron ([3], [4]), while the properties of homogeneous structures on cotangent bundle were studied by P. Stavre and the author ([5], [6], [7]). More specific, details on the homogeneous lift of a Cartan metric on cotangent bundle and on integrability conditions of homogeneous almost complex structures are given in [6], the properties of the homogeneous lift of a Riemann metric on cotangent bundle are studied in [7], and the homogeneous almost product structure case is developed in [5].

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2 Existence of metrical d-connections

Let $(M, g_{ij}(x))$ be a Riemannian space and (T^*M, π^*, M) its cotangent bundle. We introduce $g^{rs}(x)$ with $g_{ik}(x)g^{ks}(x) = \delta_i^s$.
We consider

$$\overset{c}{N}_{kr}(x,p) \overset{def}{=} p_s \gamma^s_{rk}(x), \tag{1}$$

where $\gamma^s_{rk}(x)$ are the Christoffel symbols of g. Evidently $\{\overset{c}{N}_{kr}(x,p)\}$ are the coefficients of a nonlinear connection on $\widetilde{T^*M} = T^*M \setminus \{0\}$ which is 1- homogeneous on the fibres. Using $\overset{c}{N}_{kr}$ we consider $\delta_k = \partial_k + \overset{c}{N}_{kr}(x,p)\partial^r$; $\delta p_k = dp_k - \overset{c}{N}_{ik}(x,p)dx^i$.
We have

$$\overset{*}{G} = h\overset{*}{G} + v\overset{*}{G}, \quad \overset{*}{G} = g_{ij}(x)dx^i \otimes dx^j + g^{rs}(x)\delta p_r \otimes \delta p_s. \tag{2}$$

If we define the homothety $\overset{*}{h}_t : (x,p) \to (x,tp)$, $\forall t \in \mathbf{R}$, then

$$\left(\overset{*}{G} \circ \overset{*}{h}_t\right)(x,p) = g_{ij}(x)dx^i \otimes dx^j + t^2 g^{rs}(x)\delta p_r \otimes \delta p_s \neq \overset{*}{G}(x,p). \tag{3}$$

Proposition 1 *$\overset{*}{G}$ is globally defined Riemannian metric on $\widetilde{T^*M}$ and is not homogeneous on the fibres of T^*M.*

We consider the function

$$H(x,p) = g^{rs}(x)p_r p_s. \tag{4}$$

Obviously H is 2-homogeneous on the fibres of cotangent bundle $\widetilde{T^*M}$.
If $\overset{*}{\mathbf{G}}$ is defined by

$$\overset{*}{\mathbf{G}} = g_{ij}(x)dx^i \otimes dx^j + \frac{a^2}{H} g^{rs}(x)\delta p_r \otimes \delta p_s\ , \tag{5}$$

where $a > 0$ is a constant, then we get:

Proposition 2 *The following properties hold:*

1° The pair $(\widetilde{T^*M}, \overset{*}{\mathbf{G}})$ is a Riemannian space depending only on the metric g.

2° $\overset{*}{\mathbf{G}}$ is 0-homogeneous on the fibres of $\widetilde{T^*M}$.

3° The distribution N and V are ortogonal with respect to $\overset{*}{\mathbf{G}}$

$$\overset{*}{\mathbf{G}}(hX, vY) = 0, \quad \forall X, Y \in \mathcal{X}(\mathcal{T}^*\mathcal{M}).$$

$$\overset{*}{\mathbf{G}} = g_{ij}(x)dx^i \otimes dx^j + h^{rs}(x,p)\delta p_r \otimes \delta p_s, \tag{6}$$

where

$$h^{rs}(x,p) = \frac{a^2}{H} g^{rs}(x) \tag{7}$$

From [1] we have:

Definition 1 *A linear connection D on T^*M is called metrical $d-$linear connection with respect to $\overset{*}{\mathbf{G}}$ if $D\,\overset{*}{\mathbf{G}} = 0$ and D preserves by parallelism the horizontal distribution N.*

We will prove the existence of metrical $d-$linear connections. In the adapted frame we have:

$$(8)\qquad \begin{aligned} D_{\delta_k}\delta_j &= \overset{h}{F}{}^i_{jk}\,\delta_i + \widetilde{F}_{j(r)k}\partial^r, \quad D_{\delta_k}\partial^r = -\widetilde{\widetilde{F}}{}^{i(r)}_{k}\delta_i - \overset{v}{F}{}^{(r)}_{(j)k}\,\partial^j,\\ D_{\partial^k}\delta_j &= \overset{h}{C}{}^{i(k)}_{j}\,\delta_i + \widetilde{C}^{(k)}_{j(r)}\partial^r, \quad D_{\partial^k}\partial^r = -\widetilde{\widetilde{C}}{}^{(r)i(k)}\delta_i - \overset{v}{C}{}^{(r)(k)}_{(j)}\,\partial^j,\end{aligned}$$

where $\overset{h}{F}{}^i_{jk}, \widetilde{F}_{j(r)k}, \widetilde{\widetilde{F}}{}^{i(r)}_{k}, \overset{v}{F}{}^{(r)}_{(j)k}, \overset{h}{C}{}^{i(k)}_{j}, \widetilde{C}^{(k)}_{j(r)}, \widetilde{\widetilde{C}}{}^{(r)i(k)}, \overset{v}{C}{}^{(r))k)}_{(j)})$ are the coefficients of D.

Theorem 1 *There exists a metrical $d-$ linear connection D on $\widetilde{T^*M}$ with respect to $\overset{*}{\mathbf{G}}$, which depends only on the metric tensor g; its components are*

$$(9)\qquad \begin{cases} \widetilde{F}_{j(r)k} = \widetilde{\widetilde{F}}{}^{i(r)}_{k} = \widetilde{C}^{(k)}_{j(r)} = \widetilde{\widetilde{C}}{}^{(r)i(k)} = \overset{h}{C}{}^{i(k)}_{j} = 0,\\ \overset{h}{F}{}^i_{jk} = \overset{v}{F}{}^{(i)}_{(j)k} = \gamma^i_{jk}(x),\\ \overset{v}{C}{}^{(r)(k)}_{(j)} = \dfrac{1}{H}(\delta^k_j p^r + \delta^r_j p^k - g^{rk}p_j),\end{cases}$$

where $g^{rm}p_m = p^r$.

Proof. In the general case of a vector bundle we have a canonical metrical connection given by [2],

$$\begin{cases} \overset{h}{F}{}^i_{jk} = \dfrac{1}{2}g^{is}(\delta_j g_{si} + \delta_k g_{js} - \delta_s g_{jk}),\\ \overset{v}{F}{}^{(r)}_{(j)k} = \partial^r N_{jk} + \dfrac{1}{2}h^{rs}h_{js||k},\\ \overset{h}{C}{}^{i(k)}_{j} = \dfrac{1}{2}g_{js}g^{is}||^k = \dfrac{1}{2}g_{js}\partial^k g^{is},\\ \overset{v}{C}{}^{(r)(k)}_{(j)} = -\dfrac{1}{2}h_{js}(\partial^r h^{ks} + \partial^k h^{rs} - \partial^s h^{rk}),\end{cases}$$

where „$_{||}$" and „$||$" are the $h-$, and $v-$ covariant derivative with respect to the Berwald connection $(B^r_{jk} = \partial^r N_{jk}, 0)$.

But $g = g(x)$, so $\delta_j g_{si} = \partial_j g_{si}$ and $\partial^k g^{is} = 0 \Rightarrow \overset{h}{F}{}^i_{jk} = \gamma^i_{jk}(x)$ and $\overset{h}{C}{}^{i(k)}_{j} = 0$.
From $h^{rs}(x,p) = \dfrac{a^2}{H}g^{rs}(x)$ it follows $h_{rs}(x,p) = \dfrac{H}{a^2}g_{rs}(x)$. But

$$\partial^r h^{ks}(x,p) = \partial^r\left(\frac{a^2}{g^{rm}p_m p_r}g^{ks}(x)\right) = -\frac{2a^2}{H^2}g^{ks}(x)g^{rm}p_m,$$

$$\overset{v}{C}{}^{(r)(k)}_{(j)} = -\frac{1}{2}h_{js}(\partial^r h^{ks} + \partial^k h^{rs} - \partial^s h^{rk}) =$$

$$= -\frac{1}{2}\frac{H}{a^2}g_{js}(x)\left(-\frac{2a^2}{H^2}g^{ks}g^{rm}p_m - \frac{2a^2}{H^2}g^{rs}g^{km}p_m + \frac{2a^2}{H^2}g^{rk}g^{sm}p_m\right)$$

$$\overset{v}{C}{}^{(r)(k)}_{(j)} = \frac{1}{H}(\delta^k_j p^r + \delta^r_j p^k - g^{rk} p_j) \ , \qquad g^{rm} p_m = p^r.$$

$$\overset{v}{F}{}^{(r)}_{(j)k} = \partial^r \overset{c}{N}_{jk} + \frac{1}{2} h^{rs} h_{js||k} = \partial^r \overset{c}{N}_{jk} + \frac{1}{2} h^{rs} [\delta_k h_{js} - \partial^m (\overset{c}{N}_{sk}) h_{jm} - \partial^m (\overset{c}{N}_{jk}) h_{sm}].$$

Since $N_{jk} = \gamma^r_{jk} p_r$ and

$$\overset{v}{F}{}^{(r)}_{(j)k} = \gamma^r_{jk} + \frac{1}{2}\frac{a^2}{H} g^{rs}(x)[\frac{H}{a^2}\partial_k g_{js} + \frac{1}{a^2} g_{js}\partial_k g^{ml} p_m p_l + \frac{2}{a^2}\gamma^m_{kl} p_m g_{js} g^{lm} p_m -$$

$$- \frac{H}{a^2}\gamma^m_{sk} g_{jm} - \frac{H}{a^2}\gamma^m_{jk} g_{sm}],$$

we obtain

$$\overset{v}{F}{}^{(r)}_{(j)k} = \frac{1}{2}\gamma^r_{jk} + \frac{1}{2} g^{rs}\partial_k g_{js} + \frac{1}{2H}\partial_k g^{ml} p_m p_l \delta^r_j + \frac{1}{2H} g^{ms} g^{li} p_m p_i \partial_k g_{ls} \delta^r_j +$$

$$+ \frac{1}{2H} g^{sm} g^{li} p_m p_i \partial_l g_{ks} \delta^r_j - \frac{1}{2H} g^{sm} g^{li} p_m p_i \partial_s g_{kl} \delta^r_j -$$

$$- \frac{1}{4} g^{rs}\partial_s g_{kj} - \frac{1}{4} g^{rs}\partial_k g_{sj} + \frac{1}{4} g^{rs}\partial_j g_{sk}.$$

But $g^{ms}\partial_k(g_{ls}) = -\partial_k(g^{ms}) g_{ls}$, and consequently

$$\overset{v}{F}{}^{(r)}_{(j)k} = \frac{1}{2}\gamma^r_{jk} + \frac{1}{4} g^{rs}(\partial_k g_{js} + \partial_j g_{sk} - \partial_s g_{kj}) + \frac{1}{2H}\partial_k g^{ml} p_m p_l \delta^r_j - \frac{1}{2H}\partial_k g^{ms} p_m p_s \delta^r_j -$$

$$- \frac{1}{2H} g^{li} g_{ks} p_m p_i \partial_l g^{sm} \delta^r_j + \frac{1}{2H} g^{lm} g_{ks} p_m p_i \partial_l g^{si} \delta^r_j = \frac{1}{2}\gamma^r_{jk} + \frac{1}{2}\gamma^r_{jk} = \gamma^r_{jk},$$

which ends the proof.

□

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