Kähler-Nijenhuis Manifolds

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

A Kähler-Nijenhuis manifold is a Kähler manifold M, with metric g, complex structure J and Kähler form Ω, endowed with a Nijenhuis tensor field A that is compatible with the Poisson structure defined by Ω in the sense of the theory of Poisson-Nijenhuis structures. If this happens, and if $AJ = \pm JA$, M is foliated by $\operatorname{im} A$ into non degenerate Kähler-Nijenhuis submanifolds. If A is a non degenerate $(1,1)$-tensor field on M, (M, g, J, A) is a Kähler-Nijenhuis manifold iff one of the following two properties holds: 1) A is associated with a symplectic structure of M that defines a Poisson structure compatible with the Poisson structure defined by Ω; 2) A and A^{-1} are associated with closed 2-forms. On a Kähler-Nijenhuis manifold, if A is non degenerate and $AJ = -JA$, A must be a parallel tensor field.

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1 Definition and basic formulas

A Kähler manifold is a particular case of a symplectic $2n$-dimensional manifold (M, Ω) with a symplectic form defined as

$$\Omega(X, Y) = g(JX, Y) \quad (X, Y \in \Gamma TM), \tag{1}$$

where Γ denotes the space of global cross sections, J is a complex structure on M, and g is a Hermitian metric on (M, J) [4]. Accordingly, on M one has the Poisson bivector field Π defined by the Poisson brackets computed with the symplectic form Ω. The aim of this note is to discuss Nijenhuis tensor fields A that are compatible with Π in the sense of the theory of Poisson-Nijenhuis manifolds [5, 6, 8, 9]. If this happens, the quadruple (M, g, J, A) will be called a *Kähler-Nijenhuis manifold*, and $A \in \Gamma\,\mathrm{End}(TM)$ will be a *Kähler-compatible Nijenhuis* (K.c.N.) tensor field. The interest in Poisson-Nijenhuis structures comes from their usefulness in the search of first integrals of Hamiltonian dynamical systems [6].

In what follows, we will use *musical morphisms* defined by formulas of the type

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(2) $$\beta(\sharp_\Pi\alpha) = \Pi(\alpha,\beta),\ (\flat_\Omega X)(Y) = \Omega(X,Y),$$

where Π may be any 2-contravariant and Ω any 2-covariant tensor field. In particular, (1) is equivalent with $\flat_g \circ J = \flat_\Omega$, and we have $\sharp_\Pi \circ \flat_\Omega = -Id$ and $\Pi = \sharp_\Pi\Omega = \sharp_g\Omega$ ($\sharp_g = \flat_g^{-1}$) i.e.,

(3) $$\Pi(\alpha,\beta) = \Omega(\sharp_\Pi\alpha, \sharp_\Pi\beta) = \Omega(\sharp_g\alpha, \sharp_g\beta).$$

Known results on Poisson-Nijenhuis structures [6, 9] tell that, on any symplectic manifold (M,Ω), the tensor field $A \in \Gamma\,\mathrm{End}(TM)$ defines a Poisson-Nijenhuis structure on M iff $A = \sharp_\Pi \circ \flat_\Theta$, where Θ is a closed, differential 2-form such that one of the following properties holds:
1) A is a Nijenhuis tensor field i.e.,

(4) $$\mathrm{Nij}_A(X,Y) = [AX,AY] - A[X,AY] - A[AX,Y] + A^2[X,Y] = 0;$$

2) $\sharp_\Pi\Theta$ is a Poisson bivector field, i.e.,

(5) $$[\sharp_\Pi\Theta, \sharp_\Pi\Theta] = 0,$$

where [,] is the Schouten-Nijenhuis bracket [7];
3) Θ is a complementary 2-form of Π i.e. [9],

(6) $$\{\Theta,\Theta\} = 0,$$

where { , } is the Koszul bracket [7];
4) the 2-form $\tilde\Theta$ defined by

(7) $$\flat_{\tilde\Theta} = \flat_\Theta \circ \sharp_\Pi \circ \flat_\Theta$$

is closed.

Thus, if we add the request that M is a Kähler manifold, the above conditions characterize K.c.N. tensor fields. Furthermore, a Kähler-Nijenhuis structure is also defined by the closed form Θ with the properties 1)-4). We will say that Θ is the *associated* K.c.N. form of A, which, in turn, is *associated* with Θ. Notice that

(8) $$\Theta(X,Y) = -\Omega(AX,Y) = -\Omega(X,AY)$$

(use the skew symmetry of Θ), and

(9) $$\tilde\Theta(X,Y) = -\Omega(AX,AY).$$

In the rest of the paper, all the encountered $(1,1)$-tensor fields A are supposed to satisfy the second equality (8), called the *Ω-skew-symmetry* of A. Ω-skew-symmetry ensures that $A = \sharp_\Pi \circ \flat_\Theta$, where Θ is a 2-form.

If (x^i) $(i = 1, ..., 2n)$ are local coordinates on M, characteristic property 2) becomes ([7], Proposition 1.5)

(10) $$\sum_{\mathrm{Cycl}(i,j,k)} \Omega^{uv}\Theta_{ui}\nabla_v\Theta_{jk} \overset{(8)}{=} \sum_{\mathrm{Cycl}(i,j,k)} A_i^u\nabla_u\Theta_{jk} = 0,$$

where ∇ is the Levi-Civita connection of g, and we use the Einstein summation convention. Thus, the Ω-skew-symmetric tensor field A is K.c.N. iff

$$(11) \qquad \sum_{\mathrm{Cycl}(X,Y,Z)} (\nabla_X \Theta)(Y,Z) = 0, \quad \sum_{\mathrm{Cycl}(X,Y,Z)} (\nabla_{AX}\Theta)(Y,Z) = 0,$$

$\forall X,Y,Z \in \Gamma TM$, where the first condition is equivalent to $d\Theta = 0$ and the second condition is the coordinate-free equivalent of (10). Notice also that, in view of (8), conditions (11) are equivalent to

$$(12) \qquad \sum_{Cycl(X,Y,Z)} \Omega[(\nabla_X A)(Y), Z] = 0, \quad \sum_{Cycl(X,Y,Z)} \Omega[(\nabla_{AX} A)(Y), Z] = 0,$$

respectively, and the Ω-skew-symmetric tensor field A is K.c.N. iff it satisfies (12).

On the other hand, characteristic property 3) has the interesting equivalent form [7, 9]

$$(13) \qquad \delta^C(\Theta \wedge \Theta) = 2(\delta^C\Theta) \wedge \Theta,$$

where δ is the Riemannian codifferential, and $\delta^C = C \circ \delta \circ C$, with C defined by the action of J on the arguments of a form (e.g., [2]).

On a Kähler manifold, it is natural to consider the following particular cases. We will say that a tensor field $A \in \Gamma\,\mathrm{End}(TM)$ is *complex-compatible* (c.c.) if $A \circ J = J \circ A$, and is *skew-complex-compatible* (s.c.c.) if $A \circ J = -J \circ A$. Furthermore, if $A = \sharp_\Pi \circ \flat_\Theta$ where Θ is a 2-form, A is c.c. iff Θ is of the complex type $(1,1)$ and A is s.c.c. iff Θ has components of the complex type $(2,0)$ and $(0,2)$ only. This means that $\Theta(JX, JY) = \pm\Theta(X,Y)$, respectively, and, if we denote by

$$(14) \qquad \mathcal{P} = \frac{1}{2}(Id \otimes Id + J \otimes J), \; \tilde{\mathcal{P}} = \frac{1}{2}(Id \otimes Id - J \otimes J)$$

the projectors of 2-covariant tensors onto their components of complex type $(1,1)$ and $[(2,0)+(0,2)]$ (each factor of the tensor product acts on the corresponding argument), such forms may be written as

$$(15) \qquad \Theta = \mathcal{P}\Xi, \; \Theta = \tilde{\mathcal{P}}\Xi,$$

respectively, where Ξ is an arbitrary 2-form on M. In both cases, we are speaking of a real form Θ, and we will say that Θ is c.c., in the first case, and s.c.c., in the second case.

In these two cases, the conditions that ensure the K.c.N. property may be written under specific forms. Let us denote

$$(16) \qquad \begin{gathered} E_A(X,Y) = (\nabla_X A)(Y), \; F_A(X,Y) = (\nabla_{AX} A)(Y), \\ B_A = alt(E_A), \; C_A = alt(F_A), \end{gathered}$$

where *alt* is the skew-symmetric part of a tensor. Then, we get

Proposition 1.1 *1. The Ω-skew-symmetric, c.c. tensor field A is K.c.N. iff*

$$(17) \qquad \tilde{\mathcal{P}}B_A = 0, \; \tilde{\mathcal{P}}C_A = 0.$$

2. The Ω-skew-symmetric, s.c.c. tensor field A is K.c.N. iff conditions (12) *hold* $\forall X,Y,Z \in \Gamma T^cM$ $(T^cM = TM \otimes \mathbf{C})$ *that are of the complex type* $(1,0)$, *and*

$$(18) \qquad \mathcal{P}E_A = 0, \; \mathcal{P}F_A = 0.$$

Proof. If A is c.c. the extension of A to T^cM preserves the complex type and, since $\nabla J = 0$, the same holds for the operators $\nabla_X A$, $\forall X \in \Gamma T^cM$. Now, using the fact that Ω has the complex type $(1,1)$, we see that conditions (12) are identically satisfied if X, Y, Z are of the same complex type. Furthermore, from the Ω-skew-symmetry of A it follows easily that, $\forall X \in \Gamma TM$, the tensor field $\nabla_X A$ is also Ω-skew-symmetric. This implies that, for two arguments, say X, Y, of the same complex type (e.g., $(1,0)$) and the third, Z, of opposite type ($(0,1)$), (12) is equivalent to $B_A(X,Y) = 0, C_A(X,Y) = 0$. This happens iff (17) holds.

Similarly, if A is s.c.c., A and $\nabla_X A$ change the complex type of the vectors from $(1,0)$ to $(0,1)$ and conversely. This implies that, for two arguments of the same type and the third of the opposite type, (12) is equivalent to $E_A(X,Y) = 0, F_A(X,Y) = 0$ whenever X, Y have opposite complex types. This is the same thing as conditions (18). Of course, we must still ask (12) to hold for three arguments of the same complex type. Q.e.d.

Notice also that in the c.c. and s.c.c. cases (13) becomes

$$\delta(\Theta \wedge \Theta) - 2(\delta\Theta) \wedge \Theta = 0. \tag{19}$$

We end this section by a number of examples.

Example 1.1 Any parallel 2-form Θ of a Kähler manifold is a K.c.N. form. In particular, if a Kähler manifold has a parallel Ricci tensor field, the Ricci form is a c.c. form that defines a Kähler-Nijenhuis structure.

Example 1.2 Let M be a hyper-Kähler manifold with the metric g, the parallel complex structures (J_1, J_2, J_3) that satisfy the quaternionic identities, and the respective Kähler forms $\Omega_1, \Omega_2, \Omega_3$. Then, the tensors J_2, J_3 are s.c.c., K.c.N. tensor fields on the Kähler manifold (M, g, J_1, Ω_1). The corresponding K.c.N. forms are the Kähler forms $-\Omega_3, \Omega_2$, which are parallel forms [1].

Example 1.3 On a compact Hermitian symmetric space, any real closed 2-form Θ which has no $(1,1)$-component is a K.c.N. form. Indeed, $d\Theta = 0$ implies that the $(2,0)$-component of Θ is holomorphic hence, harmonic (e.g., [2]). Therefore, Θ is harmonic, and, because the manifold is a compact Hermitian symmetric space, $\Theta \wedge \Theta$ is harmonic too (e.g., [3]). Thus, Θ satisfies condition (19). Moreover, since Θ is s.c.c., by a result that will be proven at the end of this paper, Θ is a parallel form.

Example 1.4 On a compact Hermitian symmetric space any real, harmonic $(1,1)$-form Θ is a c.c., K.c.N. form. (Use again the final argument of Example 1.3).

Example 1.5 On $M = \mathbf{C}^n$, with the flat Kähler metric and the natural complex coordinates (z^α), the $(1,1)$-form $\Theta = z^1 dz^1 \wedge d\bar{z}^2$ is closed and satisfies condition (19). Hence, Θ is a c.c., K.c.N. form. It is easy to check that Θ is not a parallel form.

2 Geometric properties

Let (M, g, J, A) be a Kähler-Nijenhuis manifold. The basic geometric object that we detect beyond the usual Kählerian objects is the differentiable, generalized distribution $\mathcal{A} = \operatorname{im} A$. It is well known that this distribution is completely integrable. For the record, we write down a straightforward proof below.

Proposition 2.1 *If A is a Nijenhuis tensor field (i.e., (4) holds), the generalized distribution $\mathcal{A} = \mathrm{im}\, A$ is completely integrable.*

Proof. Condition (4) shows that $\mathcal{A}$ is an involutive distribution. Hence, integrability will follow from the Sussmann-Stefan-Frobenius theorem (e.g., [7]) if we prove that $\mathcal{A}$ is invariant i.e., $\forall X, Y \in \Gamma TM$ one has $[\exp(tAX)]_*(AY) \in \mathcal{A}$, $\forall t \in \mathbf{R}$ such that $\exp(tAX)$ exists.

Denote $A_x(t) = [\exp(tAX)]_*(A_{\exp(-tAX)(x)})$ $(x \in M)$. Then,

$$\tag{20} \frac{dA_x(t)}{dt} = \lim_{s\to 0} \frac{1}{s}\{[\exp((t+s)AX)]_*(A_{\exp[-(t+s)AX](x)}) \\ -[\exp(tAX)]_*(A_{\exp(-tAX)(x)})\} = [L_{AX}A(t)]_x,$$

where L denotes the Lie derivative.

The required invariance of $\mathcal{A}$ will be a consequence of the local existence of a $(1,1)$-tensor field $C(t)$ such that $A(t) = A \circ C(t)$. If $C(t)$ exists, (20) implies

$$A \circ \frac{\partial C}{\partial t} = L_{AX}(A \circ C) = (L_{AX}A) \circ C + A \circ L_{AX}C$$

$$\stackrel{(4)}{=} A \circ L_X A \circ C + A \circ L_{AX}C.$$

Therefore, if $C(t)$ satisfies

$$\tag{21} \frac{\partial C}{\partial t} - (L_X A) \circ C - L_{AX}C = 0,\ C(0) = Id,$$

$\forall x \in M$, $A(t)$ and $A \circ C(t)$ satisfy the same differential equation (20) and the same initial condition, and must be equal. Since (21) has a local solution, we are done. Q.e.d.

Thus, through every point $x \in M$ one has a *characteristic leaf*, the maximal integral submanifold of the generalized distribution $\mathcal{A}$, which we denote by $\mathcal{L} = \mathcal{L}_x$, immersed in M by $\iota = \iota_{\mathcal{L}} : \mathcal{L} \hookrightarrow M$.

Proposition 2.2 *Let (M, g, J, A) be a c.c. or s.c.c. Kähler-Nijenhuis manifold. Then, each characteristic leaf $\mathcal{L}$ inherits an induced structure of a non degenerate Kähler-Nijenhuis manifold with the normal bundle $\mathcal{K}|_{\mathcal{L}}$, where $\mathcal{K} = \ker A = \ker \Theta$. Furthermore, if the structure A is regular, the decomposition $TM = \mathcal{A} \oplus \mathcal{K}$ is a complex analytic, orthogonal, locally product structure on M.*

Proof. In the case of a c.c. or s.c.c. tensor field A, the distribution $\mathcal{A}$ is J-invariant, and the characteristic leaves $\mathcal{L}$ are Kähler submanifolds of M. Then, $\forall X, Y \in \Gamma TM$, we have

$$\tag{22} g(AX, Y) \stackrel{(1)}{=} -\Omega(JAX, Y) = \mp\Omega(AJX, Y) \stackrel{(8)}{=} \mp\Theta(Y, JX),$$

and we see that $Y \in \mathcal{K} = \ker \Theta$ iff $Y \perp \mathcal{A}$. Therefore, the normal bundle of $\mathcal{L}$ is $\mathcal{K}|_{\mathcal{L}}$. Since $A = \sharp_\Pi \circ \flat_\Theta$ and $\sharp_\Pi$ is an isomorphism, we also have $\ker A = \ker \Theta$.

The field of planes $\mathcal{K}$, which, by the above result, is J-invariant, is not a differentiable distribution since its dimension is not lower semi-continuous. Differentiability occurs iff the c.c. or s.c.c. Nijenhuis tensor A (and the corresponding form Θ) is *regular* i.e., of a constant rank. Then, as it is well known, $d\Theta = 0$ implies that $\mathcal{K}$ is involutive, and the decomposition $TM = \mathcal{A} \oplus \mathcal{K}$ defines a complex analytic, orthogonal, locally product structure on M.

Of course, the distribution $\mathcal{A}$ also is invariant by A hence, $A|_{\mathcal{L}}$ is a $(1,1)$-tensor field on $\mathcal{L}$. Moreover $A|_{\mathcal{L}}$ is a Nijenhuis tensor, since the Lie brackets of condition (4) are ι-compatible. In the c.c. and s.c.c. cases $A|_{\mathcal{L}}$ has zero kernel because $\ker A \perp T\mathcal{L}$. Hence, $A|_{\mathcal{L}}$ is non degenerate, and so is the associated form $\Theta_{\mathcal{L}}$. Furthermore, formula (8) shows that $\Theta_{\mathcal{L}} = \iota^*\Theta$, and property 1) of the K.c.N. structures shows that $A|_{\mathcal{L}}$ is a K.c.N. tensor field. Q.e.d.

If the Nijenhuis tensor A is non degenerate, the manifold M itself is the only characteristic leaf. Furthermore, if (M, g, J, A) is a non degenerate Kähler-Nijenhuis manifold the corresponding K.c.N. form Θ is a symplectic form and the Poisson brackets of the latter define a Poisson bivector field Ψ.

Proposition 2.3 *Let A be a Ω-skew-symmetric, non degenerate $(1,1)$-tensor field on M. Then: 1. A is K.c.N. iff it is associated with a closed 2-form Θ and the Poisson structures defined by Ω, Θ are compatible i.e., $[\Pi, \Psi] = 0$.*
2. A is K.c.N. iff both A and A^{-1} are associated with closed 2-forms.

Proof. For *1*, by a result proven in [6] (see also [8]), the compatibility condition $[\Pi, \Psi] = 0$ implies the fact that A is K.c.N. Conversely, from $A = \sharp_\Pi \circ \flat_\Theta$ we get

$$\sharp_\Psi = A^{-1} \circ \sharp_\Pi. \tag{23}$$

Hence the Poisson structure Ψ belongs to the enlarged Poisson hierarchy of the Poisson-Nijenhuis structure (Π, A), and the required compatibility follows from the properties of the Poisson hierarchy (e.g., [8]).

For *2*, again, the theorem of the Poisson hierarchy tells us that if A is K.c.N. then A^{-1} is K.c.N. too. Hence, if we write $A^{-1} = \sharp_\Pi \circ \flat_{\Theta'}$, Θ' must be closed. Conversely, assume that Θ and Θ' are closed. From (23), we get

$$\flat_{\Theta'} = \flat_\Omega \circ \sharp_\Psi \circ \flat_\Omega, \tag{24}$$

therefore, by property 4) of the K.c.N. structures (see Section 1), (Ψ, A^{-1}) is a Poisson-Nijenhuis structure, and Π belongs to the Poisson hierarchy of the former. Therefore, $[\Psi, \Pi] = 0$ and, by part *1* of the proposition, we are done. (In fact, since Θ is a symplectic form, and in view of (23), (Ψ, A^{-1}) is a Poisson-Nijenhuis structure iff the Poisson bivector fields Ψ, Π are compatible.) Q.e.d.

Remark 2.1 In the c.c. case, conclusion *2* of Proposition 2.3 follows immediately from the first part of Proposition 1.1. Indeed, we can use $\nabla(A \circ A^{-1}) = 0$ to derive

$$C_A(X, Y) = A(B_{A^{-1}}(AX, AY)), \tag{25}$$

and conclude as required from (17). We also notice the formula

$$-A^{-1}(\mathrm{Nij}_A(X, Y)) = 2[B_{A^{-1}}(AX, AY) + B_A(X, Y)]. \tag{26}$$

Corollary 2.1 *Let $A \in \mathrm{End}(TM)$ define a c.c., orthogonal, almost product structure on M. Then, the tensor field A is associated with a $(1,1)$-form Θ, and A is K.c.N. iff Θ is closed*

Proof. The orthogonality of the structure means $g(AX, AY) = g(X, Y)$, and it implies that $\Theta(X, Y) = -\Omega(AX, Y)$ is skew symmetric. Thus, Θ is the required 2-form. Furthermore, $A^{-1} = A$, and the result follows from part *2* of Proposition 2.3. Q.e.d.

In particular, if Θ of Corollary 2.1 is closed $Nij_A = 0$ and the almost product structure A is integrable.

We finish by showing that for the non degenerate, s.c.c. tensor fields the K.c.N. condition is very restrictive.

Proposition 2.4 *A non degenerate, Ω-skew-symmetric, s.c.c. tensor field $A \in \Gamma$ $\mathrm{End}(TM)$ is K.c.N. iff A is parallel.*

Proof. The quickest way to conclude is by a local computation. Consider local, complex analytic coordinates (z^α) $(\alpha = 1, ..., n)$ on M. The s.c.c. property of A means that the only possibly non-zero components of A are $(A^{\bar\beta}_\alpha,\, A^{\beta}_{\bar\alpha})$, and Θ has no component of the complex type $(1,1)$. Since $d\Theta = 0$, the complex $(2,0)$-component of Θ is holomorphic. Accordingly, condition (10) becomes

$$A^\alpha_\lambda \nabla_\alpha \Theta_{\mu\nu} = 0, \tag{27}$$

and if A is non degenerate we get $\nabla_\alpha \Theta_{\mu\nu} = 0$. Q.e.d.

Remark 2.2 Except for Proposition 2.2, the results of this note also hold for *pseudo-Kähler manifolds* i.e., where the metric g is non degenerate but it may not be positive definite.

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