Existence of Metrics with Harmonic Curvature and Non Parallel Ricci Tensor

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

A. Derdzinki [6] gave examples of Riemannian metrics with harmonic curvature and non parallel Ricci tensor on some compact manifolds $(M, g]$. We examine their existence as well as their number which naturally depends on the geometry of the manifolds.

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1 Introduction

Let (M, g) be a Riemannian manifold of dimension $n, n \geq 3$. M is said to have harmonic curvature if the divergence of its curvature tensor $\mathcal{R}$ vanishes (in local coordonates : $\nabla^i \mathcal{R}_{ijkl} = 0$).
That means the Ricci tensor r is a Codazzi tensor $(\nabla^i \mathcal{R}_{ijkl} = \nabla_k r_{hj} - \nabla_j r_{hk} = 0)$. In other words, in the compact case of the manifold the Riemannian connection is a Yang-Mills potential in the tangent bundle.

Answering the question on the parallelism of the Ricci tensor of the Riemannian metrics, A. Derdzinski gave examples of compact manifolds with harmonic curvature but non parallel Ricci tensor: $\delta\mathcal{R} = 0$ and $\nabla r \neq 0$. Moreover, he obtains some classification results, [6].
The corresponding manifolds are bundles with fibres N over the circle S^1 (parametrized by arc length t and length $T = \int_{S^1} dt$) equipped with the warped metrics $dt^2 + h^{4/n}(t) g_0$ on the product $S^1 \times N$.
Here, (N, g_0) is an Einstein manifold of dimension $n-1, n \geq 3$, with scalar curvature R and the function $h(t)$ on the prime factor is a periodic solution of the ODE, established by Derdzinski

$$(1) \qquad h'' - \frac{nR}{4(n-1)} h^{1-4/n} = -\frac{n}{4} Ch \qquad \text{for some constant} \qquad C > 0.$$

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This function must be non constant, otherwise the corresponding metric has a parallel Ricci tensor.

The goal of this paper is to study the existence and the number of such metrics which naturally must depend on the geometry of S^1 and N.
More precisely, consider the change

$$h(t) = \alpha j(t)$$

where the constant $\alpha = \left(\frac{R}{4(n-1)C}\right)^{n/4}$.
Equation (1) becomes

$$(1') \qquad j'' - \frac{nC}{4}(1+j)^{1-4/n} = -\frac{n}{4}C(1+j)$$

$$(1') \qquad j'' - \frac{nC}{4}(1+(1-4/n)j+(-2/n)j^2+....)+\frac{n}{4}C(1+j) = 0$$

$$(1') \qquad j'' + Cj + ... = 0$$

When h is closed to α , j is closed to 0.
That means equation (1') bifurcates at $j \equiv 0$ when $C = (\frac{2\pi}{T})^2$. So equation (1) also bifurcates at $h \equiv \alpha$ when $C = (\frac{2\pi}{T})^2$.
In particular, there is a positive bound T_0 such that when $T \leq T_0$ the above equation may have only constant solutions, i.e. $h(t) \equiv \alpha = \left(\frac{R}{4(n-1)C}\right)^{n/4}$.

We prove the following
Theorem. *Let us consider the Riemannian product $(S^1 \times N, dt^2+g_0)$ where (S^1, dt^2) is a circle with length T and (N, g_0) is an Einstein manifold of dimension $n-1, n \geq 3$ with positive scalar curvature R.*
There exists a constant T_0 such that if $T \leq T_0$ this manifold does not admit warped metric $dt^2 + h^{4/n}(t)g_0$ whose Ricci tensor is not parallel.

2 Metrics with harmonic curvature

Let (M, g) be a Riemannian manifold with $n = dimM \geq 3$. $\mathcal{R}$ is its curvature tensor, r its Ricci tensor W, its Weyl conformal tensor and R its scalar curvature. According to the second Bianchi identity $d\mathcal{R} = 0$ (in local coordonates $\nabla_q \mathcal{R}_{ijkl} + \nabla_i \mathcal{R}_{jqkl} + \nabla_j \mathcal{R}_{qikl} = 0$,) we get the following relations

$$\delta\mathcal{R} = -dr \qquad i.e. \quad \nabla^i \mathcal{R}_{ijkl} = \nabla_k r_{hj} - \nabla_j r_{hk}$$

$$(n-2)\delta W = -(n-3)d[r - \frac{Rg}{2n-2}] \qquad and \quad 2\delta r = -dR.$$

The sign conventions are such that $r_{ij} = \mathcal{R}^l_{ilj}, \quad R = g^{ij} r_{ij}$.
d is the exterior differentiation and δ is its formal adjoint, viewed as differential forms on M.
(M, g) has harmonic curvature if $\delta\mathcal{R} = 0$.

A Codazzi tensor C on (M,g) is a symmetric $(0,2)$ tensor field on M verifying the Codazzi equation $dC=0$ *i.e.* $\nabla_j C_{ik} = \nabla_k C_{ij}$.
Classical properties of metrics with harmonic curvature are summarized in the following lemma, [1], [8]
Lemma 1. *Let (M,g) be a Riemannian manifold of dimension $n \geq 3$. The following holds*
(1) if $n=3$, (M,g) has harmonic curvature $\delta\mathcal{R}=0$ if and only if it is conformally flat ($W \equiv 0$) and has constant scalar curvature $R = Cte$.
(2) If $n \geq 4$, $\delta W = 0$ and M has constant scalar curvature, then $\delta\mathcal{R} = 0$.
(3) (M,g) has harmonic curvature if and only if its Ricci tensor is a Codazzi tensor (i.e. $dr=0$).
(4) If (M,g) is a Riemannian product, then it has harmonic curvature if and only if any factor manifolds has harmonic curvature.

More generally, Derdzinski established a classification of the compact n-dimensional Riemannian manifolds (M_n, g), $n \geq 3$, with harmonic curvature. If the Ricci tensor $Ric(g)$ is not parallel and has less than three distinct eigenvalues at each point, then (M,g) is covered isometrically by a manifold

$$(S^1(T) \times N, dt^2 + h^{4/n}(t) g_0),$$

where the non constant positive periodic solutions h verify the equation (1). Here (N, g_0) is a (n-1)- dimensional Einstein manifold with positive (constant) scalar curvature.

3 On the existence of Derdzinski metrics

3.1 Analysis of the Derdzinski equation

The function h defined by the above warped product $(S^1(T) \times N, dt^2 + h^{4/n}(t) g_0)$ is a solution of

$$(1) \qquad h'' - \frac{nR}{4(n-1)} h^{1-4/n} = -\frac{n}{4} Ch \qquad \text{for some constant} \qquad C > 0$$

so that its curvature is harmonic and its Ricci tensor is non parallel $Dr \neq 0$.
In fact, this may be deduced from the following result (Lemma 1 in [6]).
Lemma 2. *Let I be an interval of R, and q a C^∞ function on I such that $e^q = h^{4/n}$. Let (N, g_0) be an $(n-1)$ dimensional Riemannian manifold, r_0 its Ricci tensor, and R its scalar curvature with $n \geq 3$. Consider the warped product $I \times_{e^q} N, dt^2 + h^{4/n}(t) g_0)$ and r its Ricci tensor.*
(1) Given a local product chart $t = x^0, x^1, x^2, \ldots.., x^{n-1}$ for $I \times N$ with $g_{00} = 1, g_{0i} = 0$ and $g_{ij} = e^q g_{0_{ij}}$. The components of the covariant derivative of r are

$$\nabla_0 r_{00} = -\frac{n-1}{2}[q''' + q'q''], \quad \nabla_0 r_{i0} = \nabla_i r_{00} = 0,$$

$$\nabla_0 r_{ij} = -q' r_{0_{ij}} - \frac{1}{2} e^q [q''' + (n-1) q'q''] g_{0_{ij}}, \quad \nabla_i r_{0j} = -\frac{1}{2} q' r_{0_{ij}} - \frac{n-2}{4} e^q q'q'' g_{0_{ij}}.$$

(2) If q is non constant, then the product $I \times_{e^q} N, dt^2 + h^{4/n}(t)g_0)$ has harmonic curvature if and only if (N, g_0) is an Einstein manifold and the positive function h satisfies the ODE (1) on I .

Notice that the warped product manifold $(R \times M', dt^2 + h^{4/n}(t)g')$ is complete analytic. It thus appears from Lemma 2 that condition $\nabla r = 0$ is equivalent to $q''' + q'q'' = 0$ and $(n-1)(n-2)q''e^q + 2R = 0$.

Moreover, [6] proved (his Theorem 1) that equation (1) possesses at least one non constant positive periodic solution if and only if $R > 0$ and $-\frac{n}{4}p = C > 0$.

But, some care needs to be taken on the subject of the constant and the existence of a non trivial solution. This constant C must verify some additional condition so that the conclusion holds.
More precisely, as we shall see below, if the constant C verifies

$$0 < C \leq \frac{4\pi^2}{T^2}$$

then there is no non constant periodic (of period T) solution of (1) .

3.2 Existence conditions of the Derdzinski metrics

More precisely, considering the ODE point of view, we are able to analyse all the solutions of equation (1), which may have a non constant, positive periodic solution. We shall prove the following
Theorem 1. *Consider the warped product $(S^1(T) \times N, dt^2 + h^{4/n}(t)g_0)$. The non constant positive periodic solution $h(t)$ satisfies Equation (1) and (N, g_0) is a (n-1)-dimensional Einstein manifold with positive (constant) scalar curvature R . Then, if the circle length T satisfies the following inequalities*

$$2\pi\frac{(k-1)}{\sqrt{C}} < T \leq \frac{k}{\sqrt{C}} , \qquad \textit{where } k \textit{ is an integer} > 1,$$

*there exist at least k rotationnally invariant warped metrics $dt^2 + h^{4/n}(t)g_0$ on the product manifold $S^1(T) \times N$.
Moreover, these metrics have an harmonic curvature. Their Ricci tensor are non parallel only if $T > \frac{2\pi}{\sqrt{C}}$.
Conversely, if the warped metric of the type $dt^2 + h^{4/n}(t)g_0$ on the manifold $S^1(T) \times N$ has harmonic curvature, then the function $h(t)$ on the circle satisfies the ODE (1).*

Condition $T > \frac{2\pi}{\sqrt{C}}$ implies the existence of a non constant solution $h(t)$. Otherwise, a trivial product has a parallel Ricci tensor.

3.3 Proof of Theorem 1

All periodic orbits $\gamma_c(t)$ of the following system equivalent to Equation (1)

$$(2) \qquad \begin{cases} x' = -y \\ y' = \frac{nR}{4(n-1)}x^{1-\frac{4}{n}} - \frac{nC}{4}x, \end{cases}$$

are surrounded by the homoclinic orbit γ_{c_0} . The last one may be parametrized by $(u_0(t), v_0(t))$.
Denote the coordinates of $\gamma_c(t)$ by $(u_c(t), v_c(t)$. When the value c satisfies the condition $1 < c < c_0$, the correspondant orbit is periodic (c_0 corresponds to a periodic solution of null energy) .
The center of system (2) is $(\alpha, 0)$, where $\alpha = \left(\frac{R}{4(n-1)C}\right)^{4/n}$.
One may easily remark that two positive T-periodic solutions of (1) having the same energy are translated, and thus give rise to equivalent metrics on $(S^1(T) \times N), g_0)$.
Note that the metric corresponding to the conformal factor u_0 : $g = u_0{}^{\frac{4}{n-2}} g_0$ is non complete. Then, it is not a pseudo-cylindric metric; for this reason the constant c cannot attain the critical value c_0 .
Equation (1) may be written under the following form

$$x'' + \phi(x) = 0 \tag{3}$$

where

$$\phi(x) = \frac{n}{4}C(x-\alpha) - \frac{nR}{4(n-1)}(x-\alpha)^{1-4/n}.$$

The period of the periodic solutions depends on the energy $T \equiv T(c)$ with c the energy constant. It can be expressed by

$$T(c) = \sqrt{2}\int_a^b \frac{du}{\sqrt{c - G(u)}}$$

where $G(u)$ is an integral of $\phi(u)$, with a nondegenerate relative minimum at the origin. It verifies in addition $G(a) = G(b) = c$ and $a \le \alpha \le b$.
So, $\phi(\alpha) = 0$ and $\phi'(\alpha) = \frac{n}{4}C > 0$. Hence, the origin is a center of Equation (3). That means in the neighbourhood of the trivial solution $h(t) \equiv \alpha$ Equation (1) admits a periodic solution.
We need the following result
Lemma 3. *Under the above hypothesis the family of solutions $(T, u_T(t))$ of the ODE (3) (where T is the minimal period) has bifurcation points on the values $(T_k, u_{Tk}(t))$ where $T_k = \frac{2\pi k}{\sqrt{n-2}}$ and $u_{T_k} \equiv \alpha$ is a constant. In this family, there is a curve of non trivial solutions which bifurcates to the right of the trivial one.*

This lemma is a classical result of global bifurcation theory (for details see for example [C-R]). Indeed, let us consider a positive T-periodic solution: if $T \neq T_k$, then the linearized associate equation is non-singular.
We may also deduce from bifurcation theorem, applied to the simple eigenvalues problem, that there is an unique curve of non trivial solutions near the point (T_k, α). In fact, this uniqueness is global. The trivial curve is $u_T \equiv \alpha$.
Moreover, $(\frac{du}{dT})_{T=T_k}$ is an eigenvalue of the linearized associate equation. According to the global bifurcation theory, we assert that the non trivial curves turn off on the right of the singular solution $(T_k, u_{T_k}(t))$. Consequently, when T varies, two non trivial curves never cross.

To prove Theorem 1, we also need the following.

Lemma 4 . *The minimal period* $T(c)$ *of a (periodic, positive) solution* u_T *of the equation (2) is a monotone increasing function of its energy, when* $c \in [0, c_0]$

Chow-Wang [4] also have calculated the derivative of the period function $T(c)$ for equation (2) and have found the expression of the derivative

$$T'(c) = \frac{1}{c}\int_a^b \frac{\phi^2(w) - 2G(w)\phi'(w)}{\phi^2(w)\sqrt{c - G(w)}} dw.$$

Recall $G(w)$ is the integral of ϕ verifying $G(a) = G(b) = c$.
Consider now a function ψ defined by

$$\phi(x) = \psi(\frac{x}{\alpha})$$

Lemma 4 is in fact a consequence of the following (see corollary (2-5) in [4]) applied to the function ψ .

Lemma 5 . *Consider a smooth function* ψ *such that* $\psi(1) = 0$, $\psi'(1) > 0$. *Suppose that*

$$H(x) = \psi^2(x) - 2G(x)\psi'(x) + \frac{\psi''(1)}{3\psi'^2(1)}\psi^3(x) > 0,$$

for all $x \in [a, b]$, *where* $a < 1 < b$ *and* $x \neq 1$. *Then,* $T'(c) \geq 0$ *for all* $c \in [0, c_0]$.
Moreover, if in addition

$$\psi''(x) \geq 0 \quad \text{and} \quad \Delta(x) = (x - 1)\Big[\psi'(x)\psi''(1) - \psi'(1)\psi''(x)\Big] \geq 0$$

then $H(x) > 0$.

Notice that Lemma 4 implies Lemma 3 (see corollary (3-1) in [4]).

In order to apply the preceding lemma it is more convenient to make a change of variables in equation (1) $h(t) = \alpha f(\beta t)$, where

$$\alpha = \left(\frac{R}{4(n-1)C}\right)^{4/n} \qquad \text{and} \qquad \beta = \sqrt{\frac{nC}{4}}.$$

Note that the constant C must be positive to ensure that equation (1) has a periodic non constant solution. This change gives the equation

$$f'' - f^{1-4/n} + f = 0. \tag{4}$$

We can verify that this equation satisfies Lemmas 2 and 3 given above. They will be used to complete the analysis of the equation (1).
Indeed, we get the functions

$$g(f) = f - f^{1-\frac{4}{n}} \quad ; \quad g'(f) = 1 - (1 - \frac{4}{n})f^{-\frac{4}{n}}$$

$$\text{and} \qquad g''(f) = \frac{4}{n}(1-\frac{4}{n})f^{-1-\frac{4}{n}}.$$

Notice that the hypothesis $g''(f) \geq 0$ is only satisfied if $n \geq 4$.
Then, we calculate

$$\Delta(f) = (f-1)[g''(1)g'(f) - g'(1)g''(f)].$$

We get

$$\Delta(f) = (f-1)\frac{4}{n}(1-\frac{4}{n})[1 - f^{\frac{4}{n}} + \frac{4}{n}(f^{-\frac{4}{n}} - f^{-1-\frac{4}{n}})].$$

It follows

$$\Delta(f) = \frac{4}{n}(1-\frac{4}{n})f(1-f^{-1})[1 - f^{\frac{4}{n}} + \frac{4}{n}f^{-\frac{4}{n}}(1-f^{-1})],$$

which is obviously positive.
Concerning the case $n \leq 4$. Remark that for $n = 4$ Equation (1) becomes linear (that is the trivial case). For $n = 3$, only the following implications can be made

$$g''(x) < 0 \Rightarrow -2G(x)g''(x) \geq 0 \Rightarrow H(x) > 0$$

We are now able to complete the proof of Theorem 1.
First, we deduce the increase of the period function depending on the energy of the equation (1). This fact allows us to determine the lower bound of the number of these Derdzinski metrics. We also remark, that the bifurcation points of the solution family are

$$(T_k, u_k) \quad , \ where \quad u_k = (\frac{(n-1)C}{nR})^{-n/4} \quad and \quad T_k = \frac{2\pi k}{\sqrt{C}}.$$

It appears from the above analysis that, a non constant, periodic solution of the equation (1) exists only if the circle length satisfies the following condition

$$T > \frac{2\pi}{\sqrt{C}}. \tag{5}$$

Notice that, we get an infinity of solutions . All are obtained by rotation-translation of the variable.
Moreover, in the case where T satisfies the double inequality

$$2\pi\frac{(k-1)}{\sqrt{C}} < T \leq 2\pi\frac{k}{\sqrt{C}} \quad k \ is \ an \ integer \ > 1, \tag{6}$$

then the equation (2) may admit at least k rotationnally invariant distinct solutions. Hence, we have improved the lemma 1 of Derdzinski [6] (see also Chapter 16.33 of A. Besse [1])

4 Pseudo-cylindric and Derdzinski metrics

Notice that the Riemannian product $(S^1(T) \times N, dt^2 + h^{4/n}(t)g_0)$ is conformally flat only if the factor N has constant sectional curvature.

Let the Riemannian cylindric product $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$, where S^1 is the circle of length T and $(S^{n-1}, d\xi^2)$ is the standard sphere. Such a metric has a parallel Ricci tensor. Moreover, we know that the number of Yamabe metrics is finite in the conformal class of the cylindric metric $[dt^2 + d\xi^2]$, (see [2]).
We call *pseudo-cylindric metric* any non trivial Yamabe metric g_c in $[dt^2 + d\xi^2]$. (g_c is a Yamabe metric on a n-dimensional Riemannian manifold (M, g) if there is a C^∞ positive solution u_c of a differential equation such that the metric $g_c = u_c^{\frac{4}{n-2}} g$ has a constant scalar curvature).
For $k = 2$, there is a conformal diffeomorphism between $S^n - \{p_1, p_2\}$ and $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$, where S^1 is the circle of length T. The non trivial Yamabe metrics on $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$, are called pseudo-cylindric metrics. There are metrics of the form $g = u^{\frac{4}{n-1}}(dt^2 + d\xi^2)$ where the C^∞ function u is a non constant positive solution of the Yamabe equation, [2].

We first remark there is a conformal diffeomorphism between the manifolds $\mathbf{R}^n \setminus \{0\}$ and $\mathbf{R} \times S^{n-1}$ given by sending the point x to $(\log |x|, \frac{x}{|x|})$. By using the stereographic projection, we see easily that the manifold $\mathbf{R} \times S^{n-1}$ (which is the universal covering space of $S^1 \times S^{n-1}$) is conformally equivalent to $S^n \setminus \{0, \infty\}$.
Notice that the manifold $S^n \setminus (p, -p)$ with the standard induced metric, can be considered as the warped product

$$]0, \pi[\times S^{n-1}, \quad \text{with allowed metric} \quad dt^2 + \sin^2 t d\xi^2.$$

It has been shown (using an Alexandrov reflection argument) that any solution of

$$4\frac{n-1}{n-2}\Delta_{g_0} u + R_{g_0} u - R_g u^{\frac{n+2}{n-2}} = 0, \tag{7}$$

is in fact a spherically symmetric radial function (depending on geodesic distance from either p or $-p$). Any solution of Equation (6) which gives a complete metric on the cylinder $\Re \times S^{n-1}$ is of the form $u(t, \xi) = u(t)$, where $t \in \mathbf{R}$ and $\xi \in S^{n-1}$. The background metric on the cylinder is the product $g_0 = dt^2 + d\xi^2$. For convenience, we assume the sphere radius equal to 1.
Therefore, the partial differential equation (6) is reduced to an ODE.
The cylinder has scalar curvature $R(g_0) = (n-1)(n-2)$ and $R(u^{\frac{4}{n-2}} g_0) = n(n-1)$. Thus $u = u(t)$ satisfies

$$\frac{d^2}{dt^2}u - \frac{(n-2)^2}{4}u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0. \tag{8}$$

It follows that a pseudo-cylindric metric (constant scalar curvature metric) on the product $(S^1(T) \times S^{n-1}, g_0)$ corresponds to a T-periodic positive solution of (8) and conversely. The analysis of this equation shows us, that it has only one center $(\beta, 0)$. This corresponding to the (trivial) constant solution

$$\beta = (\frac{n-2}{n})^{\frac{n-2}{4}},$$

We proved the following, [3]

Proposition . *Consider the product manifold* $(S^1(T) \times S^{n-1}, g_0)$. *Under the condition*

$$T(c) > T_1 = \frac{2\pi}{\sqrt{n-2}},$$

on the circle length, the Riemannian curvatures of the associate pseudo-cylindric metrics $g_c = {u_c}^{\frac{4}{n-2}} g_0$ *are harmonic and their Ricci tensors are non parallel.*
Moreover, any pseudo-cylindric metric may be identified to a Derdzinski metric up to a conformal transformation.

Actually, any Derdzinski metric may be identified with a pseudo-cylindric metric up to conformal diffeomorphism, let F. Then the metrics are related

$$dt^2 + f^2(t)d\xi^2 = F^*({u^j}_c^{\frac{4}{n-2}}(dt^2 + d\xi^2)),$$

where ${u^j}_c$ are the pseudo-cylindric solutions belonging to the (same) conformal class. Indeed, for the metric $dt^2 + f^2(t)d\xi^2$ we can write

$$dt^2 + f^2(t)d\xi^2 = f^2(t)[(\frac{dt}{f})^2 + d\xi^2].$$

After a change of variables and by using the conformal flatness of the product metric, we get

$$dt^2 + f^2(t)g_0 = \phi^2(\theta)[d\theta^2 + d\xi^2]$$

which is conformally flat.
To see that , it suffices to remark that any manifold carying a warped metric product $(S^1 \times N, dt^2 + h^{4/n}(t)g_0)$ with harmonic curvature is not conformally flat unless (N, g_0) is a space of constant curvature. This manifold must be locally conformally equivalent to the trivial product $S^1 \times N$.
This product will be conformally flat only if N has constant sectional curvature.

On the other hand, we remark that any warped metric $dt^2 + h^{4/n}(t)g_0$ defined by Lemma 2 on the product manifold $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$ is conformal to a Riemannian metric product $d\theta^2 + d\xi^2$. Here θ is a new S^1-parametrisation with length $\int_{S^1} \frac{dt}{h^{2/n}(t)}$. Furthermore, we have seen in a previous paper ([2]), there exists a analytic deformation in the conformal class $[g_T]$, of any warped metric $g_T = dt^2 + f^2(t)d\xi^2$ on the manifold $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$, $n = 4$ *or* 6 , and satisfying the length condition $T = \int_{S^1} \frac{dt}{f(t)}$. Notice that the product metric (under the length condition) belongs to the conformal class $[g_T]$. This analytic family of metrics depends on two parameters $(g_{\alpha,\beta})$. They all have a constant positive scalar curvature, and satisfy the condition: $g_{\alpha,0}$ is the warped metric g_T.

Moreover, by using the same argument in [2], we are able to extend the latter result to dimension 3. More precisely, a metric on the euclidean space $\mathbf{R}^3$ for which the rotation group $SO(3)$ acts by isometries, is in fact a warped product as $g_T = dt^2 + f^2(t)d\xi^2$, where $d\xi^2$ is the standard metric on the sphere S^2. We can verify that its scalar curvature is

$$R = \frac{2 - 2f'^2 - 4ff''}{f^2} .$$

Moreover, we know that every conformally flat manifold (M,g) admits a Codazzi tensor which is not a constant multiple of the metric. Let b be such a symmetric 2-tensor field; suppose b has exactly 2 distinct eigenvalues λ , μ , constant trace $tr_g(b) = c$ and is non parallel. Following [6], these conditions give locally

$$M = I \times N \quad \text{with allowed metric} \quad g = dt^2 + e^{2\psi} g_N \tag{9}$$

and

$$b = \lambda dt^2 + \mu e^{2\psi} g_N, \quad \lambda = \frac{c}{n} + (1-n)ce^{-n\psi}, \quad \mu = \frac{c}{n} + ce^{-n\psi}. \tag{10}$$

Conversely, for any such data and for an arbitrary function ψ on I , (8) defines a Riemannian manifold (M,g) with Codazzi tensor b of this type. If we assume that the positive function $h = e^{\frac{n}{2}\psi}$ satisfies Equation (1)

$$h'' - \frac{nR}{4(n-1)} h^{1-4/n} = -\frac{n}{4} Ch \qquad \text{for some constant} \qquad C > 0,$$

then the Ricci tensor is precisely a Codazzi tensor and, thus it is non parallel. Therefore, (M,g) is isometrically covered by $(R \times N, dt^2 + h^{4/n}(t) g_N)$ where (N, g_N) is an Einstein manifold with positive scalar curvature (see [1] , 16.33).

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