Euler - Savary's Formula on Minkowski Geometry

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

We consider a base curve, a rolling curve and a roulette on Minkowski plane and give the relation between the curvatures of these three curves. This formula is a generalization of the Euler - Savary's formula of Euclidean plane.

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1 Introduction

On the Euclidean plane $\mathbf{E}^2$, we consider two curves c_B and c_R. Let P be a point relative to c_R. When c_R rolles without splitting along c_B, the locus of the point P makes a curve, say c_L. On this set of curves, c_B, c_R c_L are called the base curve, rolling curve and roulette, respectively. For example, if c_B is a straight line, c_R is a quadratic curve and P is a focus of c_R, then c_L is the Delaunay curve that are used to study surfaces of revolution with the constant mean curvature.

Since this "rolling situation" makes up three curves, it is natural to ask questions: what is the relation between the curvatures of these curves, when given two curves, can we find the third one? Many geometers studied these questions and generalized the situation [3]. Today the relation of the curvatures is called as the Euler - Savary's formula.

However, the "rolling situation" on the Minkowski geometry is not studied yet. Only the Delaunay curve is considered to study surfaces of revolution with the constant mean curvature [1]. The purpose of this paper is to give answers to the abovementioned general questions on the Minkowski geometry. After the preliminaries of section 2, in section 3, we consider the associated curve that is the key concept to study the roulette, for, the roulette is one of associated curves of the base curve. Section 4 is devoted to give the Euler - Savary's formula on the Minkowski plane. In the final section, we determine the third curve from other two.

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2 Preliminaries

Let $\mathbf{L}^2$ be the Minkowski plane with metric $g = (+, -)$. A vector X of $\mathbf{L}^2$ is said to be spacelike if $g(X, X) > 0$ or $X = 0$, timelike if $g(X, X) < 0$ and null if $g(X, X) = 0$ and $X \neq 0$.

A curve c is a smooth mapping $c : I \to \mathbf{L}^2$ from an open interval I into $\mathbf{L}^2$. Let t be a parameter of c. By $c(t) = (x(t), y(t))$, we denote the orthogonal coordinate representation of $c(t)$. The vector field $\dfrac{dc}{dt} = \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}\right) =: X$ is called the tangent vector field of the curve $c(t)$. If the tangent vector field X of $c(t)$ is a spacelike, timelike, or null, then the curve $c(t)$ is called spacelike, timelike, or null, respectively.

In the rest of this paper, we mostly consider non-null curves. When the tangent vector field X is non-null, we can have the arc length parameter s and have the Frenet formula

$$\frac{dX}{ds} = kY, \quad \frac{dY}{ds} = kX, \tag{2.1}$$

where k is the curvature of $c(s)$ (cf. [2]). The vector field Y is called the normal vector field of the curve $c(s)$. Remark that we have the same representation of the Frenet formula regardless of whether the curve is spacelike or timelike.

If $\phi(s)$ is the slope angle of the curve, then we have $\frac{d\phi}{ds} = k$.

3 Associated curve

In this section, we give general formulas of the associated curve. Let $c(s)$ be a non-null curve with the arc length parameter s, and $\{X, Y\}$ the Frenet frame of $c(s)$.

If we put

$$c_A = c(s) + u_1(s)X + u_2(s)Y, \tag{3.1}$$

then $c_A(s)$ generally makes a curve. This curve is called the associated curve of $c(s)$.

Remark that $\{u_1(s), u_2(s)\}$ is a relative coordinate of $c_A(s)$ with respect to $\{c(s), X, Y\}$.

If we put

$$\frac{dc_A}{ds} = \frac{\delta u_1}{ds}X + \frac{\delta u_2}{ds}Y,$$

then, since

$$\frac{dc_A}{ds} = \frac{dc}{ds} + \frac{du_1}{ds}X + u_1\frac{dX}{ds} + \frac{du_2}{ds}Y + u_2\frac{dY}{ds} = \left(1 + \frac{du_1}{ds} + ku_2\right)X + \left(ku_1 + \frac{du_2}{ds}Y\right),$$

by virtue of (2.1), we have

$$\begin{aligned} \frac{\delta u_1}{ds} &= \frac{du_1}{ds} + ku_2 + 1, \\ \frac{\delta u_2}{ds} &= \frac{du_2}{ds} + ku_1. \end{aligned} \tag{3.2}$$

Let s_A be the arc length parameter of c_A. Then, from

$$\frac{dc_A}{ds} = \frac{dc_A}{ds_A}\frac{ds_A}{ds} = v_1 X + v_2 Y,$$

$$v_1 := \frac{du_1}{ds} + ku_2 + 1, \quad v_2 := \frac{du_2}{ds} + ku_1,$$

the Frenet frame $\{Z, W\}$ of c_A has following equations;

$$\begin{aligned} \frac{dZ}{ds_A} &= k_A W, \\ \frac{dW}{ds_A} &= k_A Z, \end{aligned} \tag{3.3}$$

where k_A is the curvature of c_A.

Let θ (resp. ω) be the slope angle of c (resp. c_A). Then

$$k_A = \frac{d\omega}{ds_A} = \frac{d\omega}{ds_A}\frac{ds}{ds_A} = \left(k + \frac{d\phi}{ds}\right)\frac{1}{\sqrt{|v_1^2 - v_2^2|}}, \tag{3.4}$$

where $\phi = \omega - \theta$.

If c_A is space-like, then we can put

$$\cosh\phi = \frac{v_1}{\sqrt{v_1^2 - v_2^2}},$$

$$\sinh\phi = \frac{v_2}{\sqrt{v_1^2 - v_2^2}}.$$

Since

$$\frac{d\phi}{ds} = \frac{d}{ds}\left(\cosh^{-1}\frac{v_1}{\sqrt{v_1^2 - v_2^2}}\right),$$

(3.4) reduces to

$$k_A = \left(k + \frac{v_1 v_2' - v_1' v_2}{v_1^2 - v_2^2}\right)\frac{1}{\sqrt{v_1^2 - v_2^2}},$$

where dash represents the derivative with respect to s.

If c_A is time-like, since $\sinh\phi = \dfrac{v_1}{\sqrt{v_2^2 - v_1^2}}$, we have

$$k_A = \left(k + \frac{v_1' v_2 - v_1 v_2'}{v_2^2 - v_1^2}\right)\frac{1}{\sqrt{v_2^2 - v_1^2}},$$

4 Euler - Savary's formula

In this section, we consider the roulette and give the Euler - Savary's formula.

Let c_B (resp. c_R) be the base (resp. rolling) curve and k_B (resp. k_R) the curvature of c_B (resp. c_R). Let P be a point relative to c_R. By c_L, we denote the roulette of the locus of P.

We can consider that c_L is an associated curve of c_B, then the relative coordinate $\{x, y\}$ of c_L with respect to c_B satisfies

$$\frac{\delta x}{ds_B} = \frac{dx}{ds_B} + k_B y + 1, \tag{4.1}$$
$$\frac{\delta y}{ds_B} = \frac{dy}{ds_B} + k_B x,$$

by virtue of (3.2).

Since c_R rolles without splitting along c_B, at each point of contact, we can consider $\{x, y\}$ is a relative coordinate of c_L with respect to c_R for a suitable parameter s_R. In this case, the associated curve is reduced to a point P. Hence it follows that

$$\frac{\delta x}{ds_R} = \frac{dx}{ds_R} + k_R y + 1 = 0, \tag{4.2}$$
$$\frac{\delta x}{ds_R} = \frac{dx}{ds_R} + k_R y = 0.$$

Substituting these equations into (4.1), we have

$$\frac{\delta x}{ds_B} = (k_B - k_R)y, \quad \frac{\delta y}{ds_B} = (k_B - k_R)x, \tag{4.3}$$

so

$$\frac{\delta x}{\delta y} = \frac{x}{y}. \tag{4.4}$$

Proposition 4.1 Let c_R rolles without splitting along c_B from the starting time $t = 0$. Then at each time $t = t_0$ of this motion, the normal at the point $c_L(t_0)$ passes through the point of contact $c_B(t_0) = c_R(t_0)$.

Suppose that c_L is spacelike. Then, from (4.3),

$$0 < \left(\frac{\delta x}{ds_B}\right)^2 - \left(\frac{\delta y}{ds_B}\right)^2 = (k_B - k_R)^2(y^2 - x^2). \tag{4.5}$$

Hence we can put

$$x = \sinh\phi, \qquad y = \cosh\phi.$$

Differentiating these equations, we have

$$\frac{dx}{ds_R} = \frac{dr}{ds_R}\sinh\phi + r\cosh\phi\frac{d\phi}{ds_R} = -1 - k_R r\cosh\phi,$$
$$\frac{dy}{ds_R} = \frac{dr}{ds_R}\cosh\phi + r\sinh\phi\frac{d\phi}{ds_R} = -k_R r\sinh\phi,$$

by virtue of (4.2). From these equations, it follows that

$$r\frac{d\phi}{ds_R} = -\cosh\phi - k_r r.$$

Therefore, substituting this equation into (3.4), we have

$$rk_L = \pm 1 - \frac{\cosh\phi}{r|k_B - k_R|}.$$

If c_L is timelike, by similar calculation, we have

$$rk_L = \pm 1 + \frac{\sinh\phi}{r|k_B - k_R|}.$$

We can easily see that the case c_L is null makes a contradiction.

Theorem 4.1 *On the Minkowski plane $\mathbf{L}^2$, suppose that a curve c_R rolles without splitting along a curve c_B. Let c_L be a locus of a point P that is relative to c_R. Let Q be a point on c_L and R a point of contact of c_B and c_R corresponds to Q relative to the rolling relation. By (r, ϕ), we denote a polar coordinate of Q with respect to the origin R and the base line $c'_B|_R$. Then curvatures k_B, k_R and k_L of c_B, c_R and c_L, respectively, satisfies*

$$rk_L = \pm 1 - \frac{\cosh\phi}{r|k_B - k_R|} \quad (\textit{when} \ \ c_L \ \ \textit{is space like}),$$

$$rk_L = \pm 1 + \frac{\sinh\phi}{r|k_B - k_R|} \quad (\textit{when} \ \ c_L \ \ \textit{is time like}).$$

5 Determining the curve

Since the roulette is a locus of a point, it is determined by the base curve and the rolling curve. In theis section, we consider the converse problem.

First suppose that a base curve c_B and a roulette c_L is given.

Let $(x(s_B), y(s_B))$ be the orthogonal coordinates of the base curve c_B with the arc length parameter s_B. For a point Q of c_B, draw the normal to the roulette c_L. Let R be the foot of this normal with the orthogonal coordinate $(f(s_B), g(s_B))$. Then the length of QR is

$$QR = \sqrt{|(f(s_B) - x(s_B))^2 - (g(s_B) - y(s_B))^2|}. \tag{5.1}$$

If we consider (5.1) on the rolling curve c_R, this equation represents the length of the point P relative to c_R and a point of c_R. Hence the orthogonal coordinate $(u(s_B), v(s_B))$ of c_R is given by the equations

$$u(s_B)^2 - v(s_B)^2 = (f(s_B) - x(s_B))^2 - (g(s_B) - y(s_B))^2,$$

$$\left(\frac{du}{ds_B}\right)^2 - \left(\frac{dv}{ds_B}\right)^2 = \pm 1,$$

the sign of ± 1 depends on spacelike or timelike of c_R.

Next suppose that a rolling curve c_R and a roulette c_L is given.

Let $(x(s_L), y(s_L))$ be the orthogonal coordinate of c_L with arclength parameter s_L. Suppose that the polar coordinate $r(s_R)$ of c_R is given by the arc length parameter s_R of c_R.

Since the normal of c_L is $\left(\dfrac{dy}{ds_L}, \dfrac{dx}{ds_L}\right)$, a point (u, v) of the base curve c_B is given by

$$u = x(s_L) \pm r(s_R)\frac{dy}{ds_L},$$
$$v = y(s_L) \pm r(s_R)\frac{dx}{ds_L}, \tag{5.2}$$

Then, from

$$\frac{du}{ds_R} = \frac{dx}{ds_L}\frac{ds_L}{ds_R} \pm \frac{dr}{ds_R}\frac{dy}{ds_L} \pm r\frac{d}{s_L}\left(\frac{dy}{ds_L}\right)\frac{ds_L}{ds_R},$$
$$\frac{dv}{ds_R} = \frac{dy}{ds_L}\frac{ds_L}{ds_R} \pm \frac{dr}{ds_R}\frac{dx}{ds_L} \pm r\frac{d}{s_L}\left(\frac{dx}{ds_L}\right)\frac{ds_L}{ds_R},$$

we have

$$\frac{du}{ds_R} = \frac{dx}{ds_L}(1 \pm rk_L)\frac{ds_L}{ds_R} \pm \frac{dr}{ds_R}\frac{dy}{ds_L},$$
$$\frac{dv}{ds_R} = \frac{dy}{ds_L}(1 \pm rk_L)\frac{ds_L}{ds_R} \pm \frac{dr}{ds_R}\frac{dx}{ds_L},$$

where k_L is the curvature of c_L.

Since s_R is also the arc length of c_B, it follows that

$$\left(\frac{du}{ds_R}\right)^2 - \left(\frac{dv}{ds_R}\right)^2 = \left(\frac{ds_L}{ds_R}\right)^2 (1 \pm rk_L)^2 - \frac{dr}{ds_R}^2 = \pm 1,$$

where the sign of ± 1 depends on spacelike or timelike of c_B. From this differential equation, we can solve $s_L = s_L(s_R)$. Substituting this equation into (5.2), we can have the orthogonal coordinate of c_B.

The solvability of these differential equations is easily checked. For example, we have solutions like that : c_B is x-axis, c_R is quadratic curve and c_L is "Delaunay curve" (cf. [1]).

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