Combinatorics of Mumford-Morita-Miller Classes in Low Genus

Gilberto Bini

Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

Here we use elementary combinatorial arguments to give explicit formulae and relations for some cohomology classes of moduli spaces of stable curves of low genus.

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1 Introduction

Let g and n be non-negative integers such that $2g-2+n>0$. We denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed genus g curves. Its points are in one-to-one correspondence with isomorphism classes of pointed curves of arithmetic genus g curves with simple nodes and finitely many automorphisms. $\overline{\mathcal{M}}_{g,n}$ is a normal projective variety of complex dimension $3g-3+n$. It can be viewed as a complex-analytic orbifold, in fact as a quotient of a smooth complete variety by a finite group (see [3], [12]). Further details on the properties of $\overline{\mathcal{M}}_{g,n}$ can be found, for instance, in [7], [13]. More generally, if P is a set with n elements, we will denote by $\overline{\mathcal{M}}_{g,P}$ the space whose elements are stable genus g curves with marked points indexed by P.

The fascinating geometry of $\overline{\mathcal{M}}_{g,n}$ has been only partially understood. In many instances, combinatorial arguments have been essential to prove various results in a natural way (cfr. [6], [11]). However, little emphasis has been given to the development of the combinatorics involved with moduli of curves, especially with its cohomology.

Here we obtain an explicit description for cohomology classes (in fact, algebraic) of fundamental importance in all genera. In Section 3.2, we also describe such classes in genus zero via the theory of hyperplane arrangements.

2 Preliminaries

For any g and P, $|P|=n$, in the range above, the collection of all moduli spaces is naturally equipped with some relevant maps. We briefly recall their definition since they will be used in what follows: for more details see, for example, [2].

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First of all, consider the projection

$$\pi : \overline{\mathcal{M}}_{g,P\bigcup\{q\}} \to \overline{\mathcal{M}}_{g,P}, \tag{1}$$

which forgets the point q on any $(n+1)$-pointed curve in the domain and contracts unstable components, i.e., without finitely many automorphisms. We denote by σ_p, $p \in P$, the canonical section of π and by D_p the corresponding divisor in $\overline{\mathcal{M}}_{g,P\cup\{q\}}$. The relative dualizing sheaf ω_π of the map (1) yields the cohomology classes:

$$\psi_p = c_1(\sigma_p^*(\omega_\pi)),\ p \in P,$$

$$K = c_1(\omega_\pi(\sum_{p\in P} D_p)).$$

The ψ_p's are usually called *universal cotangent classes*. Following [1], the *Mumford classes* are defined to be

$$\kappa_m = \pi_*(K^{m+1}). \tag{2}$$

Note that the push-forward in the last formula is well defined since the Poincaré duality with rational coefficients holds for orbifolds. For $P = \emptyset$, the analogue of (2) was first introduced in [13]. Another generalization to the case of n-pointed curves is given by

$$\widetilde{\kappa}_m = \pi_*(c_1(\omega_\pi)^{m+1}).$$

As shown in [2], the following relations hold:

$$\kappa_m = \widetilde{\kappa}_m + \sum_{p\in P} \psi_p^m, \tag{3}$$

$$\pi_*(\psi_q^{m+1}) = \kappa_m. \tag{4}$$

The set of the ψ_p's, κ_m's, and $\widetilde{\kappa}_m$'s is called the set of *Mumford-Morita-Miller* classes.

In addition to (1), further morphisms between moduli spaces of curves are defined via the collection of *stable graphs*: see, for example, [2] for their definition and properties. Here we just observe how to associate with them cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$. With the same notation adopted in [2], for any stable graph G, choose an ordering of the $l(v)$ half-edges of G going out of each vertex v. Then consider the morphisms

$$\xi_G : \prod_{v\in V} \overline{\mathcal{M}}_{g(v),l(v)} \to \overline{\mathcal{M}}_{g,P}, \tag{5}$$

where the $g(v)$'s are non-negative integers which label the vertices of G. A point in the domain of ξ_G is the datum of an $l(v)$-pointed curve C_v for each v in the set of vertices V of G. The image point is the P-labelled genus g curve that is obtained by identifying the marked points of C_v which correspond to half-edges of G linked by an edge. By definition, the map ξ_G does not depend on the ordering chosen for the half-edges going out of each vertex. By properties of stable graphs, we have

(6) $$g = \sum_{v\in V} g(v) + 1 - |V| + \frac{1}{2}\sum_{v\in V}(l(v) - |P|).$$

For the purpose of what follows, we finally recall that for any stable graph G with at least one edge the corresponding *boundary class* is given by

$$\frac{1}{|Aut(G)|}\xi_{G,*}(1),$$

where $Aut(G)$ is the automorphism group of G. These classes are usually referred to as *boundary classes* since their Poincaré dual is supported on $\overline{\mathcal{M}}_{g,P} - \mathcal{M}_{g,P}$, i.e., the boundary of the moduli space of smooth P-labelled curves.

3 Relations among Mumford-Morita-Miller classes

In this section we show some explicit formulae for the classes ψ_p's, κ_m's in terms of boundary classes. We basically extend previous work in [8] and [9].

3.1 The genus zero case

The structure of the cohomology ring of $\overline{\mathcal{M}}_{0,n}$ has been determined in [10]. For each subset $S \subseteq \{1,...,n\}$, with $|S| \geq 2$ and $|S^c| \geq 2$, let $\delta_{0,S}$ be the cohomology class dual to the divisor of genus 0 curves with one node and two components with $|S|$ and $|S^c|$ marked points, respectively. The cohomology ring of $\overline{\mathcal{M}}_{0,n}$ is a quotient of the free $\mathbf{Z}$-module

$$\mathbf{Z}[\delta_{0,S} : S \subseteq \{1,...,n\}, |S|, |S^c| \geq 2].$$

This means that any class can be written as a linear combination of monomials in the classes $\delta_{0,S}$'s. Here we show how to use combinatorial arguments to obtain explicit expressions for Mumford-Morita-Miller classes in any codimension.

Take P to be a set with n-elements and consider the moduli space $\overline{\mathcal{M}}_{0,P}$. For the ψ_p's we recall that (see [2])

(7) $$\psi_p = \sum_{\substack{q_1,q_2 \notin S,\\ p\in S}} \delta_{0,S},$$

where p, q_1, q_2 are arbitrary elements in P.

Let $\mathcal{A}_m(n)$ denote the collection of unordered m-ples $(A_1, ..., A_m)$ of subsets $A_j \subset P$ such that the following conditions are satisfied:

- for each A_k in $(A_1, ..., A_m)$, $|A_k| \geq 2$, $|A_k^c| \geq 2$ and $p, q \notin A_k$ for any pair $p, q \in P$;
- for each $k \in \{1, ..., m\}$ and each choice of $p, q \in P$, A_k is not contained in any subset $\mathcal{S}$ of $P \backslash \{p, q\}$, with $|\mathcal{S}| = m$;
- for each pair A_k, A_l of an m-ple, one of the following conditions

$$A_k \subset A_l,\ A_l \subset A_k,\ A_k \subset A_l^c,\ A_l^c \subset A_k$$

is satisfied.

For each such m-ple we define the following coefficient

$$c(A_1,...,A_m) = \begin{cases} |A_1 \cap ... \cap A_m| + (-1)^m, & A_k \subseteq P \setminus \mathcal{S}, \\ |A_1 \cap ... \cap A_m| & \text{otherwise.} \end{cases}$$

Theorem 1 *In $H^*(\overline{\mathcal{M}}_{0,n}; \mathbf{Q})$, for each m, $1 \le m \le n-3$, we have*
i)

$$\psi_p^m = \sum_{\substack{p \in A_i,\, q_1,q_2 \notin A_i \\ A_1,...,A_m}} \delta_{0,A_1} \dots \delta_{0,A_m},$$

ii)

$$\kappa_m = \sum_{(A_1,...,A_m) \in \mathcal{A}_m(n)} c(A_1,...,A_m) \delta_{0,A_1} ... \delta_{0,A_m}.$$

Proof. i) The claim follows clearly from (7).

ii) Let $\pi_s : \overline{\mathcal{M}}_{0,P \cup \{s\}} \to \overline{\mathcal{M}}_{0,P}$ be the forgetful map. We prove the result by induction on $|P| = n$. The base of the induction follows from the fact that $\kappa_m = 0$ on $\overline{\mathcal{M}}_{0,m+2}$ by dimensional calculations. Thus, we can assume $n \ge m+2$. Since (see [2])

$$\kappa_m = \pi_s^*(\kappa_m) + \psi_q^m,$$

by induction hypothesis and the relation which expresses ψ_q in terms of boundary classes (see [1]),

$$\kappa_m = \sum_{(A_1,...,A_m) \in \mathcal{A}_m(n)} c(A_1,...,A_m) \pi_q^*(\delta_{0,A_1} ... \delta_{0,A_m}) + (\sum_{\substack{s \in B, \\ p,q \notin B}} \delta_{0,B})^m =$$

$$\sum_{(A_1,...,A_m) \in \mathcal{A}_m(n)} c(A_1,...,A_m) \pi_q^*(\delta_{0,A_1} ... \delta_{0,A_m}) + \sum_{\substack{A_1,...,A_m, \\ p,q \notin A_k, s \in A_k}} \delta_{0,A_1} ... \delta_{0,A_m}. \tag{8}$$

The claim is proved if we show that the sum in (8) can be rewritten as

$$\sum_{(A'_1,...,A'_m) \in \mathcal{A}_m(n+1)} c(A'_1,...,A'_m) \delta_{0,A'_1} ... \delta_{0,A'_m}.$$

In fact, since (see [2])

$$\pi_s^*(\delta_{0,A_l}) = \delta_{0,A_l} + \delta_{0,A_l \cup \{s\}},$$

we just distinguish two cases:

1. $s \in A'_k$, $\forall k \in \{1,...,m\}$;
2. $s \notin A'_1 \cap ... \cap A'_m$.

In case 1 we can assume that $A'_k = A_k \cup \{s\}$. Therefore, by direct computations we have

$$\begin{aligned} c(A'_1, ..., A'_m) &= c(A_1, ..., A_m) + 1 = |A_1 \cap ... \cap A_m| + (-1)^m + 1 \\ &= |A'_1 \cap ... \cap A'_m| + 1, \end{aligned}$$

or

$$c(A'_1, ..., A'_m) = c(A_1, ..., A_m) + 1 = |A_1 \cap ... \cap A_m| + 1 = |A'_1 \cap ... \cap A'_m|.$$

In case 2), we have $c(A'_1, ..., A'_m) = c(A_1, ..., A_m)$ since there is no contribution to the new coefficient coming from the expansion of ψ_q^m in terms of boundary classes; hence the result follows.

□

3.1.1 An alternative description via hyperplane arrangements

Moduli space of pointed genus 0 curves can be constructed in terms of the De Concini-Procesi models, i.e., via the theory of hyperplane arrangements. We omit their definition since it is rather technical: for a detailed presentation see [4] and references therein. Here we briefly recall the relationship between $\overline{\mathcal{M}}_{0,n}$ and these models so to express the classes ψ_i and κ_i, $0 \leq i \leq n$, in terms of the combinatorics of the corresponding hyperplane arrangement.

The moduli space $\mathcal{M}_{0,n+1}$ can be viewed as the quotient of the set

$$\left\{(p_0, ..., p_n) \in \mathbf{P}^1 \times ... \times \mathbf{P}^1 : p_i \neq p_j, \forall i \neq j\right\}$$

modulo the group $PGL(2, \mathbf{C})$ which acts componentwise - in the sequel, we consider the marked points with indices in the set $\{0, 1, ..., n\}$. Note that this action identifies the moduli space of $(n+1)$-pointed rational curves with the set

$$\left\{(q_1, ..., q_{n-2}) \in \mathbf{P}^1 \times ... \times \mathbf{P}^1 : q_i \neq q_j, q_i \neq 1, 0, \infty\right\}. \tag{9}$$

Let us now consider $\mathbf{C}^n$ with the standard scalar product denoted by $(.,.)$ and the hyperplane arrangement given by the hyperplanes $z_{ij} : x_i - x_j = 0$, where $x_i \in (\mathbf{C}^n)^*$ are the coordinate functions. Moreover, be N the intersection of all the hyperplanes and $\pi : \mathbf{C}^n \to \mathbf{C}^n/N := V$ the projection onto the quotient. In [4], and with the same notation adopted there, the set in (3.1.1) is identified with the complement of the projective arrangement $\overline{\mathcal{A}}_{n-1} := \cup_{h,k=1,...,n} H_{hk}$, where $H_{hk} = \Psi(\pi(z_{hk}))$, with Ψ the projectivization map from V to $\mathbf{P}(V)$.

This description allows constructing a De Concini-Procesi model which is denoted by $\overline{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ in the literature. With the same notation of [5], the cohomology of such a model is generated over the integers by cohomology classes c_A, where A ranges over the subsets of $\{0, 1, \ldots, n+1\}$. Furthermore, as proved in [4], $\overline{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ is isomorphic to $\overline{\mathcal{M}}_{0,n+1}$. Let Φ be this isomorphism and denote by Φ^* the map induced in cohomology. Then

$$\Phi^*(\delta_{0,A}) = c_A,$$

for $A \subset \{1, ..., n\}$, and

$$-\Phi^*\big(\sum_{\{i,j\} \subset A \subset \{1,...,n\}} \delta_{0,A}\big) = c_{\{1,...,n\}}$$

for every $\{i, j\} \subset \{1, ..., n\}$, $i \neq j$. Although this definiton may seem rather strange, it is the natural correspondence between the second cohomology groups of $\overline{\mathcal{M}}_{0,n+1}$ and $\overline{Y}_{\mathcal{F}_{\mathcal{A}_{n-1}}}$.

We can now describe the image of the classes ψ_i's and κ_i's under the map Φ^*. To this end, let us denote by τ_j the transposition of S_{n+1} given by the exchange of the two marked points with indices 0 and j. We recall that there is a natural action of S_{n+1} on $\overline{\mathcal{M}}_{0,n+1}$ given as follows:

$$\tau_j \cdot [C; p_0, p_1, ..., p_j, ..., p_n] = [C; p_j, p_1, ..., p_0, \dots, p_n].$$

Proposition 2 *Let $z \in \{0, 1, ..., n\}$ and $j \in \{1, ..., n\}$. Then*

$$\Phi^*(\psi_z) = \tau_z \left(\sum_{\{j,z\} \subset B \subset \{1,...,n\}} c_{\{1,...,n\} \setminus B} \right),$$

with $1 \leq |B| \leq n - 2$.

Proof. Since

$$\tau_z \left(\sum_{\{j,z\} \subset B \subset \{1,...,n\}} c_{\{1,...,n\} \setminus B} \right) = \sum_{j \notin A \subset \{1,...,n\}, z \in A} c_A,$$

the result follows from Proposition 1.6 in [2].

□

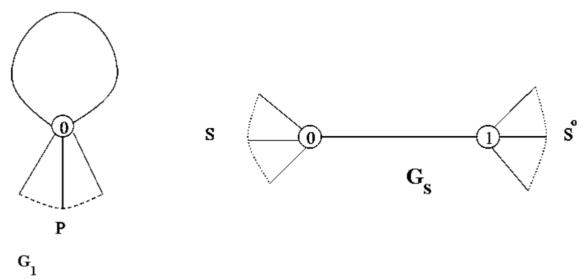
By Theorem 1, and with the same notation, we immediately get the following

Proposition 3 *For each $m \geq 0$,*

$$\Phi^*(\kappa_m) = \sum_{(A_1,...,A_m) \in \mathcal{A}_m(n)} c(A_1, ..., A_m) c_{A_1} \dots c_{A_2}.$$

3.2 The genus one case

In this section we give expressions for the Mumford classes and the powers of the universal cotangent classes in genus one. To this end, let us consider the stable graphs G_1 and G_S ($S \subset P$, $|S| \geq 2$)



As defined in (5), the morphisms associated with the graphs above will be denoted by

$$\xi_{G_1} : \overline{\mathcal{M}}_{0,P\cup\{q_1,q_2\}} \to \overline{\mathcal{M}}_{1,P}$$

and

$$\xi_{G_S} : \overline{\mathcal{M}}_{0,S\cup\{r_1\}} \times \overline{\mathcal{M}}_{1,S^c\cup\{r_2\}} \to \overline{\mathcal{M}}_{1,P}.$$

As shown in [2], we have

$$\psi_q = \frac{1}{24}\xi_{irr,*}(1) + \sum_{q\in S,|S|\geq 2} \xi_{S,*}(1),\ \forall p \in P.$$

In order to give relations in genus one we need some additional notation. If t is a stable graph such that $g(v) = 0$ for each vertex v and the half-edges are labelled by $P \cup \{a, b\}$, then we denote by $L(t)$ the stable graph obtained from t by identifying the half-edges of t labelled with a and b. Moreover, for a stable graph such that $g(v) = 0$ for each vertex v and the half-edges are labelled by $P \cup \{q_1\}$, we consider the stable graph $S(t)$ given as follows. Fix a subset S of P such that $|S| \geq 2$. For such an S, substitute in t the half-edge with label q_1 with an edge that ends with a vertex v ($g(v) = 1$) and with half-edges labelled by the elements of the set S^c, the complement of S in P. Then

Theorem 4 *i)*

$$\psi_q^{m+1} = \sum_{\substack{S_1,\dots,S_m,\\ |S_i|\geq 2}} \xi_{S_1,*}(1)\dots\xi_{S_m,*}(1) + \frac{m}{24}\xi_{irr,*} \sum_{\substack{S_1,\dots,S_{m-1},\\ |S_i|\geq 2}} \xi_{S_1,*}(1)\dots\xi_{S_{m-1},*}(1);$$

ii)

$$\begin{aligned}\kappa_m &= \frac{1}{24}\sum_{t\in G_{m-1,n+2}} C(m-1; a_1,\dots,a_m, n+2)\xi_{L(t),*}(1)\\ &+ \sum_{S,|S|\geq 2}\ \sum_{t\in G_{m-1,|S|+1}} C(m-1; a_1,\dots,a_m, |S|+1)\xi_{S(t),*}(1),\end{aligned}$$

where

$$C(m; a_1,\dots,a_{m+1}, n) = \frac{(m+1)!(a_1-1)\dots(a_{m+1}-1)a_1\dots a_{m+1}}{n(n-1)a_1(a_1+a_2)\dots(a_1+\dots+a_m)}.$$

Proof. i) By dimension computation, the Poincaré dual of the class $\xi_{irr,*}(1)$ is a point in $\overline{\mathcal{M}}_{1,1}$. Therefore, by pull-back under the map $\pi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,1}$ which forgets all marked points but the last one, we have

$$(\xi_{irr,*}(1))^2 = 0$$

on $\overline{\mathcal{M}}_{1,n}$ for any n. This completes the proof.

ii) As proved in [8], the following recursive relation holds:

$$\kappa_m = \frac{1}{24}\xi_{irr,*}(\kappa_{m-1}) + \sum_{S,|S|\geq 2} \xi_{S,*}(\kappa_{m-1}\otimes 1). \tag{10}$$

Let us denote by $G_{m,n}$ the collection of P-labelled stable graphs t ($|P| = n$) such that $g(v) = 0$ for each vertex v and with an ordering of the set of vertices. Additionally, every graph in $G_{m,n}$ has m edges - consequently $m+1$ vertices - and each vertex has at most two incident edges. As proved in [9],

$$\kappa_m = \sum_{t \in G_{m,n}} C(m; a_1, \ldots, a_{m+1}, n) \xi_{t,*}(1), \tag{11}$$

where a_i denotes the number of half-edges going out of the i-th vertex of t ($1 \leq i \leq m+1$), and ξ_t is the map corresponding to t as defined in (5).

By combining (10) and (11), we get

$$\begin{aligned} \kappa_m &= \frac{1}{24} \sum_{t \in G_{m-1,n+2}} C(m-1; a_1, \ldots, a_m, n+2) \xi_{irr,*}\left(\xi_{t,*}(1)\right) \\ &+ \sum_{S, |S| \geq 2} \sum_{t \in G_{m-1,|S|+1}} C(m-1; a_1, \ldots, a_m, |S|+1) \xi_{S(t),*}(\xi_{t,*} \otimes 1). \end{aligned} \tag{12}$$

By definition of ξ_{irr} and ξ_S, the claim follows.

□

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Korteweg-de Vries Instituut voor Wiskunde
Plantage Muidergracht 24
1018 TV Amsterdam (NL)
email:gbini@science.uva.nl